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REGULATION OF SINGULAR ORDINARY DIFFERENTIAL EQUATIONS MODELLING OSCILLATING SYSTEMS

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Non-linear, singular oscillator systems arise in various areas of the engineering and physical sciences. In particular, they occur in the areas of mechanical oscillations [8], electronic circuits [4, 7] and plasma physics [2]. The corresponding model ordinary differential equations (ODE) have been extensively investigated by Bota and Mickens [1, 4, 5]. A prototypical form for such ODE's is the WCM oscillator equation [4, 7],

$$\ddot{x} + x = \varepsilon\left[\frac{\mu - x^2}{1 - x^2}\right]\dot{x}, \tag{1}$$

where ϵ and μ are the parameters that satisfy the constraints

$$0 < \varepsilon \ll 1, \qquad 0 < \mu < 1. \tag{2}$$

Note that the phase space (x, y) of the system, modelled by equation (1), is the strip

$$|x| < 1, \qquad -\infty < y < +\infty, \tag{3}$$

where $y \equiv \mathrm{d}x/\mathrm{d}t$. For purposes of both numerical and mathematical analysis, it is more convenient to be able to use all of the phase space rather than the restricted region given by equation (3). The purpose of this Letter is to show that a non-linear transformation can be made from the variables (x, y) to new variables (u, w) such that in the new variables the whole phase space is used; i.e.,

$$-\infty < u < +\infty, \qquad -\infty < w < +\infty. \tag{4}$$

An advantage of the new variables is that they may be easily used to obtain information about possible solution behaviors by application of the qualitative theory of differential equations [6]. The use of only the transformation from x to u leads to a second order ODE which is easily seen to be a modified form of the van der Pol oscillator equation. At this point an approximation to the analytic solutions can be constructed by use of some procedure such as the method of harmonic balance [3]. The application of the inverse transformation, from u to x, then gives an approximate to the original singular oscillator equation.

In the following discussion, only equation (1) is considered; however, it should be clear from the presentation how to extend these results to other non-linear, singular systems modelled by this type of ODE.

First, consider the transformation $(x, y) \rightarrow (u, w)$, where

$$x = \frac{u}{\sqrt{1 + u^2}}, \qquad y = w. \tag{5}$$

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This transforms the strip given by equation (3) into the whole plane, as indicated by equation (4). Writing equation (1) in system form [6] gives two coupled ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -x + \varepsilon\left(\frac{\mu - x^2}{1 - x^2}\right)y. \tag{6a, b}$$

From equation (5), it follows that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\dot{u}}{[1 + u^2]^{3/2}} = y = w, \tag{7}$$

and

$$\mathrm{d}u/\mathrm{d}t = [1 + u^2]^{3/2} w. \tag{8}$$

Likewise, equation (6b) becomes, under the transformation of equation (5), the expression

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\frac{u}{\sqrt{1 + u^2}} + \varepsilon[\mu - (1 - \mu)u^2]w. \tag{9}$$

Equations (8) and (9) are the new system ODE's in the transformed phase space of variables (u, w). The trajectories in this phase space, $w = w(u)$, are solutions to the first order ODE (6),

$$\frac{\mathrm{d}w}{\mathrm{d}u} = \left\{-\frac{u}{\sqrt{1 + u^2}} + \varepsilon[\mu - (1 - \mu)u^2]w\right\}\Big/[(1 + u^2)^{3/2} w]. \tag{10}$$

Using the fact that $0 < \varepsilon$ and $0 < \mu < 1$, it can easily be shown that the phase space structure of equation (10) is topologically the same as that for the van der Pol equation,

$$\ddot{u} + u = \bar{\varepsilon}(1 - u^2)\dot{u}, \tag{11}$$

where the constant $\bar{\varepsilon} > 0$, and the system equations for (11) are

$$\mathrm{d}u/\mathrm{d}t = w, \qquad \mathrm{d}w/\mathrm{d}t = -u + \bar{\varepsilon}(1 - u^2)w. \tag{12}$$

(see Mickens [6, section I.3.3]). This immediately implies that equations (8) and (9), and by use of the inverse transformation, equation (1), have a unique limit-cycle. Previous attempts to demonstrate this result required more elaborate procedures [4].

A second way to proceed is to apply only the transformation

$$x = \frac{u}{\sqrt{1 + u^2}} \tag{13}$$

directly to equation (1). Doing this gives the result

$$\dot{x} = \frac{\dot{u}}{(1 + u)^{3/2}}, \qquad \ddot{x} = \frac{\ddot{u}(1 + u^2) - 3u\dot{u}^2}{(1 + u^2)^{5/2}}, \tag{14}$$

which, when substituted into equation (1) yields, after some simplification, the equation

$$\ddot{u} + \left[\frac{(1 + u^2)^2 - 3\dot{u}^2}{1 + u^2}\right]u = \varepsilon(1 - u)\left[\left(\frac{\mu}{1 - \mu}\right) - u^2\right]\dot{u}, \tag{15}$$

which is a modified form of the van der Pol equation.

An analytic approximation to the limit-cycle solution of equation (15) can be calculated by the use of the method of harmonic balance [3]. The first approximation is

$$u_1(t) = A \cos(\omega t), \tag{16}$$

where the amplitude A and angular frequency are to be determined by substitution of equation (16) into equation (15) and setting the coefficients of the lowest harmonics, $\cos(\omega t)$ and $\sin(\omega t)$, equal to zero, and then solving for A and ω. Carrying out this series of steps gives

$$A = 2\left(\sqrt{\frac{\mu}{1-2\mu}}\right), \qquad \omega = \sqrt{\frac{1+2\mu+2\mu^2}{(1-2\mu)(1+4\mu)}}. \tag{17a, b}$$

Further analysis shows that the parameter μ has to satisfy the restriction

$$0 < \mu < 1/4. \tag{18}$$

(The details as to how this inequality is obtained can be found in Mickens [4].) Substitution of equation (16) into equation (13), where A and ω are taken from equations (17), gives the following approximation to the limit-cycle solution of equation (1):

$$x_1(t) = [A \cos(\omega t)] \Bigg/ \sqrt{\left(1 + \frac{A^2}{2}\right) + \left(\frac{A^2}{2}\right)\cos(2\omega t)}. \tag{19}$$

In actual applications μ is rather small; i.e.,

$$0 < \mu \leqslant 1/4. \tag{20}$$

This fact implies that

$$A = 2\sqrt{\mu} + O(\mu^{3/2}), \qquad \omega = 1 + O(\mu). \tag{21}$$

Thus, for this situation, equation (19) becomes

$$x_1(t) = 2\sqrt{\mu}\cos t + O(\mu^{3/2}). \tag{22}$$

This is exactly the result obtained previously by Bota and Mickens [1] and Mickens [4]. Note that the approximate solution given by equation (19) automatically incorporates higher order corrections to $x(t)$. This clearly illustrates the major advantage of transforming to the new variables, constructing a harmonic balance solution and then transforming back to the original variables.

In summary, a method has been presented to regularize singular ODE's that occur in the modelling of certain non-linear oscillating systems. The procedure transforms the singular points out to the boundary at infinity; consequently, the usual methods of numerical, phase space and analytic analysis can be directly applied to the new ODE, which does not contain singularities.

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REFERENCES

1. K. B. BOTA and R. E. MICKENS 1984 *Journal of Sound and Vibration* **96**, 277–279. Approximate analytic solutions for singular non-linear oscillators.

2. P. K. KAW, A. SEN and E. J. VALEO 1983 *Physica* **9D**, 96–102. Coupled nonlinear stationary waves in a plasma.
3. R. E. MICKENS 1984 *Journal of Sound and Vibration* **94**, 456–460. Comments on the method of harmonic balance.
4. R. E. MICKENS 1989 *Circuits*, *Systems*, *and Signal Processing* **8**, 187–205. Investigation of the mathematical properties of a new negative resistance oscillator model.
5. R. E. MICKENS 1989 *Journal of Sound and Vibration* **134**, 187–189. Calculation of transient behavior for a nonlinear, singular oscillator equation.
6. R. E. MICKENS 1996 *Oscillations in Planar Dynamic Systems*. River Edge, New Jersey: World Scientific.
7. S. S. WALKER and J. A. CONNELLY 1983 *Circuits*, *Systems*, *and Signal Processing* **2**, 213–238. A new negative resistance oscillator model.
8. Y.-T. WANG and R. SINGH 1987 *Journal of Applied Mechanics* **54**, 209–214. Frequency response of a nonlinear pneumatic system.