Journal of Sound and Vibration (1997) **208**(2), 331–338

LETTERS TO THE EDITOR

ALTERNATIVE FORMULATIONS OF THE FREQUENCY EQUATION OF LONGITUDINALLY VIBRATING RODS COUPLED BY A DOUBLE SPRING–MASS SYSTEM

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(*Received* 25 *February* 1997, *and in final form* 15 *May* 1997)

1. INTRODUCTION

The study of the dynamical behavior of longitudinally vibrating rods has stimulated the interest of researchers for a long time. In reference [1] Laura *et al*. investigated a system consisting of a spring and a longitudinally vibrating rod with a mass attached at the other end, moving axially with constant velocity. By following a one dimensional wave approach they determined the resulting dynamic stress field if the free end of the spring is suddenly stopped.

Recently, an interesting study by Kukla *et al*. [2] was published on the problem of the natural longitudinal vibrations of two rods coupled by many translational springs, where the Green's function method was employed. Motivated by this publication, the study in [3] dealt with a similar system which was made up of two clamped–free longitudinally vibrating rods carrying tip masses to which a double spring–mass system was attached as a secondary system across the span. After setting up the frequency equation of the system using a boundary value problem formulation, the effects of the variation of some system parameters upon the natural frequencies were investigated through numerical examples.

The present study is concerned essentially with the same mechanical system described in reference [3], but here, the tip masses are ignored to simplify the formulations. That this does not mean any restriction of the main idea of the present note, will be seen clearly from the derivations in the next sections. The exact frequency equation of the system was included in [3] as a special case. The attribute 'exact' will be used in the sense that the frequency equation is obtained by means of a boundary value problem. The aim of the present work is to give two other formulations of the frequency equation of the system described above. Both formulations are based on the discretization of the elastic rods by their first n eigenfunctions, according to the assumed modes method.

The system described can be viewed as an approximate model for the calculation of the natural frequencies in the longitudinal direction of an industrial sewing machinery with parallel needles on a single shaft or a stamping press with two parallel shafts where the coupling stiffness and the mass can represent the effect of the material being sawn or stamped.

2. THEORY

The problem to be dealt with in the present note is the natural vibration problem of the system shown in Figure 1. It consists of two clamped–free axially vibrating elastic rods to which a double spring–mass secondary system is attached across the span. The length, mass per unit length, location of the spring attachment point and axial rigidity of the ith rod are L_i, m_i, $\eta_i L_i$ and $E_i A_i$ $(i = 1, 2)$, respectively. The secondary system consists of two springs of stiffnesses k_1, k_2 and the mass m_e.

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The kinetic and potential energies of the system are

$$T = \tfrac{1}{2} m_1 \int_0^{L_1} \dot{u}_1^2(x, t)\,\mathrm{d}x + \tfrac{1}{2} m_2 \int_0^{L_2} \dot{u}_2^2(x, t)\,\mathrm{d}x + \tfrac{1}{2} m_e \dot{z}_1^2 \tag{1}$$

$$V = \tfrac{1}{2} E_1 A_1 \int_0^{L_1} u_1'^2(x, t)\,\mathrm{d}x + \tfrac{1}{2} E_2 A_2 \int_0^{L_2} u_2'^2(x, t)\,\mathrm{d}x + \tfrac{1}{2} k_1 (z_1 - z_0)^2 + \tfrac{1}{2} k_2 (z_2 - z_1)^2, \tag{2}$$

where $u_1(x, t)$ and $u_2(x, t)$ represent the longitudinal displacement over the two rods and $z_1(t)$ denotes the displacement of the appended mass m_e. Finally, $z_0(t)$ and $z_2(t)$ are the displacements of the attachment points of the springs k_1 and k_2 to the rods. Dots and primes denote, as usually, partial derivates with respect to time t and position co-ordinate x, respectively.

The longitudinal displacement of the rods at point x are assumed to be expressible in the form of finite series

$$u_1(x, t) = \sum_{i=1}^{n} U_{i1}(x)\eta_{i1}(t), \qquad u_2(x, t) = \sum_{i=1}^{n} U_{i2}(x)\eta_{i2}(t), \tag{3}$$

where

$$U_{i1}(x) = \sqrt{2/m_1 L_1}\,\sin (2i - 1)\pi/2\; x/L_1, \qquad U_{i2}(x) = \sqrt{2/m_2 L_2}\,\sin (2i - 1)\pi/2\; x/L_2 \tag{4}$$

are the mass orthonormalized eigenfunctions of a clamped–free elastic rod and $\eta_{i1}(t)$, $\eta_{i2}(t)$ $(i = 1, \ldots, n)$ denote generalized co-ordinates to be determined.

If the assumed series solutions (3) are substituted into the energy equations (1) and (2), they can be expressed as

$$T = \frac{1}{2} \sum_{i=1}^{n} \dot{\eta}_{i1}^2 + \frac{1}{2} \sum_{i=1}^{n} \dot{\eta}_{i2}^2 + \frac{1}{2} m_e \dot{z}_1^2, \tag{5}$$

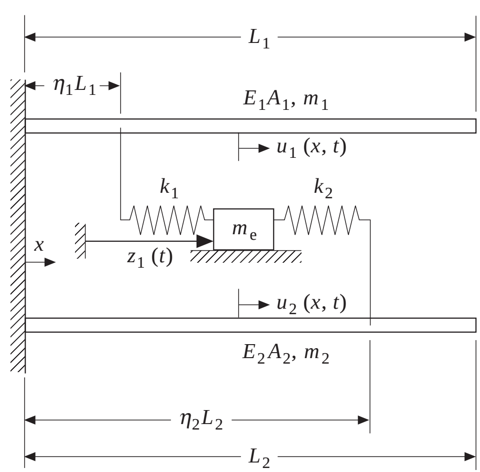

Figure 1. Two clamped-free, longitudinally vibrating elastic rods to which a double spring–mass system is attached across the span

$$V = \frac{1}{2}\sum_{i=1}^{n} \omega_{i1}^2 \eta_{i1}^2 + \frac{1}{2}\sum_{i=1}^{n} \omega_{i2}^2 \eta_{i2}^2 + \frac{1}{2} k_1 (z_1 - z_0)^2 + \frac{1}{2} k_2 (z_2 - z_1)^2, \tag{6}$$

where the orthonormalization properties of the eigenfunctions $U_{i1}(x)$ and $U_{i2}(x)$ are taken into account. Henceforth, starting with the energy expressions above, two different approaches will be used which lead to two alternative forms of the frequency equation of the system.

The first alternative form of the frequency equation follows directly from the formalism of the Langrange equations where the displacements of the attachment points of the double spring–mass system to the rods are expressed in terms of the generalized co-ordinates [4]. The formulation leads to a standard eigenvalue problem, the solution of which gives the eigenfrequency parameters of the system.

The second formulation uses the approach of Dowell [5] which is essentially based on the assumed—modes method in conjunction with the Lagrange multipliers method. The result is a simple analytical formula for the frequency equation of the system. Hence, the eigenfrequency parameters of the system are determined by solving this non-linear equation.

3. FIRST ALTERNATIVE FORM OF THE FREQUENCY EQUATION

The kinetic and potential energies of the system, i.e., expressions (5) and (6) can be written in matrix notation as

$$T = \tfrac{1}{2}\dot{\boldsymbol{\eta}}_1^T \mathbf{I}_n \dot{\boldsymbol{\eta}}_1 + \tfrac{1}{2}\dot{\boldsymbol{\eta}}_2^T \mathbf{I}_n \dot{\boldsymbol{\eta}}_2 + \tfrac{1}{2} m_e \dot{z}_1^2. \tag{7}$$

$$V = \tfrac{1}{2}\boldsymbol{\eta}_1^T \boldsymbol{\Omega}_1^2 \boldsymbol{\eta}_1 + \tfrac{1}{2}\boldsymbol{\eta}_2^T \boldsymbol{\Omega}_2^2 \boldsymbol{\eta}_2 + \tfrac{1}{2} k_1 (z_1 - z_0)^2 + \tfrac{1}{2} k_2 (z_2 - z_1)^2, \tag{8}$$

where

$$\boldsymbol{\eta}_1^T(t) = [\eta_{11}(t), \ldots, \eta_{n1}(t)], \qquad \boldsymbol{\eta}_2^T(t) = [\eta_{12}(t), \ldots, \eta_{n2}(t)],$$
$$\Omega_1^2 = \mathbf{diag}(\omega_{i1}^2),\ \Omega_2^2 = \mathbf{diag}(\omega_{i2}^2) \qquad (i = 1, \ldots, n) \tag{9}$$
$$\mathbf{I}_n : n \times n \text{ identity matrix}$$

Here, ω_{i1} and ω_{i2} represent the ith eigenfrequency of the bare upper and lower rod, respectively. The above energy expressions can be written in a more compact form as

$$T = \tfrac{1}{2}\dot{\boldsymbol{\eta}}^T \mathbf{I}_{2n} \dot{\boldsymbol{\eta}} + \tfrac{1}{2} m_e \dot{z}_1^2, \qquad V = \tfrac{1}{2}\boldsymbol{\eta}^T \boldsymbol{\Omega}^2 \boldsymbol{\eta} + \tfrac{1}{2} k_1 (z_1 - z_0)^2 + \tfrac{1}{2} k_2 (z_2 - z_1)^2, \tag{10, 11}$$

where

$$\boldsymbol{\eta}^T = [\boldsymbol{\eta}_1^T \boldsymbol{\eta}_2^T], \qquad \boldsymbol{\Omega}^2 = \mathbf{diag}(\boldsymbol{\Omega}_1^2, \boldsymbol{\Omega}_2^2) \tag{12}$$

and $\mathbf{I}_{2n}$ denotes the $2n \times 2n$ identity matrix.

The idea behind this approach is to express the displacements of the spring attachment points on to the rods, i.e., $z_0(t)$ and $z_2(t)$ in terms of the generalized co-ordinate vector $\boldsymbol{\eta}(t)$:

$$z_0(t) = \mathbf{I}_1^T \boldsymbol{\eta}(t), \qquad z_2(t) = \mathbf{I}_2^T \boldsymbol{\eta}(t). \tag{13, 14}$$

where the $2n \times 1$ vectors $\mathbf{I}_1$ and $\mathbf{I}_2$ are introduced as

$$\mathbf{I}_1^T = [\mathbf{U}_1^T(\eta_1 L_1), \mathbf{0}^T], \qquad \mathbf{I}_2^T = [\mathbf{0}^T, \mathbf{U}_2^T(\eta_2 L_2)]. \tag{15, 16}$$

Here, the $n \times 1$ vectors $\mathbf{U}_1(x)$ and $\mathbf{U}_2(x)$ are defined as

$$\mathbf{U}_1^T(x) = [U_{11}(x), \ldots, U_{n1}(x)], \qquad \mathbf{U}_2^T(x) = [U_{12}(x), \ldots, U_{n2}(x)]. \tag{17, 18}$$

Starting with the energy expressions (10) and (11), along with (13) to (18) the following matrix differential equation is obtained, by using the Lagrange equation formalism:

$$\left[\begin{array}{c:c} \mathbf{I}_{2n} & \mathbf{0} \\ \hdashline \mathbf{0}^T & m_e \end{array}\right]\left[\begin{array}{c} \ddot{\boldsymbol{\eta}} \\ \hdashline \ddot{z}_1 \end{array}\right] + \left[\begin{array}{c:c} \boldsymbol{\Omega}^2 + k_1\mathbf{l}_1\mathbf{l}_1^T + k_2\mathbf{l}_2\mathbf{l}_2^T & -(k_1\mathbf{l}_1 + k_2\mathbf{l}_2) \\ \hdashline -(k_1\mathbf{l}_1^T + k_2\mathbf{l}_2^T) & k_1 + k_2 \end{array}\right]\left[\begin{array}{c} \boldsymbol{\eta} \\ \hdashline z_1 \end{array}\right] = \mathbf{0}. \tag{19}$$

It is worth nothing that in obtaining the above form, extensive use is made of the formulas regarding the partial derivatives of bilinear forms, quadratic forms and vectors with respect to algebraic vectors [6].

By means of the transformation

$$\left[\begin{array}{c} \boldsymbol{\eta} \\ \hdashline z_1 \end{array}\right] = \left[\begin{array}{c:c} \mathbf{T} & \mathbf{0} \\ \hdashline \mathbf{0}^{\mathrm{T}} & 1/\sqrt{m_e} \end{array}\right]\left[\begin{array}{c} \mathbf{p} \\ \hdashline y \end{array}\right], \tag{20}$$

where $\mathbf{T} = \mathbf{I}_{2n}$, the equations of motion in (19) can be written as

$$\left[\begin{array}{c} \ddot{\mathbf{p}} \\ \hdashline \ddot{y} \end{array}\right] + \left[\begin{array}{c:c} \boldsymbol{\Omega}^2 + k_1\mathbf{e}_1\mathbf{e}_1^T + k_2\mathbf{e}_2\mathbf{e}_2^T & -(1/\sqrt{m_e})(k_1\mathbf{e}_1 + k_2\mathbf{e}_2) \\ \hdashline -(1/\sqrt{m_e})(k_1\mathbf{e}_1^T + k_2\mathbf{e}_2^T) & \omega_e^2 \end{array}\right]\left[\begin{array}{c} \mathbf{p} \\ \hdashline y \end{array}\right] = \mathbf{0}. \tag{21}$$

Here, the following abbreviations are introduced

$$\mathbf{e}_1 = \mathbf{T}^T\mathbf{l}_1 = \mathbf{l}_1, \qquad \mathbf{e}_2 = \mathbf{T}^T\mathbf{l}_2 = \mathbf{l}_2, \qquad \omega_e^2 = (k_1 + k_2)/m_e \tag{22}$$

Harmonic solutions of the form

$$\begin{bmatrix} \mathbf{p} \\ y \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{y} \end{bmatrix} \mathrm{e}^{\mathrm{i}\omega t} \tag{23}$$

results in a set of homogeneous equations for the amplitude vector $\bar{\mathbf{p}}$ and $\bar{y}$. A non-trivial solution of this set is possible only if the determinant of the coefficient matrix vanishes. This condition leads to the following form of the frequency equation of the mechanical system.

$$\left|\begin{array}{c:c} \boldsymbol{\Omega}^2 + k_1\mathbf{e}_1\mathbf{e}_1^T + k_2\mathbf{e}_2\mathbf{e}_2^T - \omega^2\mathbf{I}_{2n} & -(1/\sqrt{m_e})(k_1\mathbf{e}_1 + k_2\mathbf{e}_2) \\ \hdashline -(1/\sqrt{m_e})(k_1\mathbf{e}_1^T + k_2\mathbf{e}_2^T) & \omega_e^2 - \omega^2 \end{array}\right| = \mathbf{0}. \tag{24}$$

This determinantal equation can be reformulated as

$$\left|\begin{array}{c:c} \overline{\boldsymbol{\Lambda}} + 2\alpha_{k1}\bar{\mathbf{e}}_1\bar{\mathbf{e}}_1^T + 2\dfrac{\chi\alpha_{k2}}{\alpha_m\alpha_L^2}\bar{\mathbf{e}}_2\bar{\mathbf{e}}_2^T - \overline{\beta}^2\mathbf{I}_{2n} & -\sqrt{2/\alpha_{me}}\left(\alpha_{k1}\bar{\mathbf{e}}_1 + \dfrac{\chi\alpha_{k2}}{\alpha_{L\sqrt{\alpha_m\alpha_L}}}\bar{\mathbf{e}}_2\right) \\ \hdashline -\sqrt{2/\alpha_{me}}\left(\alpha_{k1}\bar{\mathbf{e}}_1^T + \dfrac{\chi\alpha_{k2}}{\alpha_{L\sqrt{\alpha_m\alpha_L}}}\bar{\mathbf{e}}_2^T\right) & \dfrac{\alpha_{k1}}{\alpha_{me}} + \dfrac{\chi\alpha_{k2}}{\alpha_{me}\alpha_L} - \overline{\beta}^2 \end{array}\right| = 0, \tag{25}$$

where

$$\overline{\beta} = \beta L_1, \qquad \lambda_k = \overline{\beta}_k^2, \qquad \omega^2 = \overline{\beta}^2 (E_1 A_1)/m_1 L_1^2, \qquad a_{ki} = \sin{(2k-1)\pi/2\, \eta_i},$$
$$\alpha_{ki} = k_i/(E_i A_i/L_i), \qquad (i = 1, 2), \qquad \alpha_{me} = m_e/m_1 L_1, \qquad \alpha_m = m_2/m_1,$$
$$\alpha_L = L_2/L_1, \qquad \chi = E_2 A_2/E_1 A_1, \qquad \delta^2 = \alpha_m \alpha_L^2/\chi, \tag{26}$$
$$\mathbf{\Lambda} = \mathbf{diag}(\lambda_i) \quad (i = 1, \ldots, n); \qquad \overline{\mathbf{\Lambda}} = \mathbf{diag}(\mathbf{\Lambda}, (1/\delta^2)\mathbf{\Lambda}),$$

are introduced. Here the $2n \times 1$ vectors $\bar{\mathbf{e}}_1$ and $\bar{\mathbf{e}}_2$ are defined as,

$$\bar{\mathbf{e}}_1^T = [a_{11}, \ldots, a_{n1}; 0, \ldots, 0], \qquad \bar{\mathbf{e}}_2^T = [0, \ldots, 0; a_{12}, \ldots, a_{n2}]. \tag{27}$$

The result above can be restated also such that the non-dimensional frequency parameters $\overline{\beta}$ of the mechanical system in Figure 1 can be obtained as the square roots of the eigenvalues of the following matrix $\mathbf{A}$:

$$\mathbf{A} = \left[\begin{array}{c:c} \overline{\mathbf{\Lambda}} + 2\alpha_{k1}\bar{\mathbf{e}}_1\bar{\mathbf{e}}_1^T + 2\,\dfrac{\chi\alpha_{k2}}{\alpha_m\alpha_L^2}\,\bar{\mathbf{e}}_2\bar{\mathbf{e}}_2^T & -\sqrt{2/\alpha_{me}}\left(\alpha_{k1}\bar{\mathbf{e}}_1 + \dfrac{\chi\alpha_{k2}}{\alpha_{L\sqrt{\alpha_m\alpha_L}}}\,\bar{\mathbf{e}}_2\right) \\ \hdashline -\sqrt{2/\alpha_{me}}\left(\alpha_{k1}\bar{\mathbf{e}}_1^T + \dfrac{\chi\alpha_{k2}}{\alpha_{L\sqrt{\alpha_m\alpha_L}}}\,\bar{\mathbf{e}}_2^T\right) & \dfrac{\alpha_{k1}}{\alpha_{me}} + \dfrac{\chi\alpha_{k2}}{\alpha_{me}\alpha_L} \end{array} \right]. \tag{28}$$

4. SECOND ALTERNATIVE FORM OF THE FREQUENCY EQUATION

The equation of the motion of the system in Figure 1 can also be established by means of the Lagrange's equations in connection with Lagrange's multipliers [5], which when considered for a system with n degrees of freedom where v redundant co-ordinates are used, as follows [7].

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_{l=1}^{v} \lambda_1 \frac{\partial f_1}{\partial q_k} \quad (k = 1, \ldots, n + v), \tag{29}$$

with the kinetic potential

$$L = T - V \tag{30}$$

and v constraint equations

$$f_1(t; q_1, \ldots, q_{n+v}) = 0 \quad (l = 1, \ldots, v). \tag{31}$$

Here, λ_1 denotes the lth Lagrangian multiplier.

In the present case, there are two constraint equations

$$f_1 = \sum_{k=1}^{n} U_{k1}(\eta_1 L_1)\eta_{k1}(t) - z_0(t) = 0; \qquad f_2 = \sum_{k=1}^{n} U_{k2}(\eta_2 L_2)\eta_{k2}(t) - z_2(t) = 0. \tag{32}$$

The evaluation of the Lagrange's equations (29) by considering the expressions (5), (6), (30) and (32) results in a set of $2n + 3$ equations. The substitution of the harmonic solutions

$$\eta_{k1} = \bar{\eta}_{k1}\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad \eta_{k2} = \bar{\eta}_{k2}\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad (k = 1, \ldots, n), \qquad z_0 = \bar{z}_0\,\mathrm{e}^{\mathrm{i}\omega t}$$
$$z_1 = \bar{z}_1\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad z_2 = \bar{z}_2\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad \lambda_1 = \overline{\lambda}_1\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad \lambda_2 = \overline{\lambda}_2\,\mathrm{e}^{\mathrm{i}\omega t} \tag{33}$$

into equations (32) and those $2n + 3$ equations yield a set of $2n + 5$ equations for the amplitudes of the harmonic functions. It can be shown that these equations result in a set of two homogeneous equations for $\bar{\lambda}_1$ and $\bar{\lambda}_2$. A non-trivial solution of this set is possible only if the determinant of the coefficient matrix vanishes. This condition in turn leads to the following frequency equation of the mechanical system shown in Figure 1:

$$\left[\sum_{k=1}^{n} \frac{U_{k1}^2(\eta_1 L_1)}{\omega_{k1}^2 - \omega^2} - \frac{k_1 - m_e\omega^2}{k_1 m_e \omega^2}\right]\left[\sum_{k=1}^{n} \frac{U_{k2}^2(\eta_2 L_2)}{\omega_{k2}^2 - \omega^2} - \frac{k_2 - m_e\omega^2}{k_2 m_e \omega^2}\right] - \frac{1}{m_e^2\omega^4} = 0, \tag{34}$$

Here, ω represents the eigenfrequency of the combined system. For further investigations, it is more suitable to rewrite the frequency equation above in terms of non-dimensional quantities as

$$\left[\sum_{k=1}^{n} \frac{2a_{k1}^2}{\lambda_k - \bar{\beta}^2} - \frac{\alpha_{k1} - \alpha_{me}\bar{\beta}^2}{\alpha_{me}\alpha_{k1}\bar{\beta}^2}\right]\left[\sum_{k=1}^{n} \frac{2a_{k2}^2}{\alpha_m\alpha_L(\lambda_k/\delta^2 - \bar{\beta}^2)} - \frac{\alpha_{k2} - (\alpha_{me}\alpha_L/\chi)\bar{\beta}^2}{\alpha_{k2}\alpha_{me}\bar{\beta}^2}\right] - \frac{1}{\alpha_{me}^2\bar{\beta}^4} = 0 \tag{35}$$

where the non-dimensional parameters are used, defined previously by (26).

After having obtained also the second form of the frequency equation of the mechanical system, it is worth emphasizing the following point. These two forms are obtained on the basis of the physical considerations. From the mathematical point of view, the focal point here is that the characteristic equation of a matrix of order $2n + 1$ can be represented in the form of an analytical expression.

It is the authors belief that this is not uninteresting. Perhaps, mathematicians can prove mathematically as well that the expansion of the determinant in (25) can actually be given by (35).

5. NUMERICAL RESULTS

This section is devoted to the numerical evaluation of the formulae established in the preceding sections. For the numerical applications, following values are chosen for the physical data of the mechanical system in Figure 1: $\eta_1 = 0{\cdot}25$, $\eta_2 = 1{\cdot}0$, $\alpha_{k1} = 1{\cdot}0$, $\alpha_{k2} = 1{\cdot}0$, $\alpha_m = 1{\cdot}0$, $\alpha_{me} = 1{\cdot}0$, $\alpha_L = 1{\cdot}5$, $\chi = 1$, $\delta = 1{\cdot}5$.

The first 21 dimensionless eigenfrequency parameters $\bar{\beta} = \beta_1 L$ are collected in Table 1. The figures in the first column are the values obtained from the exact frequency equation given in (A1). For the numerical solution of this equation on a digital computer, first the value of the determinant is obtained by the method of pivotal condensation and then, the Regula–Falsi method is applied to find the roots of equation (A1), that is, the dimensionless eigenfrequency parameters $\bar{\beta}$.

The second and third columns of Table 1 contain $\bar{\beta}$ values obtained from (28) and equation (35) for $n = 10$. In other words, the figures in the second column are the eigenvalues of the matrix **A** given by (28), whereas those of the third column are the roots of the non-linear equation in $\bar{\beta}$ given by (35). Both the eigenvalues and the roots are obtained by using MATLAB version 3.5 k for MS-DOS on a PC 386.

In order to keep the dimensions of Table 1 small, n is chosen as 10. Despite small n, it is seen from the comparison of the values from the first column with those of the second and third columns, that alternative forms of the frequency equation yield very precise approximations to the exact values. It is reasonable to expect that the dimensionless eigenfrequencies $\bar{\beta}$ obtained from (28) and (35) converge to those from the exact equation if n goes to infinity. On the other side, the $\bar{\beta}$ values from the second and third column are practically the same. This is nothing else but the numerical justification of the fact that

TABLE 1

The dimensionless eigenfrequency parameters $\bar{\beta}$ of the mechanical system in Figure 1

Solution of equation (A1)	Result from equation (28)	Solution of equation (35)
0·939349	0·941567	0·941567
1·473541	1·479536	1·479536
1·708197	1·710926	1·710926
3·284418	3·287373	3·287373
4·887227	4·888971	4·888971
5·321156	5·322953	5·322953
7·391101	7·392411	7·392412
7·965150	7·966345	7·966345
9·471977	9·473028	9·473028
11·009205	11·009356	11·009360
11·557778	11·558677	11·558680
13·646229	13·647037	13·647040
14·147440	14·147554	14·147550
15·736266	15·737029	15·737030
17·327662	17·328226	17·328230
17·827330	17·828097	17·828100
19·919095	19·919977	19·919980
20·462517	20·463045	20·463050
23·568217	23·568303	23·568300
26·708992	26·709077	26·709080
29·873562	29·874158	29·874160

both alternative forms of the frequency equation are actually identical. Thus, the argument proposed in the present note has been confirmed.

6. CONCLUSIONS

This note is concerned with the natural vibration problem of a mechanical system, consisting of two clamped–free axially vibrating elastic rods to which a double spring–mass system is attached across the span. In addition to the exact equation in a previous study, established by a boundary value problem formulation, two new alternative forms of the frequency equation are derived, starting with the discretized system. The first alternative enables one to determine the eigenfrequency parameters via the eigenvalues of a special matrix, whereas the second alternative yields the eigenfrequency parameters as the roots of a simple non-linear equation.

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APPENDIX

The exact frequency equation taken from reference [3]:

$$\begin{vmatrix} \sin\eta_1\overline{\beta} & -\sin\eta_1\overline{\beta} & -\cos\eta_1\overline{\beta} & 0 & 0 & 0 & 0 \\ \overline{\beta}\cos\eta_1\overline{\beta}+\alpha_{k1}\sin\eta_1\overline{\beta} & -\overline{\beta}\cos\eta_1\overline{\beta} & \overline{\beta}\sin\eta_1\overline{\beta} & 0 & 0 & 0 & -\alpha_{k1} \\ 0 & \cos\overline{\beta} & -\sin\overline{\beta} & 0 & 0 & 0 & 0 \\ \alpha_{k1}\sin\eta_1\overline{\beta} & 0 & 0 & \alpha_k\alpha_{k1}\sin\psi\overline{\beta} & 0 & 0 & \alpha_{me}\overline{\beta}^2-\alpha_{k1}(1+\alpha_k) \\ 0 & 0 & 0 & \sin\psi\overline{\beta} & -\sin\psi\overline{\beta} & -\cos\psi\overline{\beta} & 0 \\ 0 & 0 & 0 & \delta\overline{\beta}\cos\psi\overline{\beta}+\alpha_{k2}\sin\psi\overline{\beta} & -\delta\overline{\beta}\cos\psi\overline{\beta} & \delta\overline{\beta}\sin\psi\overline{\beta} & -\alpha_{k2} \\ 0 & 0 & 0 & 0 & \cos\delta\overline{\beta} & -\sin\delta\overline{\beta} & 0 \end{vmatrix} = 0, \tag{A1}$$

where $\psi = \delta\eta_2$, $\alpha_k = k_2/k_1$.