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USE OF QUADRATIC COMPONENTS FOR BUCKLING CALCULATIONS

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1. INTRODUCTION

The elastic stability of structures has been a major focus area of mechanics for many years. There are several standard texts on buckling including, among others, those of Timoshenko [1], Bolotin [2], and Brush and Almroth [3]. These texts include classical results for determining the buckling loads of a variety of structures with simple geometries. General purpose finite element codes such as MSC/NASTRAN [4] and ABAQUS [5] provide one with computational methods to solve certain buckling problems not amenable to closed-form solution.

Recently, the method of quadratic components [6] was developed to simulate the motion of rotating flexible structures. As the name implies, the method expresses the deformation of a structure as a quadratic function of a set of generalized co-ordinates. As an illustration, consider an inextensible cantilevered beam subjected to a transverse load at its tip. The linear part of the response consists of the transverse deformations while the quadratic part consists of the axial deformations. The second order axial deformations are required in order to satisfy the inextensibility constraint.

It turns out that second order terms in the quadratic component formulation appear as first order terms in the equations of motion if the structure undergoes significant rigid body motions. One can easily verify this fact for the cantilevered beam when its base rotates about an axis perpendicular to the transverse and axial directions. By including the second order axial deformations in the kinematics, one is able to predict correctly the spin stiffening effect. In a similar manner, the method of quadratic components can be used to model the effects of applied loads on the stiffness of a structure. In fact, the geometric stiffness matrix appears naturally through the use of quadratic components. Thus, the method provides the means to calculate buckling loads.

In the following section, the method of quadratic components is introduced and used as the basis for a buckling calculation procedure. In the third section, this procedure is applied to a cantilevered beam subjected to compressive loads of both the fixed-direction and follower types. For fixed-direction loads, the stiffness matrix of the beam is a linear function of the load and is symmetric. Thus, the buckling load can be determined by solving a symmetric eigenvalue problem. For the case of the follower-type load, it is shown that the stiffness matrix is a linear function of the load, but the matrix is non-symmetric. In this case, the stability of the beam depends on the mass distribution as well as the stiffness of the beam. Buckling calculations obtained using the new procedure are compared with classical results.

2. QUADRATIC COMPONENTS AND BUCKLING

In this section, a procedure is developed for buckling calculations based on the method of quadratic components. Under conditions of static equilibrium, the displacement field U of a structure subjected to an applied force field F can be expressed as

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$$U = N(F), \tag{1}$$

where N is a non-linear operator mapping F to U. With the method of quadratic components, the force field is expressed as a superposition of basis force fields:

$$F = s_i F^i, \tag{2}$$

where each field F^i is time-independent and summation is performed over repeated indices. The index i is assumed to have values from 1 to n. Appropriate bases of force fields can be selected to reflect either static or modal-like responses. As shown in the example problem, the basis force fields are not necessarily associated with the actual loading in the problem. The basis forces simply serve as generators for the non-linear space of displacement configurations.

Expanding the non-linear operator N as a Taylor series through quadratic terms and neglecting the higher order terms yields

$$U(\{s_i\}) = s_i\, U^i + s_i\, s_j\, G^{ij}, \tag{3}$$

where U^i and G^{ij} represent the linear and quadratic parts of the displacement field. Evaluating the displacement field given by equation (3) at the material point x and allowing the generalized co-ordinates s_i to vary with time yields

$$u(x, t) = s_i\,(t) u^i(x) + s_i\,(t) s_j\,(t) g^{ij}(x). \tag{4}$$

One can show that the symmetry $g^{ij}(x) = g^{ji}(x)$ holds. For purposes of buckling calculations, the strain energy V and kinetic energy T of the system can be expressed in matrix notation as quadratic functions of the generalized co-ordinates:

$$V = \tfrac{1}{2} s^{\mathrm{T}} K s, \qquad T = \tfrac{1}{2} \dot{s}^{\mathrm{T}} M \dot{s}, \tag{5, 6}$$

where the overdot in equation (6) denotes the time derivative and

$$s = [s_1\,(t) \ldots s_n\,(t)]^{\mathrm{T}}. \tag{7}$$

The elements in row i and column j of the matrices K and M are given by

$$k_{ij} = \int f^i(x) \cdot u^j(x)\, \mathrm{d}V, \qquad m_{ij} = \int \rho(x) u^i(x) \cdot u^j(x)\, \mathrm{d}V, \tag{8, 9}$$

where ρ is mass density and $f^i(x)$ is the basis force field F^i evaluated at x. The integrals in equations (8) and (9) are evaluated over the volume of the structure.

The load used for buckling calculations is assumed to be of the form

$$r(x, u, p) = p\hat{r}(x, u), \tag{10}$$

where p is a parameter used to scale the magnitude of the load applied to the structure. The dependence of r on the deformation field allows for the consideration of follower-type loads. Since $u(x)$ is a function of the generalized co-ordinates, the right side of equation (10) can be expressed as a Taylor series in s as

$$r(x, s, p) = p[\hat{r}_0\,(x) + a^i(x) s_i] + \mathcal{O}(s^2), \tag{11}$$

where $\mathcal{O}(s^2)$ denotes quadratic and higher order terms of the generalized co-ordinates.

The virtual work of the load is given by

$$\delta W = \int r(x, s, p) \cdot \delta u(x, t)\, \mathrm{d}V, \tag{12}$$

where δ is the variational symbol. Substituting equations (4) and (11) into equation (12) and neglecting quadratic and higher order terms yields

$$\delta W = \delta s^{\mathrm{T}}(b + pHs), \tag{13}$$

where the elements of the vector b and matrix H are given by

$$b_i = p \int \hat{r}_0\,(x) \cdot u^i(x)\,\mathrm{d}V, \qquad h_{ij} = \int [2\hat{r}_0\,(x) \cdot g^{ij}(x) + u^i(x) \cdot a^j(x)]\,\mathrm{d}V. \tag{14, 15}$$

In the present study, damping mechanisms within the structure are neglected. Thus, Hamilton's principle is written as

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W)\,\mathrm{d}t = 0. \tag{16}$$

Substituting equations (5), (6) and (13) into equation (16), integrating by parts, and setting the coefficient of δs equal to zero yields the equation of motion

$$M\ddot{s} + (K - pH)s = b. \tag{17}$$

Neglecting the right side of equation (17) and assuming a solution of the form

$$s = \mathrm{e}^{ct}\phi \tag{18}$$

yields the generalized eigenvalue problem

$$[(K - pH) - \mu M]\phi = 0, \tag{19}$$

where $\mu = -c^2$.

The system is stable provided that all the eigenvalues μ are real and greater than zero. Otherwise, there would exist a value of c with a positive real part. Thus, one can determine the stability of the system for any value of the load parameter p by examining the eigenvalues of equation (19).

For cases in which the matrix $K - pH$ is symmetric, the value of p associated with buckling can be determined by solving the generalized eigenvalue problem

$$(K - pH)\phi = 0, \tag{20}$$

where p is now considered as an eigenvalue. The lowest eigenvalue of equation (20) is the critical value of p for buckling. Symmetry of the matrix $K - pH$ is associated with conservative loads while non-symmetry of this matrix connotes a non-conservative loading.

The buckling calculation procedure is summarized as follows:

(1) Select a set of basis forces $f^i(x)$.
(2) Obtain expressions for the terms $u^i(x)$ and $g^{ij}(x)$ (see equation (4)).
(3) Obtain expressions for the matrices K, M and H (see equations (8), (9) and (15)).
(4) If the matrix $K - pH$ is symmetric, the critical value of the load parameter is the smallest eigenvalue of equation (20).
(5) If the matrix $K - pH$ is non-symmetric, the critical value of the load parameter is the smallest value of p for which at least one of the eigenvalues of equation (19) is not positive and real.

An appealing feature of this procedure is that it can be applied to a wide variety of structures by making use of existing finite element codes to aid in Steps (1)–(3). Guidelines on the use of such codes for this purpose are available [7]. While the same code used to aid in Steps (1)–(3) would likely have a buckling calculation capability, this capability may be limited to problems for which the H matrix is symmetric.

3. EXAMPLES

As an example application of the buckling calculation procedure, consider the cantilevered beam shown in Figure 1. The beam is inextensible and has uniform mass and stiffness distribution along its length. Deformations are restricted to the $(n_1 - n_2)$ plane and the load is applied at the beam tip. The beam length, bending stiffness and mass per unit length are denoted by L, EI and $\hat{m}$, respectively.

Two load cases are considered. For Case 1, the direction of the load remains in the negative n_1 direction. For Case 2, the load follows the rotation of the beam and remains parallel to the neutral axis at the tip. The latter case is an example of a follower load and is non-conservative.

Step (1): The basis forces are chosen as those associated with the eigenmodes of the beam. That is, the distributed forces which statically deform the beam into the shapes of the eigenmodes.

Step (2): The terms $u^i(x)$ are the linear eigenmodes of the system and given by Blevins [8]:

$$u^i(x) = \{\cosh(\lambda_i x/L) - \cos(\lambda_i x/L) - \sigma_i [\sinh(\lambda_i x/L) - \sin(\lambda_i x/L)]\} n_2, \qquad (21)$$

where the coefficients λ_i and σ_i for $i = 1, \ldots, 5$ are given in Table 1. The terms $g^{ij}(x)$ are determined by the constraint that the beam is inextensible. Thus,

$$g^{ij}(x) = \left\{ -\frac{1}{2} \int_0^x \frac{\mathrm{d}u^i(\tau)}{\mathrm{d}\tau} \frac{\mathrm{d}u^j(\tau)}{\mathrm{d}\tau} \, \mathrm{d}\tau \right\} n_1 = L g^{ij}(x) n_1. \qquad (22)$$

Substituting equation (21) into equation (22), setting x equal to L, and using integration tables (Blevins [8]), one obtains

$$g^{ii}(L) = -\tfrac{1}{2}[\sigma_i \lambda_i (2 + \sigma_i \lambda_i)] \qquad (23)$$

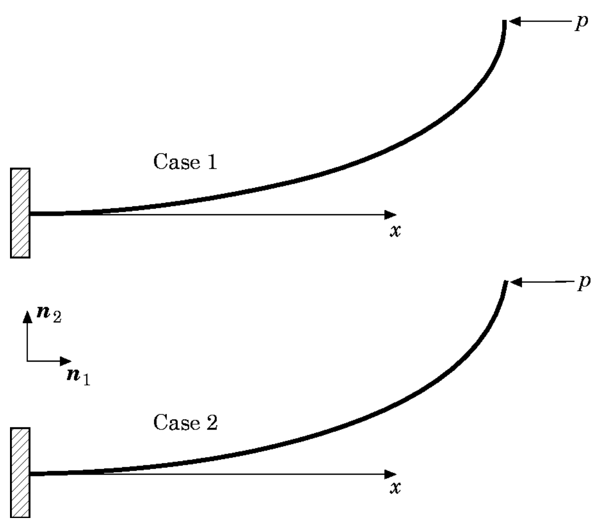

Figure 1. Load cases for cantilevered beam examples.

TABLE 1

Beam coefficients used in equation (21).

i	λ_i	σ_i
1	1·87510407	0·734095514
2	4·69409113	1·018467319
3	7·85475744	0·999224497
4	10·99554073	1·000033553
5	14·13716839	0·999998550

and for $i \neq j$,

$$g^{ij}(L) = -2\lambda_i \lambda_j / (\lambda_i^4 - \lambda_j^4)\, [(-1)^{i+j}(\sigma_j \lambda_i^3 - \sigma_i \lambda_j^3) - \lambda_i \lambda_j (\sigma_i \lambda_i - \sigma_j \lambda_j)]. \qquad (24)$$

Step (3): The elements of the stiffness and mass matrices are given by

$$k_{ij} = EI \int_0^l \left[\frac{d^2u^i}{dx^2}\frac{d^2u^j}{dx^2}\right] dx, \qquad m_{ij} = \hat{m} \int_0^L [u^i \cdot u^j]\, dx. \qquad (25,\ 26)$$

Performing the integrations in equations (25) and (26), one obtains

$$k = (EI/L)\, \mathrm{diag}\,(\lambda_1^4, \ldots, \lambda_n^4), \qquad M = \hat{m} L^3 I_n, \qquad (27,\ 28)$$

where diag denotes a diagonal matrix and I_n is the identity matrix of dimension n.

Since the beam is one-dimensional, equation (15) simplifies to

$$h_{ij} = \int_0^L [2\hat{r}_0(x) \cdot g^{ij}(x) + u^i(x) \cdot a^j(x)]\, dx. \qquad (29)$$

For both loading cases, the term $\hat{r}_0$ is given by

$$\hat{r}_0(x) = -\delta(x - L) n_1, \qquad (30)$$

where δ is the Dirac delta function. The term $a^i(x)$ is equal to zero for Case 1. For Case 2,

$$a^j(L) = -2\sigma_j \lambda_j (-1)^{j+1} \delta(x - L) n_2. \qquad (31)$$

The leading coefficient of $\delta(x - L)n_2$ on the right side of equation (31) is the slope of the jth eigenmode at the beam tip. Substituting equations (21), (22), (30) and (31) into equation (29) yields

$$h_{ij} = -2L[g^{ij}(L) + 2\sigma_j \lambda_j (-1)^{i+j}]. \qquad (32)$$

Step (4): For Case 1 the matrix $K - pH$ is symmetric, therefore, the critical value of the load parameter p is the smallest eigenvalue of equation (20).

Step (5): For Case 2 the matrix $K - pH$ is non-symmetric. Thus, the critical value of the load parameter is the smallest value of p for which at least one of the eigenvalues of equation (19) is not positive and real.

3.1. *Results*

The critical value of the load parameter p can be expressed in terms of the dimensionless variable α as

$$p = \alpha EI/L^2. \qquad (33)$$

TABLE 2

Results for cases 1 *and* 2

n	α Case 1	α Case 2
1	2·6598	∞
2	2·4817	20·105
3	2·4740	20·113
4	2·4697	20·052
5	2·4688	20·061

Table 2 shows the values of α for $n = 1, \ldots, 5$ for both load cases. Recall that n is the number of generalized co-ordinates. The exact value of α for Case 1 is equal to $\pi^2/4 \approx 2{\cdot}4674$ (reference [1]). The exact value of α for Case 2 is equal to approximately 20·05 [2]. Notice that in the table, α converges monotonically from above to a constant value for Case 1. For Case 2, α appears to be converging to a constant value, but the convergence is non-monotonic.

4. CONCLUSIONS

The method of quadratic components has been shown to be applicable to buckling problems with both conservative and non-conservative loads. The method has been used to develop a buckling calculation procedure which was applied to an example problem. The results from the example problem were in excellent agreement with classical results. The procedure developed can be applied to a wide variety of different structures by making use of existing finite element software.

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