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DYNAMICS OF FLEXIBLE SLIDING BEAMS—NON-LINEAR ANALYSIS PART I: FORMULATION

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Equations of motion for the geometrically non-linear analysis of flexible sliding beams, deployed or retrieved through a rigid channel, are derived through an extension of Hamilton's principle. Based on the assumptions of Euler–Bernoulli beam theory, the equations of motion account for small strains but large rotations. Also provided is an alternative formulation wherein by superposition of a prescribed axial velocity the beam is brought to rest and the channel assumes the prescribed velocity. The consistency of the two formulations is shown through an appropriate transformation of the governing equations to a fixed domain. The fixed domain provides a very convenient frame work for numerical solution of the equations of motion. Discretization procedures using Galerkin's method, and numerical examples involving large amplitude vibrations of the flexible sliding beam are presented in part II.

1. INTRODUCTION

The flexible extendible beam problem falls under the broad topic of axially moving solid continua. Axially moving materials arise in problems associated with cable tramways, spacecraft antennas, band saws, cold and hot rolling processes, magnetic tape drives and fan belts. Since these systems have one large dimension (along the axis of motion) and two smaller ones, they are usually analysed as one-dimensional string or beam problems. In some applications, however, account must be taken of a second large dimension. For instance, the deployment of solar arrays in space applications requires modelling the array as a moving membrane or plate. A recent survey of these problems is given by Wickert and Mote [1].

A related problem is that of pipes conveying fluid. Literature on pipes conveying fluids is very extensive and an excellent survey of these problems is given by Païdoussis [2]. From a materials perspective, these problems fall within the domain of solid mechanics, although the flow aspect of these problems gives them a flavour of fluid mechanics. For instance,

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in the motion of a computer tape between two reels, it is not practical to follow the individual particles of tape in time, since some tape particles leave (while others enter) the domain of interest. One must then use the Eulerian *window* over the domain of interest. If the tracking of particles is relinquished, then conservation of mass will not be automatically satisfied. That is, new mass elements enter the domain and, depending on the boundary conditions, some mass elements may leave the domain, with the result that the mass in the domain may change with time. In these problems, the rate at which mass elements enter and leave the domain is prescribed.

Most investigations of axially moving beams deal with beams supported at two fixed points, and it is the transverse motion of the beam within the span that is of interest (Wickert and Mote [1]). If the beam is assumed to be axially rigid, then under these conditions, the mass of the system within the domain of interest may be conserved for small amplitude motions. But for cantilevered sliding beams mass is not conserved, as new mass elements enter the domain of interest (the protruding part of the beam) even for small amplitude oscillations.

A derivation of the non-linear, coupled longitudinal and transverse equations of motion of the flexible extendible beam has been provided by Tabarrok *et al*. [3] through Newton's second law. In addition, it was shown that for a constant axial velocity, oscillatory motions dominate the response during the initial stage of deployment and that, at least within the linear theory, the transverse deflection becomes unbounded with time. Their findings were confirmed by simulations using the assumed-modes technique. The same technique was employed in an investigation by Cherchas and Gossain [4] of the dynamics of a large flexible solar array as it deploys from a spinning spacecraft. Several investigators have also examined the stability of beams under harmonic longitudinal motion for beams of constant length (Elmaraghy and Tabarrok [5]) and variable length (Zajaczkowski and Lipinski [6], Zajaczkowski and Yamada [7]). Regions of stability and instability in the excitation amplitude and frequency parameter space were identified. Although such excitation does not occur in most band-and-wheel systems, many robotic and mechanism components execute periodic axial motions.

Recently, flexible extendible beams have gained prominence due to new applications in the area of robotics, specifically in the modelling of flexible links travelling through prismatic joints. Wang and Wei [8] used a modified Galerkin method to solve the equation of motion of an axially moving beam. However, their derivation of the governing equation, through Newton's second law, leaves out certain terms. Yuh and Young [9] used the assumed-modes method and compared their simulation results with those obtained experimentally. Buffinton [10] also used the assumed-modes technique to model the moving beam as an unconstrained body, and treated the beam's finite number of supports as kinematical constraints. Kim and Gibson [11] used the finite element approach to model a sliding flexible link. However, the derivation of the complementary kinetic energy of the sliding flexible link, outlined by Kim [12], also leaves out certain terms. Stylianou and Tabarrok [13] used the finite element method and develop elements with time-varying domains to investigate the dynamics of the flexible extendible beam under more general configurations. Most of these works are concerned with linear axially inextensible sliding beams. In an important paper, Vu-Quoc and Li [14] present a very comprehensive study of the axially translating beam and introduce novel ideas for the analysis of this system. They employ the so called geometrically exact beam theory (Simo and Vu-Quoc [15]) and consider large angle maneuvers. They used the finite element method and shed new light on the mechanics of this problem.

In the present work, equations of motion for geometrically non-linear flexible sliding beams, deployed or retrieved through prismatic joints, are derived. The beams can undergo large rotations in a plane. Based on the assumptions of Euler–Bernoulli beam theory, the equations of motion are derived for small deformations through an extension of Hamilton's principle. Following the approach outlined by Vu-Quoc and Li, an alternative formulation wherein the beam is brought to rest and the channel assumes a prescribed velocity is outlined. Using an appropriate transformation, the two formulations are shown to be consistent.

2. SLIDING BEAM FORMULATION

The natural way to formulate the flexible sliding beam problem is to view the beam sliding through a rigid channel and undergoing large overall motions when it has emerged from the channel. Thus at any instant, as noted by Vu-Quoc and Li [14], three different configurations can be distinguished, namely: the initial undeformed or material configuration in the channel, the spatially fixed intermediate configuration and the sliding deformed configuration.

The intermediate configuration is an artifice introduced for purposes of formulation. That is, the beam does not follow a sequence of configurations from within the channel to the intermediate and finally to the deformed configuration. Rather to describe the deformed configuration we need an underformed reference configuration and it is for this purpose that the intermediate configuration is introduced.

The intermediate configuration can be seen as an Eulerian domain with respect to the translating undeformed beam and a Lagrangian domain with respect to the current deformed beam. The intermediate configuration is a fixed spatial configuration used as reference for the deformed state. In this configuration the boundary extends in time, that is this configuration has a moving boundary.

The mapping from the material configuration to the intermediate configuration is a prescribed sliding rigid body motion along the channel axis. This can be seen by looking at the material point M that passes by inertially fixed points and at time t coincides with $\bar{m}$. Then, the deformation from the intermediate configuration to the current configuration is a Lagrangian description from the inertially fixed point $\bar{m}$ to the spatial point m; see Figure 1.

Viewing the sliding beam as a system of changing mass, one assumes that the part of the beam inside the channel is non-deformable and has a prescribed motion. Thus, the task is to determine the motion of the beam as it emerges from the channel.

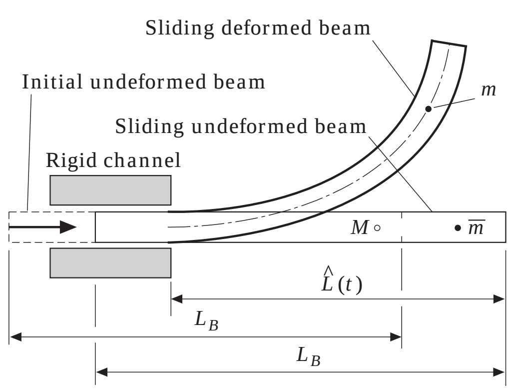

Figure 1. Sliding beam: initial undeformed, sliding undeformed and deformed configurations.

2.1. KINEMATIC DESCRIPTION

The distance δS_0, between two points lying on the centerline of the beam in the undeformed state may be written as

$$(\delta S_0)^2 = (\delta\chi_1)^2 + (\delta\chi_2)^2 \tag{1}$$

and the distance, δS, between the same two points after deformation is given by

$$(\delta S)^2 = (\delta x_1)^2 + (\delta x_2)^2, \tag{2}$$

where $x_1 = x_1(\chi_1, \chi_2)$ and $x_2 = x_2(\chi_1, \chi_2)$.
Therefore, the change of distance between these two points is given by

$$(\delta S)^2 - (\delta S_0)^2 = (\delta x_1)^2 + (\delta x_2)^2 - [(\delta\chi_1)^2 + (\delta\chi_2)^2]. \tag{3}$$

For the beam initially lying along the $\mathbf{I}_1$-axis ($\delta S_0 = \delta\chi_1$, $\delta\chi_2 = 0$):

$$\delta x_1 = (\partial x_1/\partial\chi_1)\delta\chi_1, \qquad \delta x_2 = (\partial x_2/\partial\chi_1)\delta\chi_1. \tag{4}$$

Hence, equation (2) may be expressed as

$$\delta S = \sqrt{(\partial x_1/\partial\chi_1)^2 + (\partial x_2/\partial\chi_1)^2}\,\delta\chi_1 \tag{5}$$

and equation (3) may be written as

$$(\delta S)^2 - (\delta S_0)^2 = \left[\left(\frac{\partial x_1}{\partial\chi_1}\right)^2 + \left(\frac{\partial x_2}{\partial\chi_1}\right)^2 - 1\right](\delta\chi_1)^2. \tag{6}$$

For the axially inextensible beam, $\delta S = \delta S_0$, and for this case the inextensibility condition from equation (6) is found as

$$(\partial x_1/\partial\chi_1)^2 + (\partial x_2/\partial\chi_1)^2 = 1. \tag{7}$$

It will be also useful to express the above conditions in terms of displacement components of the centerline u_1, u_2; see Figure 2.

Noting that

$$u_1 = x_1 - \chi_1, \qquad u_2 = x_2, \tag{8, 9}$$

where for the present case along the centerline $\chi_2 = 0$, then

$$\partial x_1/\partial\chi_1 = \partial u_1/\partial\chi_1 + 1 \qquad \text{and} \qquad \partial x_2/\partial\chi_1 = \partial u_2/\partial\chi_1. \tag{10, 11}$$

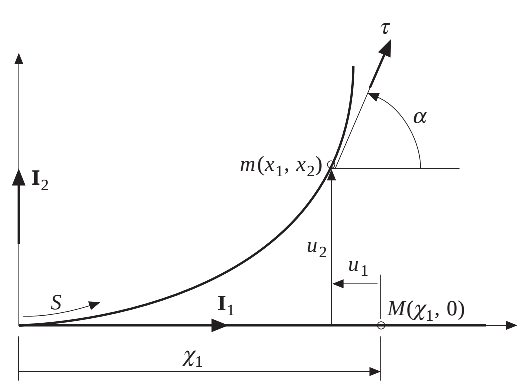

Figure 2. Centerline kinematics of a deformed beam.

Therefore, equations (5–7) may be rewritten in terms of the displacements as

$$\delta S = \sqrt{(\partial u_1/\partial \chi_1 + 1)^2 + (\partial u_2/\partial \chi_1)^2}\,\delta \chi_1, \tag{12}$$

$$(\delta S)^2 - (\delta S_0)^2 = [(\partial u_1/\partial \chi_1 + 1)^2 + (\partial u_2/\partial \chi_1)^2 - 1](\delta \chi_1)^2, \tag{13}$$

and

$$(\partial u_1/\partial \chi_1 + 1)^2 + (\partial u_2/\partial \chi_1)^2 = 1. \tag{14}$$

In general for the axially extensible beam, $\delta S \neq \delta S_0$, and the axial strain ε of the centerline may be defined through the engineering strain definition as

$$\delta S = (1 + \varepsilon)\delta \chi_1 \qquad \text{or} \qquad \partial \chi_1/\partial S = 1/(1 + \varepsilon). \tag{15, 16}$$

Substituting the above expression into equations (5) and (12), the following relations are obtained:

$$1 + \varepsilon = \sqrt{(\partial x_1/\partial \chi_1)^2 + (\partial x_2/\partial \chi_1)^2}, \qquad 1 + \varepsilon = \sqrt{(\partial u_1/\partial \chi_1 + 1)^2 + (\partial u_2/\partial \chi_1)^2}. \tag{17, 18}$$

If α is the angle between the tangent at a point along the centerline of the beam and the $\mathbf{I}_1$-axis, Figure 2, then the curvature of the beam's centerline may be expressed as

$$\kappa = \partial \alpha/\partial S. \tag{19}$$

Now, using equation (16), one may write

$$\kappa = (\partial \alpha/\partial \chi_1)(\partial \chi_1/\partial S) = (1/[1 + \varepsilon])\partial \alpha/\partial \chi_1. \tag{20}$$

To express κ in terms of the displacement components, one proceeds to relate $\partial \alpha/\partial \chi_1$ to the displacements. From the centerline curve, one has that

$$\cos \alpha = \partial x_1/\partial S = (\partial x_1/\partial \chi_1)(\partial \chi_1/\partial S) = 1/(1 + \varepsilon)(1 + \partial u_1/\partial \chi_1) \tag{21}$$

and

$$\sin \alpha = \partial x_2/\partial S = (\partial x_2/\partial \chi_1)(\partial \chi_1/\partial S) = 1/(1 + \varepsilon)\, \partial u_2/\partial \chi_1. \tag{22}$$

Differentiating equations (22) with respect to χ_1, one obtains

$$(\partial \alpha/\partial \chi_1) \cos \alpha = [(\partial^2 u_2/\partial \chi_1^2)(1 + \varepsilon) - (\partial u_2/\partial \chi_1)\, \partial \varepsilon/\partial \chi_1]/(1 + \varepsilon)^2. \tag{23}$$

Substituting for $\cos \alpha$ from equation (21) and using the relation (18), in equation (23) one finds

$$\frac{\partial \alpha}{\partial \chi_1} = \left(\frac{\partial^2 u_2}{\partial \chi_1^2}\left[\left(\frac{\partial u_1}{\partial \chi_1} + 1\right)^2 + \left(\frac{\partial u_2}{\partial \chi_1}\right)^2\right] - \frac{\partial u_2}{\partial \chi_1}\left[\frac{\partial^2 u_1}{\partial \chi_1^2}\left(\frac{\partial u_1}{\partial \chi_1} + 1\right)\right.\right.$$

$$\left.\left. + \frac{\partial u_2}{\partial \chi_1}\frac{\partial^2 u_1}{\partial \chi_1^2}\right]\right/ (1 + \varepsilon)^2\left(\frac{\partial u_1}{\partial \chi_1} + 1\right), \tag{24}$$

which upon simplification results in

$$\frac{\partial \alpha}{\partial \chi_1} = \left(\frac{1}{1 + \varepsilon}\right)^2\left[\frac{\partial^2 u_2}{\partial \chi_1^2}\left(1 + \frac{\partial u_1}{\partial \chi_1}\right) - \frac{\partial u_2}{\partial \chi_1}\frac{\partial^2 u_1}{\partial \chi_1^2}\right]. \tag{25}$$

For the axially inextensible beam where $\varepsilon = 0$, the use of relations (10) and (11), in equation (25) leads to

$$\partial \alpha/\partial \chi_1 = (\partial^2 x_2/\partial S^2)(\partial x_1/\partial S) - (\partial^2 x_1/\partial S^2)(\partial x_2/\partial S). \tag{26}$$

Further, using the inextensibility condition (7) and noting that in this case $\partial S = \partial \chi_1$, it can be shown that

$$\frac{\partial x_1}{\partial S} = \sqrt{1 - \left(\frac{\partial x_2}{\partial S}\right)^2}, \qquad \frac{\partial^2 x_1}{\partial S^2} = -\frac{\partial^2 x_2}{\partial S^2}\frac{\partial x_2}{\partial S} \Bigg/ \sqrt{1 - \left(\frac{\partial x_2}{\partial S}\right)^2}. \tag{27}$$

2.2. EQUATION OF MOTION FOR AXIALLY INEXTENSIBLE FLEXIBLE SLIDING BEAMS

Deflection of the sliding beam may be large, hence non-linear terms of up to third order will be retained in the governing equation of the beam.

2.2.1. *Extended Hamilton's Principle*

Evidently sliding beams are systems of changing mass, that is, generally the number of particles in the system at time t_1 are different from that at time t_2. Hamilton's principle, in its classical form applies to systems of particles that retain their identity and number, i.e., the same particle system is considered at times t_1 and t_2. Hence, to apply Hamilton's principle to the flexible sliding beam problem some modifications, related to the identity and aggregate of particles, are necessary.

It should be noted that the use of Hamilton's principle for the axially rigid sliding cantilever beams has been discussed by Tabarrok *et al*. [3] and the following is an alternative approach in formulating the motion of sliding beams. The new approach sheds further light on this dynamically rich problem.

As noted earlier, the sliding beam problem has some features of flow, conventionally described by Eulerian formulations. There have been a number of studies to extend Hamilton's principle to "flow" problems, e.g., Veubeke [16], Leech [17] and Dost and Tabarrok [18]. In these studies the difficult task of following the particles, as against monitoring changes at specific points in space, is recognized and discussed. For the sliding beam problem, the Lagrangian description, with its focus on the particles, can still be used even though the number of particles in the system is not fixed. This problem of changing mass was addressed by McIver [19] in a remarkable development. In the following the salient points in McIver's formulation is reviewed.

From D'Alembert principle, one has for a system of N particles:

$$\sum_{i=1}^{N} \left(m_i \frac{\mathrm{D}^2 \mathbf{r}_i}{\mathrm{D}t^2} + \frac{\partial \Pi}{\partial \mathbf{r}_i} - \mathbf{F}_i \right) \cdot \delta \mathbf{r}_i = 0, \tag{28}$$

where $\Pi = \Pi(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N)$ is the potential energy of the particles, $\mathbf{F}_i$ denotes the forces without potentials acting on the ith particle, $\mathbf{r}_i$ is the position vector of this particle of mass m_i and $\delta \mathbf{r}_i$ is a virtual displacement. $\mathrm{D}()/\mathrm{D}t$ is the material time derivative following the particle. One notes that

$$\sum_{i=1}^{N} \left(\frac{\partial \Pi}{\partial \mathbf{r}_i} \right) \cdot \delta \mathbf{r}_i = \delta \Pi \tag{29}$$

and

$$\sum_{i=1}^{N} (\mathbf{F}_i) \cdot \delta \mathbf{r}_i = \delta W \tag{30}$$

and

$$\sum_{i=1}^{N}\left(m_i\frac{\mathrm{D}^2\mathbf{r}_i}{\mathrm{D}t^2}\right)\cdot\delta\mathbf{r}_i=\frac{\mathrm{D}}{\mathrm{D}t}\left[\sum_{i=1}^{N}\left(m_i\frac{\mathrm{D}\mathbf{r}_i}{\mathrm{D}t}\right)\cdot\delta\mathbf{r}_i\right]-\sum_{i=1}^{N}\left(m_i\frac{\mathrm{D}\mathbf{r}_i}{\mathrm{D}t}\right)\cdot\delta\frac{\mathrm{D}\mathbf{r}_i}{\mathrm{D}t} \tag{31}$$

or

$$\sum_{i=1}^{N}\left(m_i\frac{\mathrm{D}^2\mathbf{r}_i}{\mathrm{D}t^2}\right)\cdot\delta\mathbf{r}_i=\frac{\mathrm{D}}{\mathrm{D}t}\left[\sum_{i=1}^{N}\left(m_i\frac{\mathrm{D}\mathbf{r}_i}{\mathrm{D}t}\right)\cdot\delta\mathbf{r}_i\right]-\delta T^*, \tag{32}$$

where T^* is the kinetic co-energy of the particles. Substituting from equations (29), (30) and (32) into equation (28), one may express D'Alembert's principle as

$$\delta\mathscr{L}+\delta W-\frac{\mathrm{D}}{\mathrm{D}t}\left[\sum_{i=1}^{N}\left(m_i\frac{\mathrm{D}\mathbf{r}_i}{\mathrm{D}t}\right)\cdot\delta\mathbf{r}_i\right]=0, \tag{33}$$

where $\mathscr{L}=T^*-\Pi$ is the Lagrangian of the system.

Starting from this point, in a remarkable paper McIver [19], considered a generalization from a discrete to a continuous system. In keeping with the precepts of conservation of mass and preservation of the identity of particles, McIver expressed equation (33) as

$$\delta\mathscr{L}_c+\delta W-\frac{\mathrm{D}}{\mathrm{D}t}\left(\int_{v_c(t)}(\rho\mathscr{U})\cdot\delta\mathbf{r}\,\mathrm{d}v\right)=0, \tag{34}$$

where ρ denotes the density and $\mathscr{U}$ is the velocity of the particle at time t and $\mathscr{L}_c$ and δW are the Lagrangian of the system and the virtual work performed by the generalized forces undergoing virtual displacements. The subscript c denotes a fixed material system enclosed in a volume v_c.

Hamilton's principle is obtained by integrating equation (34) with respect to time over a time interval t_1 to t_2, yielding

$$\delta\int_{t_1}^{t_2}\mathscr{L}_c\,\mathrm{d}t+\int_{t_1}^{t_2}\delta w\,\mathrm{d}t-\int_{v_c(t)}(\rho\mathscr{U})\cdot\delta\mathbf{r}\,\mathrm{d}v\bigg|_{t_1}^{t_2}=0. \tag{35}$$

If one now imposes the requirement that at time t_1 and t_2 the configuration be prescribed, i.e., $\delta\mathbf{r}=0$, then the last term in equation (35) drops out, leaving

$$\delta\int_{t_1}^{t_2}\mathscr{L}_c\,\mathrm{d}t+\int_{t_1}^{t_2}\delta W\,\mathrm{d}t=0. \tag{36}$$

Now to proceed from a closed material system to an open system Reynold's transport theorem is invoked which states that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{v_o(t)}(-)\,\mathrm{d}v=\frac{\mathrm{D}}{\mathrm{D}t}\int_{v_c(t)}(-)\,\mathrm{d}v+\int_{s_o(t)}((\ )(\mathscr{V}-\mathscr{U})\cdot\mathbf{n})\,\mathrm{d}S. \tag{37}$$

In the above expression, $v_o(t)$ is the open control volume with the moving control surface $s_o(t)$ which advances with the velocity $\mathscr{V}\cdot\mathbf{n}$, in the outward normal direction $\mathbf{n}$ and across which mass is transported. The control volume $v_o(t)$ is pervious to the particles and thus the system is not necessarily of constant mass or, if of constant mass, it need not always contain the same set of particles. At instant t, the open control volume, $v_o(t)$, coincides with the closed control volume $v_c(t)$ with the boundary $s_c(t)$ for which $\mathscr{V} \cdot \mathbf{n} = \mathscr{U} \cdot \mathbf{n}$; see Figure 3.

It is important to be clear about the operators d( )/dt and D( )/Dt. The operator d( )/dt refers to the time derivative following a control volume whether of constant mass or not. It is obvious that when the control volume contains the same particles, i.e., it is the so called material control volume, then this derivative is equivalent to the material time derivative D( )/Dt.

Using Reynold's transport theorem (37), the virtual work equation (37), for a system of changing mass, can be written as

$$\delta \mathscr{L}_o + \delta W - \frac{\mathrm{d}}{\mathrm{d}t} \int_{v_o(t)} \rho(\mathscr{U} \cdot \delta \mathbf{r}) \, \mathrm{d}v + \int_{s_o(t)} [\rho(\mathscr{U} \cdot \delta \mathbf{r})(\mathscr{V} - \mathscr{U}) \cdot \mathbf{n}] \, \mathrm{d}s = 0, \tag{38}$$

where in the Lagrangian $\mathscr{L}_o$ of the open system the mass is not fixed.

Now integrating with respect to time over the interval t_1, t_2 and again requiring the system configurations at t_1, t_2 be prescribed, the extended form of Hamilton's principle for a system of changing mass can be expressed as

$$\delta \int_{t_1}^{t_2} \mathscr{L}_o \, \mathrm{d}t + \int_{t_1}^{t_2} \delta W \, \mathrm{d}t + \int_{t_1}^{t_2} \mathrm{d}t \int_{s_o(t)} [\rho(\mathscr{U} \cdot \delta \mathbf{r})(\mathscr{V} - \mathscr{U}) \cdot \mathbf{n}] \, \mathrm{d}s = 0, \tag{39}$$

where δW is the virtual work performed by the non-potential forces acting on the same system. If one only considers the contribution of virtual work due to surface tractions acting on the open and closed boundary of the system, then the extended Hamilton's principle for a system of changing mass becomes

$$\delta \int_{t_1}^{t_2} \mathscr{L}_o \, \mathrm{d}t + \int_{t_1}^{t_2} \mathrm{d}t + \int_{s_o(t)} (\sigma \cdot \mathbf{n}) \cdot \delta \mathbf{r} \, \mathrm{d}s + \int_{t_1}^{t_2} \mathrm{d}t \int_{s_o(t)} \\ \times ([\sigma + \rho \mathscr{U}(\mathscr{V} - \mathscr{U})] \cdot \mathbf{n}) \cdot \delta \mathbf{r} \, \mathrm{d}s = 0, \tag{40}$$

where σ is the stress tensor and $\sigma \cdot \mathbf{n}$ represents the surface traction vector.

It should be noted that not all virtual displacement distributions are permissible. The virtual displacements should not only satisfy the imposed geometrical constraints ensuring

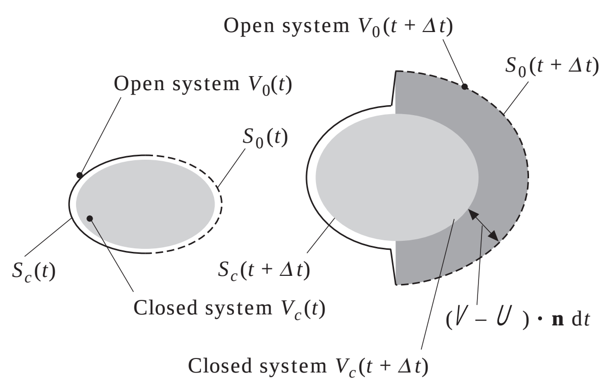

Figure 3. System of changing mass: open and closed control volumes.

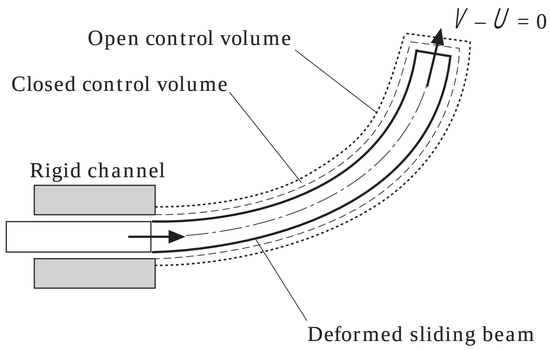

Figure 4. Sliding beam: open and closed control volumes: $\mathscr{U} \neq 0$, $\mathscr{V} = 0$, $\delta \mathbf{r} = 0$.

that there is no contribution of virtual work due to constraint forces, but must also satisfy the conservation of mass.

It is also worth noting that Hamilton's principle for closed systems for which there are no non-potential forces, is an extremum principle. This is not the case for systems of changing mass. This follows from the term accounting for changing mass in equation (39) which is in the form of a virtual work expression. In certain cases, by choosing particular boundary velocities, it is possible to render Hamilton's principle for a system of changing mass, a stationary principle, see McIver [19].

Returning to the sliding beam problem which is a system of changing mass, one applies the extended Hamilton's principle to obtain the governing equation of motion.

Figure 4 shows open and closed control volumes chosen for the sliding beam. At the tip of the beam the velocity of the material point is equal to the velocity of the moving boundary, i.e., $\mathscr{V}_{Tip} - \mathscr{U}_{Tip} = 0$. For simplicity, the only type of surface traction considered is a prescribed tip load independent of displacements. Hence, its virtual work can be expressed as a total differential.

At the root, the control surface does not move, but the material point on the centerline of the beam has the velocity $\mathscr{U}_{root}$. Here, one invokes the previous assumption on the motion of the beam being prescribed inside the channel where the beam is considered rigid. This precludes the existence of a non-conservative force at the root (see equation (40)). Now, since the lateral deflection and the slope of the beam (for a cantilever support) at the root are prescribed as zero, virtual displacements vanish at the wall and there is no virtual work contribution from the wall reaction shear force and moment.

Implementing the above statements into equation (40), the extended Hamilton's principle for the sliding beam, outside the channel, with the variable length $L(t)$ can be expressed as

$$\delta \int_{t_1}^{t_2} \mathscr{L}_o \mathrm{d}t = 0, \tag{41}$$

which is the form of Hamilton's principle used by Tabarrok *et al*. [3].

2.2.2. *Complementary kinetic energy*

In the absence of rotary inertia effects, the velocity of point m, on the centerline of the sliding beam, Figure 5, is made of two parts:

(1) Due to rigid body sliding motion of the beam:

$$\mathbf{V}_{rb} = V\boldsymbol{\tau} = V(\partial x_1/\partial S)\mathbf{I}_1 + V\,\partial x_2/\partial S\mathbf{I}_2, \tag{42}$$

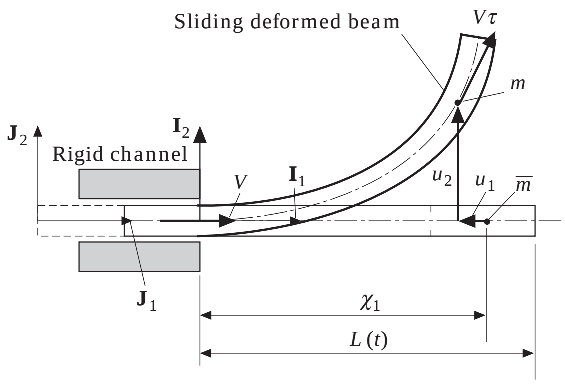

Figure 5. Axially inextensible sliding beam undergoing large overall motion: kinematics of deformation.

where V is the prescribed sliding velocity of the beam and $\boldsymbol{\tau}$ is the unit vector tangent to the centerline of the beam at the same material point.

(2) Due to elastic deformation:

$$\mathbf{V}_e = (\partial u_1/\partial t)\mathbf{I}_1 + (\partial u_2/\partial t)\mathbf{I}_2 \tag{43}$$

or

$$\mathbf{V}_e = (\partial x_1/\partial t)\mathbf{I}_1 + (\partial x_2/\partial t)\mathbf{I}_2. \tag{44}$$

Therefore, the kinetic co-energy for the sliding beam may be written as

$$T^* = \int_0^{L(t)} \tfrac{1}{2}\rho A\left[\left(\frac{\partial x_1}{\partial t} + V\frac{\partial x_1}{\partial S}\right)^2 + \left(\frac{\partial x_2}{\partial t} + V\frac{\partial x_2}{\partial S}\right)^2\right]\mathrm{d}S. \tag{45}$$

2.2.3. *Strain energy*

For an axially inextensible beam ($\varepsilon = 0$) in the case of large rotations and small strains, the strain energy correct to $\mathcal{O}(\varepsilon^4)$ was given by Stoker [20] as

$$\Pi = \int_0^{L(t)} \frac{EI}{2}\kappa^2\,\mathrm{d}S, \tag{46}$$

where I is the appropriate second moment of area and E is Young's modulus of the beam material. Now, substituting for the curvature from equation (26), the strain energy may be expressed as

$$\Pi = \int_0^{L(t)} \frac{EI}{2}\left(\frac{\partial^2 x_2}{\partial S^2}\right)^2\left[1 + \left(\frac{\partial x_2}{\partial S}\right)^2\right]\mathrm{d}S. \tag{47}$$

2.2.4. *Lagrangian*

For a flexible sliding beam which undergoes large overall motion and is axially inextensible, using equations (45) and (47), the Lagrangian can be expressed as

$$\mathscr{L}_o = T^* - \Pi$$

$$= \int_0^{L(t)} \tfrac{1}{2}\rho A\left[\left(\frac{\partial x_1}{\partial t} + V\frac{\partial x_1}{\partial S}\right)^2 + \left(\frac{\partial x_2}{\partial t} + V\frac{\partial x_2}{\partial S}\right)^2\right]\mathrm{d}S$$

$$- \int_0^{L(t)} \frac{EI}{2}\left(\frac{\partial^2 x_2}{\partial S^2}\right)^2\left[1 + \left(\frac{\partial x_2}{\partial S}\right)^2\right]\mathrm{d}S. \qquad (48)$$

2.2.5. *Equation of motion*

After many straight forward but tedious manipulations, the governing equation of motion for an axially inextensible flexible sliding beam, in the absence of traction and body forces, may be obtained as the stationary condition of the extended Hamilton's principle (41) as follows (see Behdinan [21]):

In the domain:

$$\rho A\left(\frac{\partial^2 x_2}{\partial t^2} + 2V\frac{\partial^2 x_2}{\partial t\partial S}\left[1 + \left(\frac{\partial x_2}{\partial S}\right)^2\right] + V^2\frac{\partial^2 x_2}{\partial S^2}\left[1 + \left(\frac{\partial x_2}{\partial S}\right)^2\right]\right.$$

$$\left. + \frac{\partial V}{\partial t}\frac{\partial^2 x_2}{\partial S^2}(L - S)\left[1 + \tfrac{3}{2}\left(\frac{\partial x_2}{\partial S}\right)^2\right]\right)$$

$$+ EI\left(\frac{\partial^4 x_2}{\partial S^4}\left[1 + \left(\frac{\partial x_2}{\partial S}\right)^2\right] + \left(\frac{\partial^2 x_2}{\partial S^2}\right)^3 + 4\frac{\partial x_2}{\partial S}\frac{\partial^2 x_2}{\partial S^2}\frac{\partial^3 x_2}{\partial S^3}\right)$$

$$+ \rho A\frac{\partial x_2}{\partial S}\int_0^S\left[\frac{\partial x_2}{\partial S}\frac{\partial^3 x_2}{\partial t^2\partial S} + \left(\frac{\partial^2 x_2}{\partial t\partial S}\right)^2\right]\mathrm{d}S$$

$$- \rho A\frac{\partial^2 x_2}{\partial S^2}\int_S^{L(t)}\int_0^S\left[\frac{\partial x_2}{\partial S}\frac{\partial^3 x_2}{\partial t^2\partial S} + \left(\frac{\partial^2 x_2}{\partial t\partial S}\right)^2\right]\mathrm{d}S\,\mathrm{d}S$$

$$- \rho A\frac{\partial^2 x_2}{\partial S^2}\int_S^{L(t)}\left[\frac{1}{2}\frac{\partial V}{\partial t}\left(\frac{\partial x_2}{\partial S}\right)^2 + 2V\frac{\partial x_2}{\partial S}\frac{\partial^2 x_2}{\partial t\partial S} + V^2\frac{\partial x_2}{\partial S}\frac{\partial^2 x_2}{\partial S^2}\right]\mathrm{d}S = 0. \qquad (49)$$

At the boundaries:

$$EI\left\{\frac{\partial^3 x_2}{\partial S^3}\left[1 + \tfrac{1}{2}\left(\frac{\partial x_2}{\partial S}\right)^2\right] + \left(\frac{\partial^2 x_2}{\partial S^2}\right)^2\frac{\partial x_2}{\partial S}\right\}\delta x_2\Bigg|_0^{L(t)} = 0,$$

$$EI\frac{\partial^2 x_2}{\partial S^2}\left[1 + \tfrac{1}{2}\left(\frac{\partial x_2}{\partial S}\right)^2\right]\delta\left(\frac{\partial x_2}{\partial S}\right)\Bigg|_0^{L(t)} = 0. \qquad (50)$$

Equation (49) is a non-linear partial integro-differential equation. Equations (50) provide all the boundary conditions, namely the essential boundary conditions for prescribed kinematic variables and the corresponding natural boundary conditions for force variables.

Removing terms of second and higher order, one obtains the following linear equation of motion:

$$\rho A\left[\frac{\partial^2 x_2}{\partial t^2} + 2V\frac{\partial^2 x_2}{\partial t\partial S} + V^2\frac{\partial^2 x_2}{\partial S^2} + \frac{\partial V}{\partial t}\frac{\partial^2 x_2}{\partial S^2}(L-S)\right] + EI\frac{\partial^4 x_2}{\partial S^4} = 0, \tag{51}$$

which expresses the dynamic equilibrium for the linear axially inextensible flexible sliding beam first derived by Tabarrok *et al*. [3].

2.2.6. *Sliding beam in uniform gravitational field*

The potential energy of the flexible sliding beam in a uniform gravitational field g along x_1 is given by

$$G = \rho Ag\int_0^{L(t)} x_1\,\mathrm{d}S. \tag{52}$$

For the inextensible case, the inclusion of the potential energy due to gravity gives rise to two additional terms on the left side of the governing equation of motion (49), namely:

$$\rho Ag\frac{\partial x_2}{\partial S}\left[1 + \tfrac{1}{2}\left(\frac{\partial x_2}{\partial S}\right)^2\right] - \rho Ag\frac{\partial^2 x_2}{\partial S^2}(L-S)\left[1 + \tfrac{3}{2}\left(\frac{\partial x_2}{\partial S}\right)^2\right].$$

2.3. EQUATION OF MOTION FOR AXIALLY EXTENSIBLE FLEXIBLE SLIDING BEAMS

In this case, the beam is axially extensible, i.e., $\varepsilon \neq 0$, and one needs to account for transverse *and* axial motions. Thus for the axially flexible sliding beam two non-linear coupled partial differential equations are to be derived. It is important to note that the discussion in the previous section (inextensible sliding beams) remains valid and the equations of motion may be obtained from the extended Hamilton's principle as given in equation (41).

2.3.1. *Complementary kinetic energy*

The velocity of a material point on the centerline of the beam can be expressed as, Figure 6:

$$\mathrm{d}x/\mathrm{d}t = (\mathrm{d}x_1/\mathrm{d}t)\mathbf{I}_1 + (\mathrm{d}x_2/\mathrm{d}t)\mathbf{I}_2. \tag{53}$$

Now, substituting the relations (8) and (9) into equation (53), one obtains

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(\frac{\mathrm{d}\chi_1}{\mathrm{d}t} + \frac{\partial u_1}{\partial\chi_1}\frac{\partial\chi_1}{\partial t} + \frac{\partial u_1}{\partial t}\right)\mathrm{I}_1 + \left(\frac{\partial u_2}{\partial\chi_1}\frac{\partial\chi_1}{\partial t} + \frac{\partial u_2}{\partial t}\right)\mathrm{I}_2, \tag{54}$$

where $\partial\chi_1/\partial t$ is the prescribed velocity V of the beam. Equation (54) may be simplified to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left[\frac{\partial u_1}{\partial t} + V\left(1 + \frac{\partial u_1}{\partial\chi_1}\right)\right]\mathbf{I}_1 + \left[\frac{\partial u_2}{\partial t} + V\frac{\partial u_2}{\partial\chi_1}\right]\mathbf{I}_2, \tag{55}$$

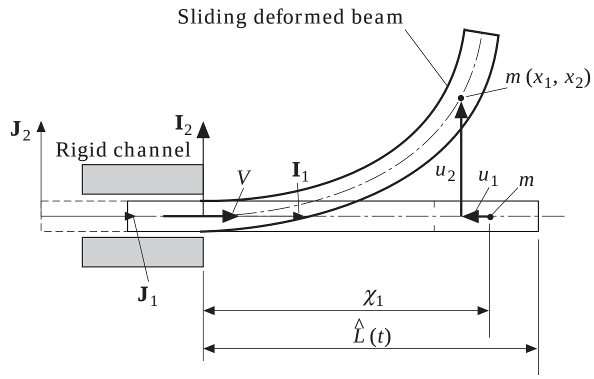

Figure 6. Flexible sliding beam undergoing large overall motion: kinematics of deformation.

which describes the velocity of a material point on the centerline of the beam. Using equation (55), the kinetic co-energy of the sliding beam may be expressed as

$$T^* = \int_0^{\hat{L}(t)} \tfrac{1}{2}\rho_0 A\left\{\left[\frac{\partial u_1}{\partial t} + V\left(1 + \frac{\partial u_1}{\partial \chi_1}\right)\right]^2 + \left[\frac{\partial u_2}{\partial t} + V\frac{\partial u_2}{\partial \chi_1}\right]^2\right\} \mathrm{d}\chi_1, \tag{56}$$

where $\hat{L}(t)$ is the length of the protruding part of the undeformed sliding beam and ρ_0 is the mass density of the undeformed beam.

2.3.2. *Strain energy*

For an axially flexible beam undergoing large overall motions with finite rotations and small strains, Stoker [20] has shown that if shear is neglected, the axial strain may be expressed as

$$\varepsilon_{\chi_1} = \varepsilon - (1 + \varepsilon)\kappa\chi_2. \tag{57}$$

Then an increment of the strain energy is taken to be

$$\mathrm{d}\Pi = (Eb/2)\varepsilon_{\chi_1}^2 \,\mathrm{d}\chi_1\,\mathrm{d}\chi_2, \tag{58}$$

where b is the width of the beam. Now one can express the strain energy of the beam as

$$\Pi = \int_{-h/2}^{h/2}\int_0^{\hat{L}(t)} \frac{Eb}{2}[\varepsilon - (1 + \varepsilon)\kappa\chi_2]^2 \,\mathrm{d}\chi_1\,\mathrm{d}\chi_2, \tag{59}$$

where h is the height of the beam's cross section. After simplifying the above relation, the strain energy becomes

$$\Pi = \int_0^{\hat{L}(t)} \frac{E}{2}[A\varepsilon^2 + I(1 + \varepsilon)^2\kappa^2] \,\mathrm{d}\chi_1, \tag{60}$$

where A is the undeformed cross-sectional area of the beam. Expressing ε and κ via equations (18), (20) and (25), one may write the strain energy in terms of the displacement components as follows:

$$\Pi = \int_0^{\hat{L}(t)} \tfrac{1}{2} EA\left[\frac{\partial u_1}{\partial \chi_1} + \tfrac{1}{2}\left(\frac{\partial u_2}{\partial \chi_1}\right)^2\right]^2 \mathrm{d}\chi_1$$

$$+ \int_0^{\hat{L}(t)} \tfrac{1}{2} EI\left[\left(\frac{\partial^2 u_2}{\partial \chi_1^2}\right)^2 - 2\left(\frac{\partial^2 u_2}{\partial \chi_1^2}\right)^2 \frac{\partial u_1}{\partial \chi_1} - 2\left(\frac{\partial^2 u_2}{\partial \chi_1^2}\right)^2 \left(\frac{\partial u_2}{\partial \chi_1}\right)^2 - 2\frac{\partial u_2}{\partial \chi_1}\frac{\partial^2 u_2}{\partial \chi_1^2}\frac{\partial^2 u_1}{\partial \chi_1^2}\right] \mathrm{d}\chi_1. \quad (61)$$

2.3.3. *Equations of Motion*

With expression for T^* and Π at hand the equations of motion from Hamilton's principle are derived as follows. In this case one has two independent variables (χ_1, t) and two dependent variables and (u_1, u_2).

In the domain:

$$\rho_0 A\left[\frac{\partial^2 u_1}{\partial t^2} + \frac{\partial V}{\partial t}\left(1 + \frac{\partial u_1}{\partial \chi_1}\right) + 2V\frac{\partial^2 u_1}{\partial t \partial \chi_1} + V^2 \frac{\partial^2 u_1}{\partial \chi_1^2}\right]$$

$$- EA\left[\frac{\partial^2 u_1}{\partial \chi_1^2} + \frac{\partial u_2}{\partial \chi_1}\frac{\partial^2 u_2}{\partial \chi_1^2}\right] - EI\left[\frac{\partial^2 u_2}{\partial \chi_1^2}\frac{\partial^3 u_2}{\partial \chi_1^3} + \frac{\partial u_2}{\partial \chi_1}\frac{\partial^4 u_2}{\partial \chi_1^4}\right] = 0, \quad (62)$$

$$\rho_0 A\left[\frac{\partial^2 u_2}{\partial t^2} + \frac{\partial V}{\partial t}\frac{\partial u_2}{\partial \chi_1} + 2V\frac{\partial^2 u_2}{\partial t \partial \chi_1} + V^2 \frac{\partial^2 u_2}{\partial \chi_1^2}\right] - EA\left[\frac{\partial u_1}{\partial \chi_1}\frac{\partial^2 u_2}{\partial \chi_1^2} + \frac{\partial u_2}{\partial \chi_1}\frac{\partial^2 u_1}{\partial \chi_1^2} + \tfrac{3}{2}\left(\frac{\partial u_2}{\partial \chi_1}\right)^2 \frac{\partial^2 u_2}{\partial \chi_1^2}\right]$$

$$- EI\left[3\frac{\partial^3 u_1}{\partial \chi_1^3}\frac{\partial^2 u_2}{\partial \chi_1^2} + 2\frac{\partial u_1}{\partial \chi_1}\frac{\partial^4 u_2}{\partial \chi_1^4} + \frac{\partial u_2}{\partial \chi_1}\frac{\partial^4 u_1}{\partial \chi_1^4} + 2\left(\frac{\partial u_2}{\partial \chi_1}\right)^2 \frac{\partial^4 u_2}{\partial \chi_1^4} + 8\frac{\partial u_2}{\partial \chi_1}\frac{\partial^2 u_2}{\partial \chi_1^2}\frac{\partial^3 u_2}{\partial \chi_1^3}\right.$$

$$\left. + 2\left(\frac{\partial^2 u_2}{\partial \chi_1^2}\right)^3 + 4\frac{\partial^3 u_2}{\partial \chi_1^3}\frac{\partial^2 u_1}{\partial \chi_1^2} - \frac{\partial^4 u_2}{\partial \chi_1^4}\right] = 0. \quad (63)$$

At the boundaries:

$$\left\{EA\left[\frac{\partial u_1}{\partial \chi_1} + \tfrac{1}{2}\left(\frac{\partial u_2}{\partial \chi_1}\right)^2\right] + EI\frac{\partial u_2}{\partial \chi_1}\frac{\partial^3 u_2}{\partial \chi_1^3}\right\}\delta u_1\Bigg|_0^{\hat{L}(t)} = 0,$$

$$EI\frac{\partial u_2}{\partial \chi_1}\frac{\partial^2 u_2}{\partial \chi_1^2}\,\delta\left(\frac{\partial u_1}{\partial \chi_1}\right)\Bigg|_0^{\hat{L}(t)} = 0, \quad (64)$$

$$\left\{EA\frac{\partial u_2}{\partial \chi_1}\left[\frac{\partial u_1}{\partial \chi_1}+\tfrac{1}{2}\left(\frac{\partial u_2}{\partial \chi_1}\right)^2\right]-EI\left[\frac{\partial^3 u_2}{\partial \chi_1^3}-2\frac{\partial^3 u_2}{\partial \chi_1^3}\frac{\partial u_1}{\partial \chi_1}-2\frac{\partial^2 u_2}{\partial \chi_1^2}\frac{\partial^2 u_1}{\partial \chi_1^2}-2\frac{\partial^3 u_2}{\partial \chi_1^3}\left(\frac{\partial u_2}{\partial \chi_1}\right)^2\right.\right.$$

$$\left.\left.-2\left(\frac{\partial^2 u_2}{\partial \chi_1^2}\right)^2\frac{\partial u_2}{\partial \chi_1}-\frac{\partial u_2}{\partial \chi_1}\frac{\partial^3 u_1}{\partial \chi_1^3}\right]\right\}\delta u_2\Bigg|_0^{\hat{L}(t)}=0,$$

$$\left\{EI\left[\partial^2 u_2/\partial \chi_1^2-2\frac{\partial^2 u_2}{\partial \chi_1^2}\frac{\partial u_1}{\partial \chi_1}-2\frac{\partial^2 u_2}{\partial \chi_1^2}\left(\frac{\partial u_2}{\partial \chi_1}\right)^2-\frac{\partial u_2}{\partial \chi_1}\frac{\partial^2 u_1}{\partial \chi_1^2}\right]\right\}\delta\left(\frac{\partial u_2}{\partial \chi_1}\right)\Bigg|_0^{\hat{L}(t)}=0. \quad (65)$$

Equations (62) and (63) are two non-linear, coupled partial differential equations which describe the dynamics of an axially extensible flexible sliding beam undergoing large overall motion. Through numerical techniques one can obtain approximate solutions for these governing equations. These will be discussed in part II of this paper. Equations (64) and (65) express the kinematic and natural boundary conditions of the beam.

In the next section, an alternative formulation wherein the beam remains fixed (no axial motion) and the channel moves with a prescribed velocity along the beam is considered.

3. ALTERNATIVE FORMULATION—THE SLIDING CHANNEL

In the sliding beam problem the beam emerges with a prescribed velocity from a rigid channel which is at rest. The inertial reference frame is attached to the channel and the motion is observed by an inertial observer placed, say, on the channel. Now a translational velocity is superimposed on the system with the intent of bringing the beam to rest. Then the channel and the observer will move with this imposed velocity. In the following the equations of motion of the system as seen by the same observer which is now moving will be obtained.

It is important to note that in this case the reference axes attached to the moving channel is not, in general, an inertial frame. The exception occurs when the superimposed translational velocity is constant. Then the moving axes is an acceptable inertial frame and it is related to the original fixed inertial frame through a Galilean transformation. Now the initial undeformed configuration which is fixed in space will be taken as the material configuration. The part inside the channel is assumed to be rigid and the protruding part undergoes large overall motion with a moving boundary with respect to the material frame; see Figure 7.

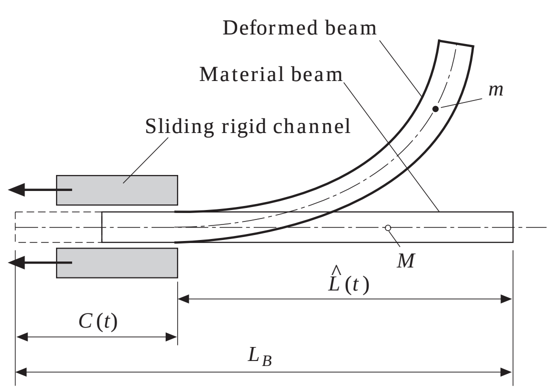

Figure 7. Sliding channel: material and spatial configurations.

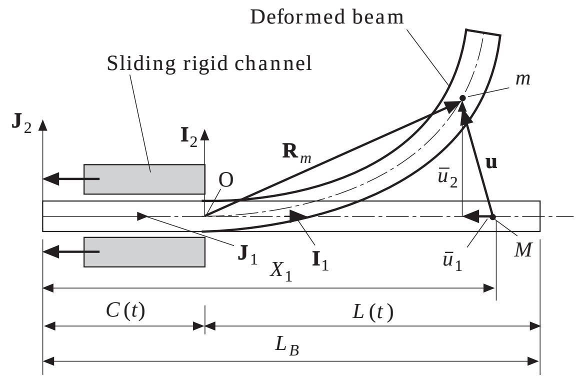

Figure 8. Axially inextensible beam with sliding channel undergoing large overall motion: kinematics of the deformation.

Evidently, the defined material configuration has the role of the Lagrangian domain with respect to the deformed configuration. In this case, the structural deformation is described by a mapping, which is time dependent, from the material configuration to the deformed configuration.

3.1. KINEMATIC DESCRIPTIONS

As noted earlier one will obtain the differential equation of motion as seen by an observer O moving with the channel.

If one considers the beam to be initially lying along the $\mathbf{J}_1$ axis (Figure 8), then the length of the beam inside the channel is given by

$$\mathbf{C}(t) = [L_B - L(t)]\mathbf{J}_1, \tag{66}$$

where L_B is the total length of the beam and $L(t)$ is the time varying length of the beam outside the channel.

The deformation from the material configuration to the deformed configuration is a Lagrangian description; thus a point M on the centerline of the undeformed beam maps to a new position m in the deformed configuration. Since the part of the beam inside the channel is considered to be rigid, one may refer the displacement vector $\bar{\mathbf{u}}(X_1, t)$, from M to m, to the spatial frame $\mathbf{I}_1$ and $\mathbf{I}_2$. Now the position vector of a point m on the centerline of the deformed beam may be expressed as (Figure 8):

$$\mathbf{R}_m = [X_1 - C(t)]\mathbf{I}_1 + \bar{\mathbf{u}}. \tag{67}$$

3.2. EQUATION OF MOTION OBTAINED FROM ALTERNATIVE FORMULATION

In this section, only terms up to the third order will be retained in the governing equations of motion.

3.2.1. *Extended Hamilton's principle*

Since the boundary is moving and the length of the beam outside the channel changes with time, this problem can also be viewed as a system of changing mass. Following the discussion for axially inextensible sliding beams, the governing equation of motion for the moving boundary problem is obtained via the extended Hamilton's principle.

Figure 9 shows the open and closed control volumes for the problem at hand. At the tip of the beam the velocity of the moving open control surface is equal to the velocity

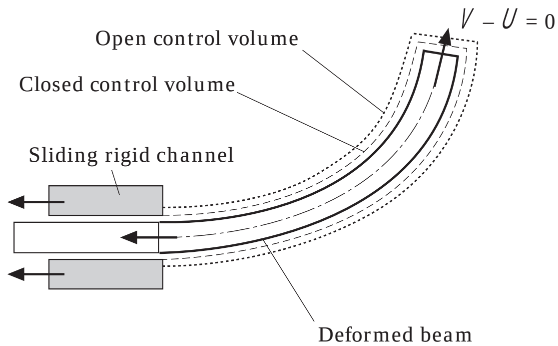

Figure 9. Sliding channel: open and closed control volumes; $\mathscr{U} = 0$, $\mathscr{V} \neq 0$, $\delta \mathbf{r} = 0$.

of the material points. Once again, one assumes that the virtual work of possible prescribed traction forces at the tip is expressed as a total differential.

At the root, one has a moving boundary with a prescribed sliding velocity; therefore the open control surface has the prescribed velocity $\mathscr{V}_{root}$. Since the beam is considered axially fixed, the material point on the centerline at the root of the beam is motionless. Also, it should be noted that the beam is considered to be axially rigid inside the channel and the lateral deflection and slope are prescribed at the root. Hence, there are no virtual work terms from the reaction forces at the root. Under these conditions the extended Hamilton's principle takes the form given in equation (41).

3.2.2. *Complementary kinetic energy*

As seen by the observer on the moving channel (see Figure 8), the velocity of a mass point $m(x_1, x_2)$ on the centerline of the beam in the deformed configuration may be obtained by differentiating equation (67) i.e.:

$$\mathbf{V}_m = \partial \mathbf{R}_m/\partial t = (\partial C/\partial t)\mathbf{I}_1 + (\partial/\partial t)\bar{\mathbf{u}} \tag{68}$$

or

$$\mathbf{V}_m = (V + \partial \bar{u}_1/\partial t)\mathbf{I}_1 + (\partial \bar{u}_2/\partial t)\mathbf{I}_2. \tag{69}$$

Thus, the kinetic co-energy with reference to the moving axes $\mathbf{I}_1$, $\mathbf{I}_2$ for the part of the beam outside the channel can be expressed as

$$T^* = \int_{C(t)}^{L_B} \tfrac{1}{2}\rho A[(V + \partial \bar{u}_1/\partial t)^2 + (\partial \bar{u}_2/\partial t)^2]\, \mathrm{d}X_1, \tag{70}$$

or in terms of the spatial co-ordinates $(\bar{x}_1, \bar{x}_2)$, using the relations

$$\bar{x}_1 = X_1 - C(t) + \bar{u}_1, \qquad \bar{x}_2 = \bar{u}_2. \tag{71}$$

The kinetic co-energy may be expressed as

$$T^* = \int_{C(t)}^{L_B} \tfrac{1}{2}\rho A\left[\left(\frac{\partial \bar{x}_1}{\partial t}\right)^2 + \left(\frac{\partial \bar{x}_2}{\partial t}\right)^2\right] \mathrm{d}X_1. \tag{72}$$

3.2.3. *Strain energy*

For an axially inextensible beam, the expression for the strain energy is as given in equation (47), namely

$$\Pi = \int_{C(t)}^{L_B} \frac{EI}{2}\left(\frac{\partial^2 \bar{x}_2}{\partial S^2}\right)^2\left[1 + \left(\frac{\partial \bar{x}_2}{\partial S}\right)^2\right] \mathrm{d}S. \tag{73}$$

3.2.4. *Equation of motion*

Now obtaining the Lagrangian $L = T^* - \Pi$ and substituting the Lagrangian into the extended Hamilton's principle and carrying out the indicated variation, one obtains the governing equation of motion of the axially inextensible flexible beam with time dependent moving boundary as follows.

In the domain:

$$\begin{aligned}
&\rho A\,\frac{\partial^2 \bar{x}_2}{\partial t^2} + EI\left(\frac{\partial^4 \bar{x}_2}{\partial S^4}\left[1 + \left(\frac{\partial \bar{x}_2}{\partial S}\right)^2\right] + \left(\frac{\partial^2 \bar{x}_2}{\partial S^2}\right)^3 + 4\,\frac{\partial \bar{x}_2}{\partial S}\,\frac{\partial^2 \bar{x}_2}{\partial S^2}\,\frac{\partial^3 \bar{x}_2}{\partial S^3}\right) \\
&\quad + \rho A\,\frac{\partial \bar{x}_2}{\partial S}\int_{C(t)}^{S}\left[\frac{\partial \bar{x}_2}{\partial S}\,\frac{\partial^3 \bar{x}_2}{\partial t^2 \partial S} + \left(\frac{\partial^2 \bar{x}_2}{\partial t \partial S}\right)^2\right]\mathrm{d}S \\
&\quad - \rho A\,\frac{\partial^2 \bar{x}_2}{\partial S^2}\int_{S}^{L_B}\int_{C(t)}^{S}\left[\frac{\partial \bar{x}_2}{\partial S}\,\frac{\partial^3 \bar{x}_2}{\partial t^2 \partial S} + \left(\frac{\partial^2 \bar{x}_2}{\partial t \partial S}\right)^2\right]\mathrm{d}S\,\mathrm{d}S = 0.
\end{aligned} \tag{74}$$

At the boundaries:

$$\begin{aligned}
&EI\left\{\frac{\partial^3 \bar{x}_2}{\partial S^3}\left[1 + \tfrac{1}{2}\left(\frac{\partial \bar{x}_2}{\partial S}\right)^2\right] + \left(\frac{\partial^2 \bar{x}_2}{\partial S^2}\right)^2 \frac{\partial \bar{x}_2}{\partial S}\right\}\delta \bar{x}_2\Bigg|_{C(t)}^{L_B} = 0, \\
&EI\,\frac{\partial^2 \bar{x}_2}{\partial S^2}\left[1 + \tfrac{1}{2}\left(\frac{\partial \bar{x}_2}{\partial S}\right)^2\right]\delta\left(\frac{\partial \bar{x}_2}{\partial S}\right)\Bigg|_{C(t)}^{L_B} = 0.
\end{aligned} \tag{75}$$

Equation (74) is a non-linear partial integro-differential equation. It will be noted that this equation appears simpler than that obtained for the sliding beam given in equation (49). This justifies the advantages of the new formulation. Since the beam is fixed, there are no convective terms in the kinetic co-energy. Accordingly this formulation may be considered as a full Lagrangian formulation. Equation (75) gives the kinematic and natural boundary conditions of the system. Details of the derivation of the above equations may be found in Behdinan [21].

3.3. EQUATION OF MOTION FOR AXIALLY EXTENSIBLE FLEXIBLE BEAMS WITH SLIDING BOUNDARIES

For the axially extensible beam, $\varepsilon \neq 0$, two non-linear coupled equations are expected. The extended Hamilton's principle in the form used in the previous section will also be used for this problem.

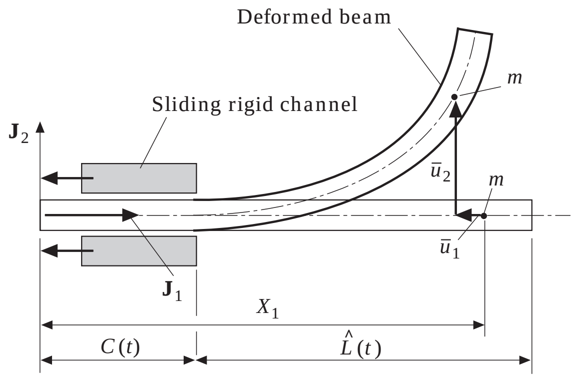

Figure 10. Flexible beam with sliding channel undergoing large overall motion: kinematics of deformation.

3.3.1. *Complementary kinetic energy*

The kinetic co-energy may be written as, see Figure 10:

$$T^* = \int_{C(t)}^{L_B} \tfrac{1}{2}\rho_0 A\left[\left(V + \frac{\partial \bar{u}_1}{\partial t}\right)^2 + \left(\frac{\partial \bar{u}_2}{\partial t}\right)^2\right] \mathrm{d}X_1. \tag{76}$$

It should be noted that in this case, $\bar{u}_1(X_1, t)$ and $\bar{u}_2(X_1, t)$ are independent.

3.3.2. *Strain energy*

The expression in equation (61) may also be used for the strain energy of the deformed beam outside the moving channel:

$$\Pi = \int_{C(t)}^{L_B} \tfrac{1}{2}EA\left[\frac{\partial \bar{u}_1}{\partial X_1} + \tfrac{1}{2}\left(\frac{\partial \bar{u}_2}{\partial X_1}\right)^2\right]^2 \mathrm{d}X_1 + \int_{C(t)}^{L_B} \tfrac{1}{2}EI\left[\left(\frac{\partial^2 \bar{u}_2}{\partial X_1^2}\right)^2 - 2\left(\frac{\partial^2 \bar{u}_2}{\partial X_1^2}\right)^2 \frac{\partial \bar{u}_1}{\partial X_1}\right.$$

$$\left. - 2\left(\frac{\partial^2 \bar{u}_2}{\partial X_1^2}\right)^2\left(\frac{\partial \bar{u}_2}{\partial X_1}\right)^2 - 2\,\frac{\partial \bar{u}_2}{\partial X_1}\frac{\partial^2 \bar{u}_2}{\partial X_1^2}\frac{\partial^2 \bar{u}_1}{\partial X_1^2}\right] \mathrm{d}X_1. \tag{77}$$

3.3.3. *Equation of motion*

Since the Lagrangian $L = T^* - \Pi$ is a second order functional with two independent variables (X_1, t) and two dependent functions $(\bar{u}_1(X_1, t), \bar{u}_2(X_1, t))$, two coupled Euler–Lagrange equations are obtained from the stationary conditions of Hamilton's principle [21]. Using the extended Hamilton's principle one finds the stationary conditions as follows:

In the domain:

$$\rho_0 A\left[\frac{\partial^2 \bar{u}_1}{\partial t^2} + \frac{\partial V}{\partial t}\right] - EA\left[\frac{\partial^2 \bar{u}_1}{\partial X_1^2} + \frac{\partial \bar{u}_2}{\partial X_1}\frac{\partial^2 \bar{u}_2}{\partial X_1^2}\right] - EI\left[\frac{\partial^2 \bar{u}_2}{\partial X_1^2}\frac{\partial^3 \bar{u}_2}{\partial X_1^3} + \frac{\partial \bar{u}_2}{\partial X_1}\frac{\partial^4 \bar{u}_2}{\partial X_1^4}\right] = 0, \tag{78}$$

$$\rho_0 A\left[\frac{\partial^2 \bar{u}_2}{\partial t^2}\right] - EA\left[\frac{\partial \bar{u}_1}{\partial X_1}\frac{\partial^2 \bar{u}_2}{\partial X_1^2} + \frac{\partial \bar{u}_2}{\partial X_1}\frac{\partial^2 \bar{u}_1}{\partial X_1^2} + \tfrac{3}{2}\left(\frac{\partial \bar{u}_2}{\partial X_1}\right)^2\frac{\partial^2 \bar{u}_2}{\partial X_1^2}\right]$$

$$- EI\left[3\,\frac{\partial^3 \bar{u}_1}{\partial X_1^3}\frac{\partial^2 \bar{u}_2}{\partial X_1^2} + 2\,\frac{\partial \bar{u}_1}{\partial X_1}\frac{\partial^4 \bar{u}_2}{\partial X_1^4} + \frac{\partial \bar{u}_2}{\partial X_1}\frac{\partial^4 \bar{u}_1}{\partial X_1^4}\right.$$

$$\left. + 2\left(\frac{\partial \bar{u}_2}{\partial X_1}\right)^2\frac{\partial^4 \bar{u}_2}{\partial X_1^4} + 8\,\frac{\partial \bar{u}_2}{\partial X_1}\frac{\partial^2 \bar{u}_2}{\partial X_1^2}\frac{\partial^3 \bar{u}_2}{\partial X_1^3} + 2\left(\frac{\partial^2 \bar{u}_2}{\partial X_1^2}\right)^3 + 4\,\frac{\partial^3 \bar{u}_2}{\partial X_1^3}\frac{\partial^2 \bar{u}_1}{\partial X_1^2} - \frac{\partial^4 \bar{u}_2}{\partial X_1^4}\right] = 0. \quad (79)$$

At the boundaries:

$$\left\{EA\left[\frac{\partial \bar{u}_1}{\partial X_1} + \tfrac{1}{2}\left(\frac{\partial \bar{u}_2}{\partial X_1}\right)^2\right] + EI\,\frac{\partial \bar{u}_2}{\partial X_1}\frac{\partial^3 \bar{u}_2}{\partial X_1^3}\right\}\delta\bar{u}_1\Bigg|_{C(t)}^{L_B} = 0,$$

$$EI\,\frac{\partial \bar{u}_2}{\partial X_1}\frac{\partial^2 \bar{u}_2}{\partial X_1^2}\,\delta\left(\frac{\partial \bar{u}_1}{\partial X_1}\right)\Bigg|_{C(t)}^{L_B} = 0, \quad (80)$$

$$\left\{EA\,\frac{\partial \bar{u}_2}{\partial X_1}\left[\frac{\partial \bar{u}_1}{\partial X_1} + \tfrac{1}{2}\left(\frac{\partial \bar{u}_2}{\partial X_1}\right)^2\right] - EI\left[\frac{\partial^3 \bar{u}_2}{\partial X_1^3} - 2\,\frac{\partial^3 \bar{u}_2}{\partial X_1^3}\frac{\partial \bar{u}_1}{\partial X_1} - 2\,\frac{\partial^2 \bar{u}_2}{\partial X_1^2}\frac{\partial^2 \bar{u}_1}{\partial X_1^2}\right.\right.$$

$$\left.\left. - 2\,\frac{\partial^3 \bar{u}_2}{\partial X_1^3}\left(\frac{\partial \bar{u}_2}{\partial X_1}\right)^2 - 2\left(\frac{\partial^2 \bar{u}_2}{\partial X_1^2}\right)^2\frac{\partial \bar{u}_2}{\partial X_1} - \frac{\partial \bar{u}_2}{\partial X_1}\frac{\partial^3 \bar{u}_1}{\partial X_1^3}\right]\right\}\delta\bar{u}_2\Bigg|_{C(t)}^{L_B} = 0,$$

$$\left\{EI\left[\frac{\partial^2 \bar{u}_2}{\partial X_1^2} - 2\,\frac{\partial^2 \bar{u}_2}{\partial X_1^2}\frac{\partial \bar{u}_1}{\partial X_1} - 2\,\frac{\partial^2 \bar{u}_2}{\partial X_1^2}\left(\frac{\partial \bar{u}_2}{\partial X_1}\right)^2 - \frac{\partial \bar{u}_2}{\partial X_1}\frac{\partial^2 \bar{u}_1}{\partial X_1^2}\right]\right\}\delta\left(\frac{\partial \bar{u}_2}{\partial X_1}\right)\Bigg|_{C(t)}^{L_B} = 0. \quad (81)$$

Equations (78) and (79) are non-linear coupled partial differential equations describing the dynamics of the flexible beam with a moving boundary. Equations (80) and (81) provide the consistent kinematic and force boundary conditions for the same problem.

4. COMPARISON OF THE TWO FORMULATIONS

Since the two formulations describe the motion of the same physical system, they are related through a transformation. The required transformation relates the material and intermediate co-ordinates (Figures 5 and 10) and is given by

$$X_1 = \chi_1 + C(t), \qquad X_2 = \chi_2. \quad (82)$$

In one formulation, namely the sliding beam, the equations contain convective acceleration terms whereas in the other there are no convective accelerations.

Also, one has seen that the two formulations lead to non-linear equations of motion defined on the variable time domain $(0, L(t))$ for the sliding beam and $(C(t), L_B)$ for the alternative formulation. It is interesting to map the equations of the two formulations to a fixed domain at time t, i.e.:

$$\mathbf{u}(X_1, t)\rightarrow\hat{\mathbf{u}}(\eta_1(X_1, t), t) \quad (83)$$

and

$$\bar{\mathbf{u}}(\chi_1(X_1, t), t) \to \hat{\mathbf{u}}(\eta_1(X_1, t), t). \tag{84}$$

To this end one writes for the sliding beam formulation:

$$\begin{aligned} &\eta_1 = \chi_1/L(t), \qquad \text{where} \qquad (0 < \eta_1 \leqslant 1); \\ &\eta_2 = \chi_2, \qquad \text{where} \qquad (-h/2 < \eta_2 \leqslant h/2); \end{aligned} \tag{85}$$

or for mapping back:

$$\begin{aligned} &\chi_1 = L(t)\eta_1, \qquad \text{where} \qquad (0 < \chi_1 \leqslant L(t)); \\ &\chi_2 = \eta_2, \qquad \text{where} \qquad (-h/2 < \chi_2 \leqslant h/2). \end{aligned} \tag{86}$$

For the alternative formulation, to obtain the governing equation of motion in the fixed domain, one uses the transformation:

$$\begin{aligned} &\eta_1 = [X_1 - C(t)]/L(t), \qquad \text{where} \qquad (0 < \eta_1 \leqslant 1); \\ &\eta_2 = X_2, \qquad \text{where} \qquad (-h/2 < \eta_2 \leqslant h/2); \end{aligned} \tag{87}$$

or for mapping back:

$$\begin{aligned} &X_1 = L(t)\eta_1 + C(t), \qquad \text{where} \qquad (C(t) < X_1 \leqslant L_B); \\ &X_2 = \eta_2, \qquad \text{where} \qquad (-h/2 < X_2 \leqslant h/2). \end{aligned} \tag{88}$$

Substituting transformation (85) for the sliding beam formulation into equations (62) and (63), one finds the transformed equations as follows:

$$\begin{aligned} &\rho_0 A\left[L^2\frac{\partial^2 \hat{u}_1}{\partial t^2} + L^2\frac{\partial V}{\partial t} + 2LV(1-\eta_1)\frac{\partial^2 \hat{u}_2}{\partial t\partial \eta_1} + V^2(1-\eta_1)^2\frac{\partial^2 \hat{u}_1}{\partial \eta_1^2}\right. \\ &\qquad\left. + (1-\eta_1)\left(L\frac{\partial V}{\partial t} - 2V^2\right)\frac{\partial \hat{u}_1}{\partial \eta_1}\right] - EA\left[\frac{\partial^2 \hat{u}_1}{\partial \eta_1^2} + \frac{1}{L}\left(\frac{\partial \hat{u}_2}{\partial \eta_1}\frac{\partial^2 \hat{u}_2}{\partial \eta_1^2}\right)\right] \\ &\qquad - \frac{EI}{L^3}\left[\frac{\partial^2 \hat{u}_2}{\partial \eta_1^2}\frac{\partial^3 \hat{u}_2}{\partial \eta_1^3} + \frac{\partial \hat{u}_2}{\partial \eta_1}\frac{\partial^4 \hat{u}_2}{\partial \eta_1^4}\right] = 0 \end{aligned} \tag{89}$$

and

$$\begin{aligned} &\rho_0 A\left[L^2\frac{\partial^2 \hat{u}_2}{\partial t^2} + 2LV(1-\eta_1)\frac{\partial^2 \hat{u}_2}{\partial t\partial \eta_1} + V^2(1-\eta_1)^2\frac{\partial^2 \hat{u}_2}{\partial \eta_1^2} + (1-\eta_1)\left(L\frac{\partial V}{\partial t} - 2V^2\right)\frac{\partial \hat{u}_2}{\partial \eta_1}\right] \\ &\qquad - \frac{EA}{L}\left[\frac{\partial \hat{u}_1}{\partial \eta_1}\frac{\partial^2 \hat{u}_2}{\partial \eta_1^2} + \frac{\partial \hat{u}_2}{\partial \eta_1}\frac{\partial^2 \hat{u}_1}{\partial \eta_1^2} + \frac{3}{2L}\left(\frac{\partial \hat{u}_2}{\partial \eta_1}\right)^2\frac{\partial^2 \hat{u}_2}{\partial \eta_1^2}\right] \\ &\qquad - \frac{EI}{L^2}\left[\frac{3}{L}\frac{\partial^3 \hat{u}_1}{\partial \eta_1^3}\frac{\partial^2 \hat{u}_2}{\partial \eta_1^2} + \frac{2}{L}\frac{\partial \hat{u}_1}{\partial \eta_1}\frac{\partial^4 \hat{u}_2}{\partial \eta_1^4} + \frac{1}{L}\frac{\partial \hat{u}_2}{\partial \eta_1}\frac{\partial^4 \hat{u}_1}{\partial \eta_1^4} + \frac{2}{L^2}\left(\frac{\partial \hat{u}_2}{\partial \eta_1}\right)^2\frac{\partial^4 \hat{u}_2}{\partial \eta_1^4} + \frac{8}{L^2}\frac{\partial \hat{u}_2}{\partial \eta_1}\frac{\partial^2 \hat{u}_2}{\partial \eta_1^2}\frac{\partial^3 \hat{u}_2}{\partial \eta_1^3}\right. \\ &\qquad\left. + \frac{2}{L^2}\left(\frac{\partial^2 \hat{u}_2}{\partial \eta_1^2}\right)^3 + \frac{4}{L}\frac{\partial^3 \hat{u}_2}{\partial \eta_1^3}\frac{\partial^2 \hat{u}_1}{\partial \eta_1^2} - \frac{\partial^4 \hat{u}_2}{\partial \eta_1^4}\right] = 0. \end{aligned} \tag{90}$$

It remains now to use equation (87) for the alternative formulation into equations (78) and (79) to obtain the equation of motion for this problem in the fixed domain. Not surprisingly, these latter equations are identical to equations (89) and (90). This result confirms the self consistency of the two formulations. Details of derivations can be found in Behdinan [21].

5. CONCLUSIONS

A comprehensive formulation for the sliding beam problem has been outlined and an alternative formulation provided. Non-linearities due to large deflections have been considered in these formulations. The governing equations and the related boundary conditions were obtained as stationary conditions of the extended Hamilton's principle.

The Euler–Lagrange equations for both problems yield non-linear partial differential equations for which closed form solutions do not exist. Consistency in the two formulations has been demonstrated.

In both formulations, the special case of the axially inextensible beams has been considered. Such a constraint leads to a single non-linear partial integro-differential equation which in the case of the sliding beam problem, after eliminating the higher order terms, yields the well known linear sliding beam equation of motion.

Clearly the governing equations of motion are too complicated to solve analytically. In part II of this paper, numerical solutions for such dynamical systems will be explored.

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