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AXISYMMETRIC STRAINING MODES IN THE VIBRATION OF CIRCULAR PLATES

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Axisymmetric straining modes, which are different from the axisymmetric flexural modes of vibrating circular plates and cannot be revealed by conventional plate theory-based approaches, are investigated in the present study. Mode shapes of an example problem with clamped and four different simply-supported boundary conditions are presented to show what straining modes may look like. The straining modes are classified into three types and a criterion is set up to predict the appearance of straining modes. Some observations about straining modes are obtained and discussions are made. Results for different Poisson ratios are also shown.

1. INTRODUCTION

In a previous paper [1] concerning the axisymmetric vibration of circular and annular plates, some modes, which are different from conventional, flexural vibration modes, were found and named as straining modes. They are so called because, from their mode shapes, the tensile and compressive behaviors of the plate itself in the radial or axial direction are more obvious than the flexural motion of the plate. Some similar types of vibration modes have been shown to appear in the vibration analyses of cantilevered rectangular parallelpipeds, and rods and beams with uniform boundary conditions using the three-dimensional method of Leissa *et al*. [2, 3]. However, to the authors' knowledge, such modes for circular plates have not been reported in the literature before, [4–9] since, e.g., the boundary conditions specified in this paper have not been mentioned, and it is also impossible for them to appear in the vibration analysis of circular plates using plate theory-based approaches. In the present study, the first ten mode shapes under five different boundary conditions are shown and the straining modes are investigated and classified. Also, a criterion is established to judge whether a mode is a straining mode.

2. APPROACH

The axisymmetric finite element is employed in the present study. For axisymmetric vibration of a circular plate, the displacement field is

$$u = u(r, z, t), \qquad w = w(r, z, t) \tag{1}$$

where u and w are the displacements in the radial and the thickness directions. r and z denote the co-ordinates in these directions, respectively, and t is the time variable. The circumferential displacement and co-ordinates do not enter into the formulation.

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Proceeding with the above displacements, one uses the following strain–displacement relations,

$$\varepsilon_r = \frac{\partial u}{\partial r}, \qquad \varepsilon_z = \frac{\partial w}{\partial z}, \qquad \varepsilon_\theta = \frac{u}{r}, \qquad \gamma_{Iz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \qquad \gamma_{r\theta} = \gamma_{z\theta} = 0, \tag{2}$$

and the stress–strain relations,

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 \\ c_{12} & c_{22} & c_{23} & 0 \\ c_{13} & c_{23} & c_{33} & 0 \\ 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix}, \tag{3}$$

where

$$c_{11} = c_{22} = c_{33} = \frac{E(1-v)}{(1+v)(1-2v)}, \qquad c_{12} = c_{13} = c_{23} = \frac{Ev}{(1+v)(1-2v)}, \qquad c_{66} = G, \tag{4}$$

and then substitute (2) and (3) into Hamilton's principle

$$0 = \int_0^t \left[\iint_{vol} [(\sigma_r\,\delta\varepsilon_r + \sigma_\theta\,\delta\varepsilon_\theta + \sigma_z\,\delta\varepsilon_z + \tau_{rz}\,\delta\gamma_{rz}) - (\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w})]\,\mathrm{d}V \right] \mathrm{d}t \tag{5}$$

The above variational form can be expressed in terms of the displacements u and w. These primary variables are then expressed as the products of their nodal values u_i, w_i and shape functions N_i as is done in the usual finite element procedures,

$$(u, w) = \sum_{i=1}^{n} N_i\,(u_i, w_i) \tag{6}$$

One can obtain the element equation of motion

$$[\mathbf{m}]\{\ddot{\mathbf{U}}\} + [\mathbf{k}]\{\mathbf{U}\} = \mathbf{0} \tag{7}$$

where $\{\mathbf{U}\}^{\mathrm{T}} = [u_1, u_2, \ldots, u_n; w_1, w_2, \ldots, w_n]$, n is the number of nodes in an element, and $[\mathbf{m}]$, $[\mathbf{k}]$ are the elemental mass and stiffness matrices.

Assembling all the element equations, one derives the global equation of motion

$$[\mathbf{M}]\{\ddot{\mathbf{X}}\} + [\mathrm{K}]\{\mathrm{X}\} = 0 \tag{8}$$

which then corresponds to an eigenvalue equation of the following form:

$$[\mathbf{K}]\{\mathbf{X}\} = \lambda[\mathrm{M}]\{\mathrm{X}\} \tag{9}$$

The eigenvalue λ denotes the square of vibration frequency ω. For more details of the formulation, please refer to [1].

3. EXAMPLES AND STRAINING MODES

To show what straining modes may look like, the vibrations of circular plates with $E = 210$ GPa, $v = 0{\cdot}3$, $\rho = 7810$ kg/m^3 and radius to thickness ratio $a/h = 5$ are analysed for five different boundary conditions (see Fig. 1). The finite element used is the 8-node quadratic isoparametric element [10], square in shape with side length of the finite element equal to the thickness of the plate, so that there are 5 elements in the mesh. The first ten

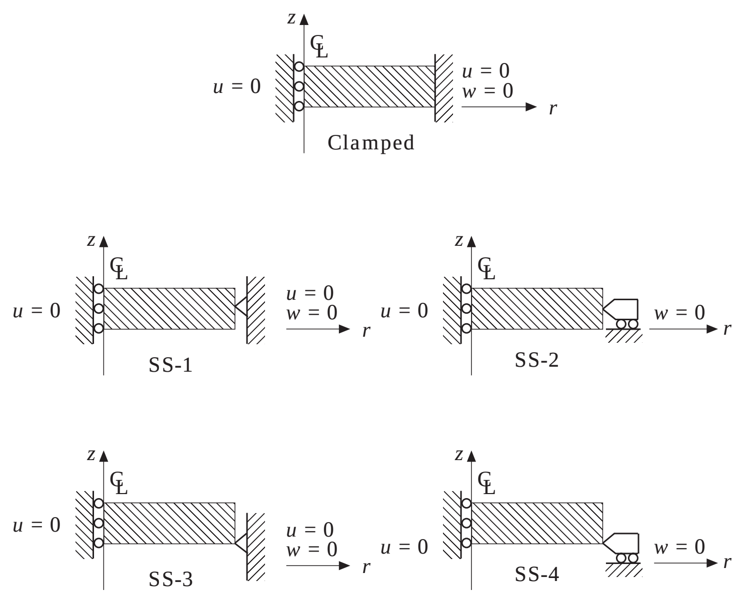

Figure 1. Boundary conditions used.

vibration modes for clamped, SS-1, SS-2, SS-3 and SS-4 boundary conditions are shown in Fig. 2–6, respectively. From these figures, one may observe that straining modes are in general not difficult to identify, and may be classified into the following three types: (1) radial extension modes, (2) thickness extension modes, (3) radial shear modes.

Radial extension modes are mainly of radial tension and compression such as the third mode of the SS-2 case (Fig. 4), while the thickness extension modes represent mainly the tension and compression in the thickness (axial) direction. The eighth mode of the clamped

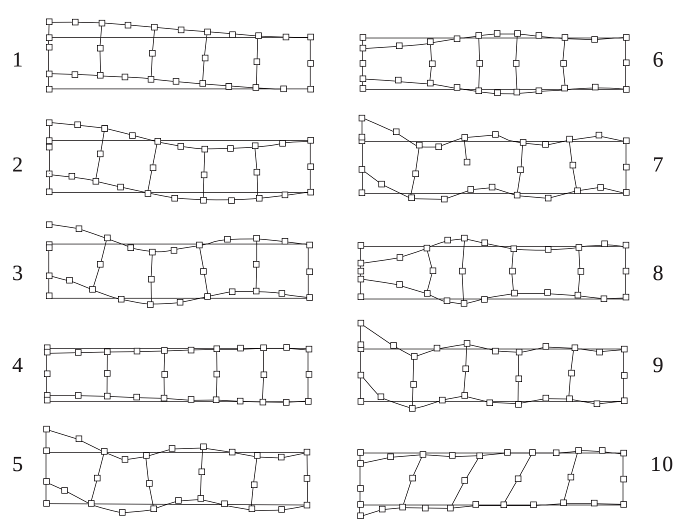

Figure 2. The first ten modes for $a/h = 5$, $\nu = 0{\cdot}3$ and the clamped boundary condition.

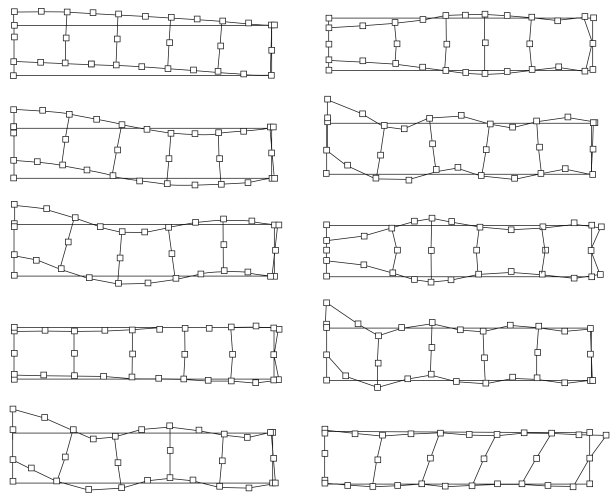

Figure 3. The first ten modes for $a/h = 5$, $\nu = 0{\cdot}3$ and the SS-1 boundary condition.

case (Fig. 2) is an example of the thickness extension mode. As to the radial shear modes, an example is the tenth mode of the SS-1 case in Fig. 3 in which the overall deformation of the top surface in the radial direction is opposite to that of the bottom surface.

However, it is noteworthy that very few straining modes belong to any single one of the above three modes. In general, the radial extension mode are mixed with the thickness extension mode to form a straining mode with little flexural motion, and the radial shear

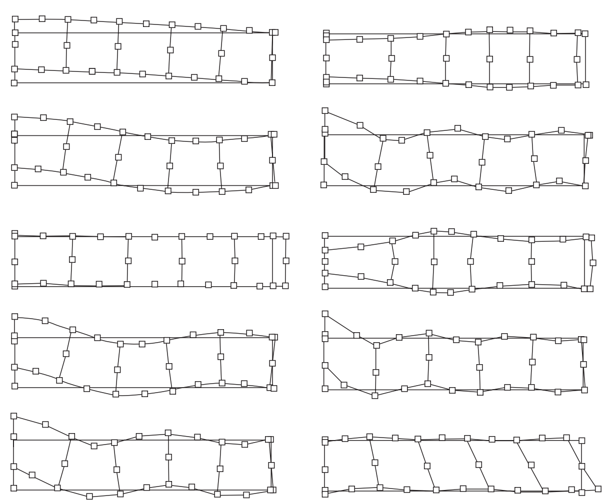

Figure 4. The first ten modes for $a/h = 5$, $\nu = 0{\cdot}3$ and the SS-2 boundary condition.

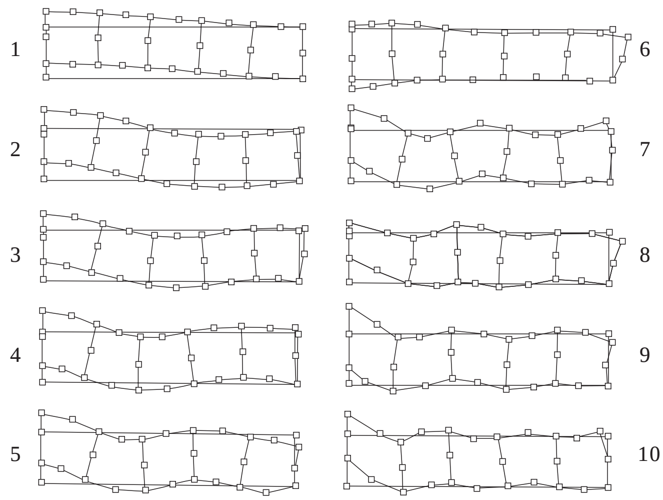

Figure 5. The first ten modes for $a/h = 5$, $\nu = 0{\cdot}3$ and the SS-3 boundary condition.

modes tend to occur with more or less flexural deformation. Therefore, one may consider that all vibration modes could be combinations of straining modes and flexural modes. Flexural motion dominates in some modes while straining deformation does in other modes.

But, how can one tell if a mode is a straining mode except by showing the mode shapes, which is sometimes not so convenient? If one examines these mode shapes more carefully, one finds that straining modes have the characteristics of less flexural motion and more

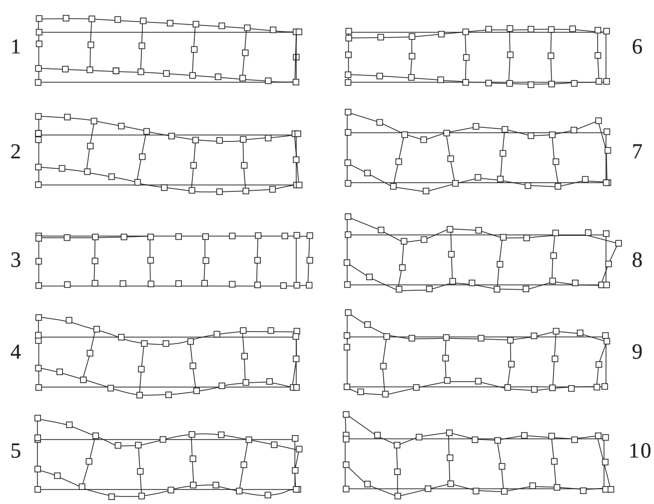

Figure 6. The first ten modes for $a/h = 5$, $\nu = 0{\cdot}3$ and the SS-4 boundary condition.

radial motion. A quantity, called RMD (ratio of modal displacements), is therefore set up and defined as the ratio of averaged absolute transverse displacement per node along the midplane to averaged absolute radial displacement per node along the top and bottom surfaces.

Although displacements are not characteristic in vibration analysis, the ratios of the absolute displacements in the transverse direction and the radial direction could be a meaningful quantity in making the judgement whether a mode is a straining mode. This is justified in Table 1 where, for each small value of RMD, one can correspondingly find a straining mode appearing in the figures of the mode shapes, and vice versa. The lower the value of RMD, the more obvious is the mode a straining mode. For the cases analysed in the present study, RMD for most straining modes is zero within the arithmatic accuracy of the computer. However, the appearance of some values of RMD, which are not small enough to be considered as showing straining modes, nor do they seem to be flexural modes, may represent the existence of mixed modes.

From Table 1, one may also observe that constraints along the bottom edge (SS-3 and SS-4 cases) reduce the number of straining modes in the first ten modes, and the absence of constraints in the radial direction (SS-2 and SS-4 cases) seems to trigger the first straining mode to appear earlier. The third and fourth modes of SS-3 are also noticeable. They have similar flexural deformation but different radial straining. These are the modes which are unlikely to be obtained simultaneously in the vibration analysis by the conventional plate theory-based approaches.

During the present investigation, one also finds that changing the radius-to-thickness ratio, Poisson ratio, or finite element mesh may make straining modes appear earlier or later. However, for the sake of brevity, only the results for $v = 0{\cdot}2$ and $0{\cdot}4$ with the same a/h and finite element mesh as the previous $v = 0{\cdot}3$ cases are shown in Tables 2 and 3.

4. DISCUSSIONS AND CONCLUSION

In the present study, some aspects of the straining modes which were found during the vibration analysis of circular plates by the axisymmetric finite element have been shown. The approach is based on three-dimensional elasticity and no assumptions or approximations have been made. From the present results, one may observe that not only do such straining modes exist, but they can also appear at a mode as low as the third one.

TABLE 1

The values of the non-dimensional frequencies $\bar{\omega}$ *and* RMD *for* $a/h = 5$ *and* $v = 0{\cdot}3$, $\bar{\omega} = \omega(\rho h a^4/D)^{1/2}$, $D = Eh^3/12\ (1 - v^2)$

	Clamped		SS-1		SS-2		SS-3		SS-4	
Mode	$\bar{\omega}$	RMD	$\bar{\omega}$	RMD	$\bar{\omega}$	RMD	$\bar{\omega}$	RMD	$\bar{\omega}$	RMD
1	9·563	6·583	4·802	6·446	4·802	6·446	7·082	6·478	4·762	6·537
2	31·82	3·430	25·60	3·191	25·60	3·191	26·41	3·020	24·88	3·363
3	60·74	3·032	55·15	2·720	35·42	0·000	46·41	1·485	35·42	0·042
4	66·59	0·000	63·12	0·000	55·15	2·720	58·53	1·961	51·75	3·067
5	93·86	2·926	89·88	2·566	89·88	2·566	80·21	2·529	79·60	2·495
6	120·9	0·000	114·9	0·000	92·47	0·000	98·10	0·262	92·47	0·051
7	131·1	3·021	128·8	2·758	128·8	2·758	109·4	2·591	108·5	2·398
8	173·0	0·000	165·3	0·000	145·1	0·000	144·6	1·599	144·2	2·631
9	175·8	5·573	175·4	5·510	175·4	5·510	152·9	3·649	145·3	0·955
10	193·3	0·257	181·0	0·162	181·0	0·162	185·3	2·538	177·1	2·695

TABLE 2

The values of the non-dimensional frequencies $\bar{\omega}$ *and* RMD *for* $a/h = 5$ *and* $\nu = 0{\cdot}2$, $\bar{\omega} = \omega(\rho h a^4/D)^{1/2}$, $D = Eh^3/12(1 - \nu^2)$

	Clamped		SS-1		SS-2		SS-3		SS-4	
Mode	$\bar{\omega}$	RMD	$\bar{\omega}$	RMD	$\bar{\omega}$	RMD	$\bar{\omega}$	RMD	$\bar{\omega}$	RMD
1	9·479	6·539	4·661	6·420	4·661	6·420	6·938	6·460	4·628	6·503
2	31·81	3·331	25·63	3·134	25·63	3·134	26·25	2·979	24·93	3·304
3	61·17	2·878	55·52	2·613	34·35	0·000	46·28	1·380	34·35	0·029
4	66·45	0·000	63·08	0·000	55·52	2·613	57·97	2·015	52·13	2·982
5	94·99	2·721	90·61	2·397	90·61	2·397	80·56	2·415	80·26	2·388
6	121·4	0·000	115·2	0·000	92·73	0·000	98·09	0·300	92·73	0·032
7	132·4	2·683	129·2	2·410	129·2	2·410	110·3	2·386	108·9	2·215
8	175·7	0·000	166·7	0·000	147·3	0·000	146·4	1·546	145·0	2·540
9	177·9	5·029	177·3	5·109	177·3	5·109	155·4	3·201	147·4	0·175
10	197·8	0·297	185·6	0·093	185·6	0·093	187·8	2·874	179·6	2·995

TABLE 3

The values of the non-dimensional frequencies $\bar{\omega}$ *and* RMD *for* $a/h = 5$ *and* $\nu = 0{\cdot}4$, $\bar{\omega} = \omega(\rho h a^4/D)^{1/2}$, $D = Eh^3/12(1 - \nu^2)$

	Clamped		SS-1		SS-2		SS-3		SS-4	
Mode	$\bar{\omega}$	RMD	$\bar{\omega}$	RMD	$\bar{\omega}$	RMD	$\bar{\omega}$	RMD	$\bar{\omega}$	RMD
1	9·785	6·643	4·930	6·486	4·930	6·486	7·229	6·515	4·885	6·579
2	32·08	3·611	25·50	3·280	25·50	3·280	26·55	3·065	24·79	3·443
3	60·32	3·243	54·55	2·884	36·37	0·000	46·47	1·617	36·37	0·057
4	66·94	0·000	63·27	0·000	54·55	2·884	59·33	1·937	51·31	3·183
5	92·25	3·264	88·74	2·868	88·74	2·868	80·16	2·695	78·97	2·695
6	120·1	0·000	114·2	0·000	91·62	0·000	97·85	0·168	91·63	0·081
7	129·7	3·732	128·2	3·441	128·2	3·441	108·8	2·937	108·3	2·726
8	169·7	0·000	162·5	0·000	141·3	0·000	141·3	1·517	141·1	0·932
9	173·6	6·854	172·6	3·216	172·6	3·216	149·6	4·393	143·4	3·006
10	186·9	0·179	174·8	1·051	174·8	1·051	181·2	1·850	173·1	2·213

Therefore, it is very possible that the straining modes can contribute to the dynamic stresses developed during vibration. In that case, straining modes may play a significant role in some particular occasions, e.g., the straining modes can enhance the possibility of crack extension during vibration of a circular plate, especially when excitation is near the frequency of a straining mode. It is also noteworthy that straining modes do not show up when using the conventional plate theory-based methods. It is therefore suggested that, to show the complete facets of the vibration of circular plates, three-dimensional analyses should be considered.

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