Journal of Sound and Vibration (1997) **208**(4), 575–586

UNIFORM MOTION OF A CONSTANT LOAD ALONG A STRING ON AN ELASTICALLY SUPPORTED MEMBRANE

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(*Received* 23 *September* 1996, *and in final form* 24 *June* 1997)

The uniform motion of a constant load along an infinite string on a Winkler supported membrane is studied for several ratios of the load velocity and the wave velocity in the membrane and the string. To study the displacements of the system, deflection profiles of the membrane and the string are calculated and presented as graphs. Also, expressions for the equivalent stiffness of the Winkler supported membrane interacting with a string are derived analytically.

1. INTRODUCTION

With the development of high speed trains, it is necessary to take into account the coupling between the wave processes in the track and the supporting soil due to a train. In references [1–3] the critical velocities and the steady state behavior of a uniformly moving load along a beam on elastic half-space have been investigated, showing the relevance of this problem for the interaction of a high-speed train with tracks on softer subsoils. The references further show that the critical velocities of the load are located near the surface-wave velocity of the half-space.

For research purposes and for a qualitative understanding of the dynamic process both in the track and the supporting surface it is reasonable to consider a more simple model of the "track–subsoil" interaction. Hence the track will be modelled as a string and the half-space as an elastically supported membrane. The waves in the Winkler-supported membrane model the relevant surface waves in the half-space.

The steady state behavior of the system due to a uniformly moving constant load along the string is to be determined. This model shows the principal features of a field generated by a moving object over a one-dimensional system and the supporting surface and the dependency of this field on the load velocity. The results show that this model may be used to investigate the interaction problem in sub-, trans- and supercritical cases qualitatively as follows from a comparison with the results of more complex models used in references [1–3].

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2. MODEL

Consider an infinite string on an elastically supported membrane. A constant load is uniformly moving along the string as depicted in Figure 1.

It is assumed that the string is in continuous contact with the membrane at the line $y = 0$ all the time. Their vertical motion is described by well-known equations; see reference [4]. Both the membrane and the string have mass, no bending stiffness and the restoring force is due to tension.

The equations for the vertical displacement of the coupled system of the membrane and string are

$$U^m_{tt} - c_m^2\,(U^m_{xx} + U^m_{yy}) + \mu^2 U^m = -\frac{\delta(y)}{\rho}\,\{U^s_{tt} - c_s^2\,U^s_{xx} + \tilde{P}\,\delta(x - Vt)\},$$

$$-\infty < x, y, t < +\infty, \qquad \text{with} \qquad U^m_{ff} = \frac{\partial^2}{\partial f^2}\,U^m, \tag{1}$$

$$U^m(x, y = 0, t) = U^s(x, t);$$

$$c_m^2 = N^m/\rho^m, \qquad c_s^2 = N^s/\rho^s, \qquad \mu^2 = k/\rho^m, \qquad \rho = \rho^m/\rho^s, \qquad \tilde{P} = P/\rho^s,$$

where $U^m(x, y, t)$ and $U^s(x, t)$ are the vertical displacements of the membrane and the string respectively, N^m and N^s are the membrane and the string tensions respectively, c_m and c_s are the wave speeds in the membrane and in the string respectively, k is the stiffness of the elastic foundation of the membrane per unit square, ρ^m is the mass of the membrane per unit area, ρ^s is the mass of the string per unit length and P is the constant load.

3. GENERAL SOLUTION

To derive the general solution for the displacements in the system for all load velocities, one applies the following exponential Fourier transforms over time and spatial co-ordinates to equations (1):

$$W^m(\omega, k_1, k_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U^m(x, y, t)\exp\{\mathrm{i}(\omega t - k_1 x - k_2 y)\}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y,$$

$$W^s(\omega, k_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U^s(x, t)\exp\{\mathrm{i}(\omega t - k_1 x)\}\,\mathrm{d}t\,\mathrm{d}x. \tag{2}$$

This yields

$$W^m = -\frac{D^s(\omega, k_1)}{\rho D^m(\omega, k_1, k_2)}\,W^s + \frac{2\pi\tilde{P}\,\delta(\omega - k_1 V)}{\rho D^m(\omega, k_1, k_2)}, \tag{3}$$

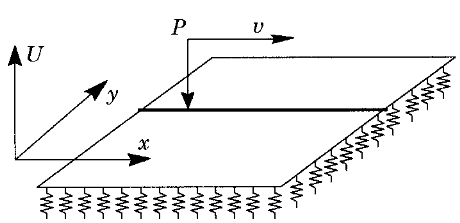

Figure 1. The model and reference system.

where $D^m(\omega, k_1, k_2) = \omega^2 - c_m^2 (k_1^2 + k_2^2) - \mu^2$ represents the dispersion relation of the membrane on the elastic foundation and $D^s(\omega, k_1) = \omega^2 - c_s^2 k_1^2$ the dispersion relation of the string.

The relation between the transform of the membrane displacements and the transform of the string displacement, representing the compatibility condition in equations (1), yields

$$\int_{-\infty}^{\infty} W^m(\omega, k_1, k_2)\, \mathrm{d}k_2 = 2\pi W^s(\omega, k_1). \tag{4}$$

Substituting the expression for $W^m(\omega, k_1, k_2)$ into the compatibility condition results in

$$W^s = -D^s(\omega, k_1) W^s \chi(\omega, k_1) + 2\pi \tilde{P}\, \delta(\omega - k_1 V) \chi(\omega, k_1), \tag{5}$$

where

$$\chi(\omega, k_1) = \frac{1}{2\pi\rho} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_2}{D^m(\omega, k_1, k_2)}.$$

For interpretation purposes it is convenient to rewrite equation (5) in the form

$$W^s(\omega, k_1) = \frac{2\pi \tilde{P}\, \delta(\omega - k_1 V)}{\omega^2 - c_s^2 k_1^2 - I(\omega, k_1)}, \qquad \text{where} \qquad I(\omega, k_1) = -\chi(\omega, k_1)^{-1}. \tag{6}$$

The expression in the denominator in equation (6) is the dispersion relation of the string interacting with the elastically supported membrane, in which the first two terms represent the dispersion relation of the string and the last $I(\omega, k_1)$ the equivalent stiffness of the membrane interacting with the infinite string. Evaluating the integral in the expression for $\chi(\omega, k_1)$ by contour integration, one finds that

$$I(\omega, k_1) = -\mathrm{i}\gamma \sqrt{\omega^2 - c_m^2 k_1^2 - \mu^2}, \tag{7}$$

with $\operatorname{Im} \sqrt{(\omega^2 - c_m^2 k_1^2 - \mu^2)} > 0$ and $\gamma = 2 c_m \rho$. Substitution of the expression for $W^s(\omega, k_1)$ from equations (6) and (7) into equation (3) results in

$$W^m(\omega, k_1, k_2) = \frac{2\pi \mathrm{i}\gamma \tilde{P}\, \delta(\omega - k_1 V) \sqrt{\omega^2 - c_m^2 k_1^2 - \mu^2}}{D^m(\omega, k_1, k_2)\, (\omega^2 - c_s^2 k_1^2 + \mathrm{i}\gamma \sqrt{\omega^2 - c_m^2 k_1^2 - \mu^2})}. \tag{8}$$

Hence the steady state solution of the membrane displacements (1) has the form

$$U^m(x - Vt, y) = \frac{\tilde{P}}{4\pi^2 \rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{i}\gamma \exp\{\mathrm{i}[k_1 (x - Vt) + k_2 y]\} \sqrt{k_1^2 (V^2 - c_m^2) - \mu^2}\, \mathrm{d}k_1\, \mathrm{d}k_2}{D_m (k_1 V, k_1, k_2)\, [k_1^2 (V^2 - c_s^2) + \mathrm{i}\gamma \sqrt{k_1^2 (V^2 - c_m^2) - \mu^2}]}$$

$$= \frac{\tilde{P}}{2\pi} \int_{-\infty}^{\infty} \frac{\exp\{\mathrm{i}[k_1 \xi + (|y|/c_m) \sqrt{k_1^2 (V^2 - c_m^2) - \mu^2}]\}}{[k_1^2 (V^2 - c_s^2) + \mathrm{i}\gamma \sqrt{k_1^2 (V^2 - c_m^2) - \mu^2}]}\, \mathrm{d}k_1, \tag{9}$$

with $\operatorname{Im} \sqrt{k_1^2 (V^2 - c_m^2) - \mu^2} > 0$ and $\xi = x - Vt$. Here ξ is a co-ordinate related to the moving object. Note that U^m depends on the modulus of the y-value, since the system is symmetric relative to the x-axis.

From this expression for the displacement of the system of membrane and string one can derive the critical velocity of the load. The integral (9) diverges for $\xi = x - Vt = 0$, $y = 0$ when the velocity of the load is equal to c_s ($V = c_s$, $c_s < c_m$). Then the displacement

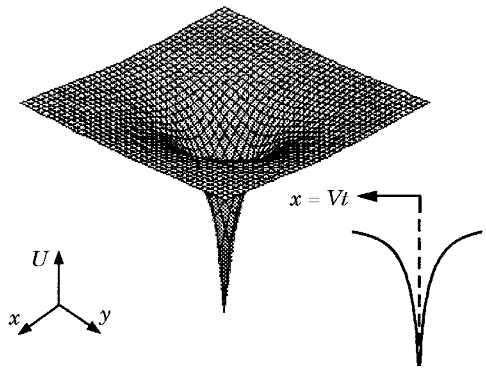

Figure 2. The membrane and string displacements for $V/c_s = 0{\cdot}9$, $c_m/c_s = 1{\cdot}5$.

of the elastic system under the load is infinite. In this case the integral (9) can be rewritten in the form

$$U^m(\xi = x - Vt, y) = \frac{\tilde{P}}{2\pi\gamma\sqrt{c_m^2 - V^2)}}\int_{-\infty}^{\infty} \exp\{iar \sinh(\chi + i\vartheta)\}\, d\chi, \tag{10}$$

where $a = \mu/c_m$, $r^2 = (\xi^2/\lambda^2) + y^2$, $\sin(\vartheta) = y/r$, $\cos(\vartheta) = \xi/r\lambda$, $\lambda = \sqrt{1 - V^2/c_m^2}$, and $V = c_s$. Evidently, the integral (10) diverges when $r = 0$ ($\xi = 0$, $y = 0$). Moreover, if $c_s = c_m$, $V = c_s$ the displacement of the whole system is infinite (for $t \to \infty$).

3.1. SUBCRITICAL CASE ($V < c_s, c_m$)

If the velocity of the load is smaller than c_s, c_m ($V < c_s, c_m$), then equation (9) can be rewritten in the form

$$U^m(\xi = x - Vt, y) = -\frac{\tilde{P}}{2\pi}\int_{-\infty}^{\infty} \frac{\exp\{ik_i\,\xi - (|y|/c_m)\sqrt{k_1^2\,(c_m^2 - V^2) + \mu^2}\}}{(c_s^2 - V^2)k_1^2 + \gamma\sqrt{k_1^2\,(c_m^2 - V^2) + \mu^2}}\, dk_1, \tag{11}$$

with $\operatorname{Re}\sqrt{k_1^2\,(c_m^2 - V^2) + \mu^2} > 0$. Note that in general, the integrand of equation (9) is a multiple-valued function because of the presence of the radical $R = \sqrt{k_1^2\,(V^2 - c_m^2) - \mu^2}$. Branch points occur when $R = 0$. Therefore for the calculation of the integral in equation (11) one has to investigate the location of the singular points of the integrand in the complex k_1-plane. The denominator of the integrand in equation (11) has two imaginary zeros

$$\tilde{k}_1 = \pm i\,\frac{\sqrt{\sqrt{\gamma^4\alpha^4 + 4\beta^4\mu^2\gamma^2} - \gamma^2\alpha^2}}{\sqrt{2}\beta^2}, \qquad \text{where } \alpha = \sqrt{(c_m^2 - V^2)}, \quad \beta = \sqrt{(c_s^2 - V^2)}, \tag{12}$$

and two branch points $\pm i\mu/\sqrt{(c_m^2 - V^2)}$ in the complex k_1-domain. Therefore one can integrate equation (11) along the real k_1-axis by using a standard numerical method. The results are shown in Figures 2 and 3 for $c_m/c_s = 1{\cdot}5$ and $c_m/c_s = 0{\cdot}5$ respectively. As shown in the figures, the moving load does not radiate any elastic waves in this range, but it excites a localized eigenfield moving with the load and the deflection profile of the string and the membrane depends on the relation between c_s and c_m.

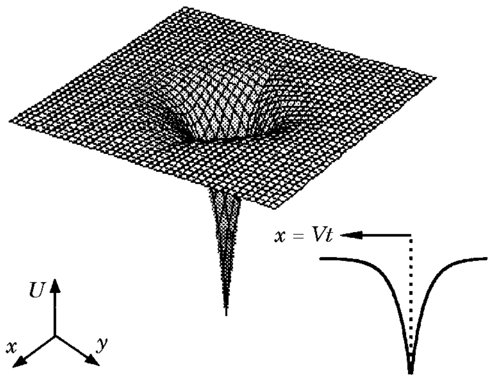

Figure 3. The membrane and string displacements for $V/c_s = 0{\cdot}4$, $c_m/c_s = 0{\cdot}5$.

3.2. TRANSCRITICAL CASES ($c_m < V < c_s$, $c_s < V < c_m$)

Case 1: $c_m < V < c_s$. In this case equation (9) can be rewritten in the form

$$U^m(\xi = x - Vt, y) = -\frac{\tilde{P}}{2\pi}\int_{-\infty}^{\infty} \frac{\exp\{\mathrm{i}[k_1\,\xi + (|y|/c_m)\sqrt{k_1^2\,(V^2 - c_m^2) - \mu^2}]\}}{k_1^2\,(c_s^2 - V^2) - \mathrm{i}\gamma\sqrt{k_1^2\,(V^2 - c_m^2) - \mu^2}}\,\mathrm{d}k_1, \qquad (13)$$

with $\mathrm{Im}\,\sqrt{k_1^2\,(V^2 - c_m^2) - \mu^2} > 0$. The denominator of the integrand in equation (13) also has two imaginary zeros in the complex k_1-domain,

$$\tilde{k}_1 = \pm\mathrm{i}\,\frac{\sqrt{\sqrt{\gamma^4\alpha^4 + 4\beta^4\mu^2\gamma^2} + \gamma^2\alpha^2}}{\sqrt{2}\beta^2}, \qquad \text{where } \alpha = \sqrt{(V^2 - c_m^2)}, \quad \beta = \sqrt{(c_s^2 - V^2)}, \qquad (14)$$

and two branch points $\eta_{1,2} = \pm\,\mu/\sqrt{(V^2 - c_m^2)}$ located on the real k_1-*axis*. To evaluate the integral by contour integration one has to investigate the location of the branch points after introducing a small dissipation [5]. One therefore introduces the member $2\delta U_t$ $(\delta \to 0)$ in equation (1) which describes an additional viscous dissipation in the foundation. After some simple tranforms, one can rewrite equation (13) in the form

$$U_\delta^m\,(\xi = x - Vt, y) = -\frac{\tilde{P}}{2\pi}\int_{-\infty}^{\infty} \frac{\exp\{\mathrm{i}[k_1\,\xi + (|y|/c_m)\sqrt{k_1^2\,(V^2 - c_m^2) + 2\mathrm{i}\delta V k_1 - \mu^2}]\}}{(c_s^2 - V^2)k_1^2 - \mathrm{i}\gamma\sqrt{k_1^2\,(V^2 - c_m^2) + 2\mathrm{i}\delta V k_1 - \mu^2}}\,\mathrm{d}k_1, \qquad (15)$$

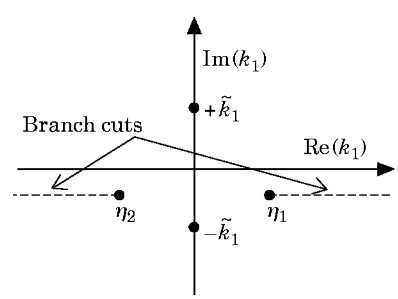

Figure 4. The branch points, poles and cuts in the case $c_s > V > c_m$.

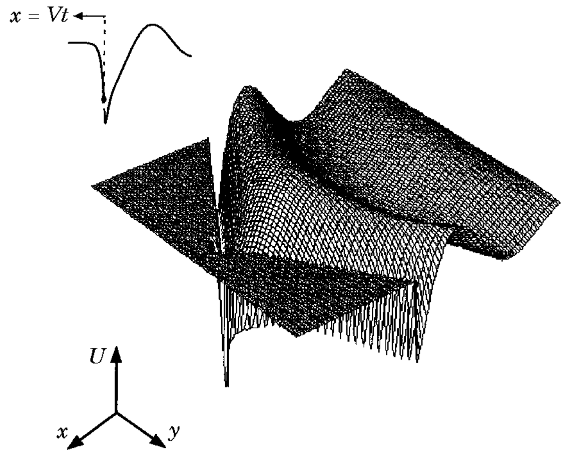

Figure 5. The membrane and string displacements for $V/c_s = 0{\cdot}9$, $c_m/c_s = 0{\cdot}5$.

where $\mathrm{Im}\sqrt{R(k_1)} = \sqrt{k_1^2(V^2 - c_m^2) + 2i\delta V k_1 - \mu^2} > 0$, and $U^m(\xi, y) = \lim_{\delta\to+0} U^m_\delta(\xi, y)$. The branch points are determined by $R(k_1) = 0$. Thus, in the complex k_1-plane, the branch points are given by the expression

$$\eta_{1,2} = -\frac{i\delta V}{(V^2 - c_m^2)} \pm \left(\frac{\mu}{\sqrt{(V^2 - c_m^2)}} + O(\delta^2)\right),$$

and are illustrated in Figure 4.

Therefore it is appropriate to cut the plane as shown in the figure. Then the radical $\sqrt{R(k_1)}$ has a positive imaginary part everywhere on the path of integration. Thus, following Jordan's lemma [6], for $\arg(\xi, y) > 0$, one closes the path of integration (along the real axis) in the upper half-plane and for $\arg(\xi, y) < 0$ one closes the path of integration in the lower half-plane, where $\arg(\xi, y)$ is defined as $\arg(\xi, y) = \{\xi + |y|\sqrt{(V^2/c_m^2) - 1}\}$.

One may reduce the integral (13) to the form

$$U^m(\xi = x - Vt, y) = -2\pi i \left\{ \frac{\tilde{P}}{2\pi} \frac{\exp\{i[k_1\xi + (|y|c_m)\,F(k_1)]\}}{\dfrac{\mathrm{d}}{\mathrm{d}k_1}[(c_s^2 - V^2)k_1^2 + i\gamma F(k_1)]} \right\}_{k_1 = +\tilde{k}_1}, \qquad (16)$$

for $\arg(\xi, y) > 0$, $\quad F(k_1) = \sqrt{k_1^2(V^2 - c_m^2) - \mu^2}$, and

$$U^m(\xi = x - Vt, y) = \frac{\tilde{P}}{2\pi} \left\{ 2\pi i \frac{\exp\{i[k_1\xi + (|y|/c_m)F(k_1)]\}}{\dfrac{\mathrm{d}}{\mathrm{d}k_1}[(c_s^2 - V^2)k_1^2 + i\gamma F(k_1)]} \right\}_{k_1 = -\tilde{k}_1}$$

$$+ \frac{2\tilde{P}}{\pi}\int_\eta^\infty \frac{\sin(k_1\xi)}{(c_s^2 - V^2)^2k_1^4 + \gamma^2F^2(k_1)}\{(c_s^2 - V^2)k_1^2\sin(\arg y) - \gamma F(k_1)\cos(\arg y)\}\,\mathrm{d}k_1, \quad (17)$$

for $\arg(\xi, y) < 0$, where $\arg y = (|y|/c_m)\sqrt{(V^2 - c_m^2)k_1^2 - \mu^2}$ and $n = \mu/\sqrt{(V^2 - c_m^2)}$

The results of a numerical evaluation of equations (16) and (17) for relevant parameters are represented qualitatively in Figure 5. The figure shows that the transcritically moving load ($c_m < V < c_s$) radiates elastic waves into the membrane. The wave field is located inside the cone analogous to the Mach cone in acoustics, which satisfies the equation $\xi^2 = y^2(V^2/c_m^2 - 1)$. Also, one can see in Figure 5 that the waves in the membrane excite wave motion in the spring and the amplitude of this motion is decreasing with increasing distance from the source point. Note further that the maximum displacement takes place behind the load.

Case 2: $c_s < V < c_m$. In this case the displacement of the system is described by the expression

$$U^m_\delta(\xi = x - Vt, y) = \frac{\tilde{P}}{2\pi}\int_{-\infty}^{\infty} \frac{\exp[\mathrm{i}k_1\xi - (|y|/c_m)\sqrt{k_1^2(c_m^2 - V^2) - 2\mathrm{i}\delta V k_1 + \mu^2}]}{(V^2 - c_s^2)k_1^2 - \gamma\sqrt{k_1^2(c_m^2 - V^2) - 2\mathrm{i}\delta V k_1 + \mu^2}}\,\mathrm{d}k_1 \tag{18}$$

with $\mathrm{Re}\sqrt{k_1^2(c_m^2 - V^2) - 2\mathrm{i}\delta V k_1 + \mu^2} > 0$ and $U^m(\xi, y) = \lim_{\delta \to +0} U^m_\delta(\xi, y)$. The integrand of equation (18) has two branch cuts which are located in the upper and lower complex half-planes of k_1:

$$\eta_{1,2} = \mathrm{i}\frac{\delta V}{(c_m^2 - V^2)} \pm \mathrm{i}\left(\frac{\mu}{\sqrt{(c_m^2 - V^2)}} + O(\delta^2)\right),$$

$$\tilde{k}_1 = \pm\frac{\sqrt{\sqrt{\gamma^4\alpha^4 + 4\beta^4\mu^2\gamma^2} + \gamma^2\alpha^2}}{\sqrt{2}\beta^2},$$

where $\alpha = \sqrt{(V^2 - c_m^2)}$ and $\beta = \sqrt{(c_s^2 - V^2)}$, and two poles $\pm\tilde{k}_1$ in the lower complex half-plane, as follows from the denominator of equation (15) ($\delta \to 0$), as depicted in Figure 6.

One may now integrate expression (15) using contour integration. Note that for $\xi > 0$ the path of integration (along the real k_1-*axis*) can be closed in the upper half-plane and for $\xi < 0$ in the lower complex k_1-half-plane. So one finds, for the displacement of the system,

$$U^m(\xi = x - Vt, y) = -\frac{\tilde{P}}{2\pi}\int_{\eta_1}^{\infty} \frac{\exp\{-k_1\xi\}}{(V^2 - c_s^2)^2 k_1^4 + \gamma^2 F^2(k_1)}\{(V^2 - c_s^2)k_1^2$$

$$\times \sin\left(\frac{y}{c_m}F(k_1)\right) + \gamma F(k_1)\cos\left(\frac{y}{c_m}F(k_1)\right)\Big\}, \qquad \text{for} \qquad \xi > 0 \tag{19}$$

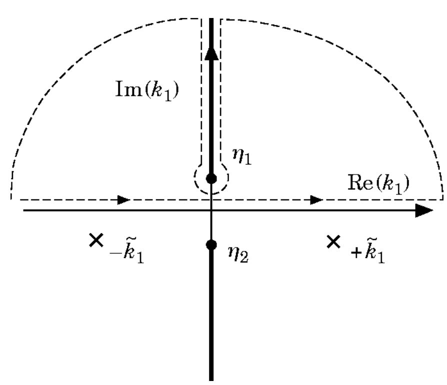

Figure 6. The branch points and cuts in the case $c_m > V > c_s$; contour of integration in the upper half-plane.

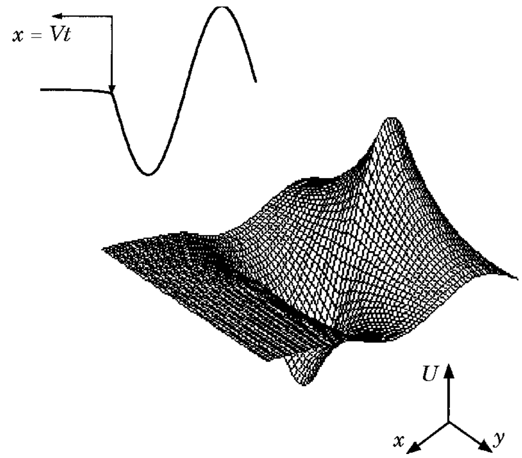

Figure 7. The membrane and string displacements for $V/c_s = 1{\cdot}8$, $c_m/c_s = 2{\cdot}5$.

and

$$U^m(\xi = x - Vt, y) = \frac{\tilde{P}}{2\pi}\left\{-2\pi i\left(\frac{\exp\{ik_1\xi - (|y|/c_m)F(k_1)\}}{\frac{d}{dk_1}[V^2 - c_s^2)k_1^2 - \gamma\sqrt{(c_m^2 - V^2)k_1^2 + \mu^2}]}\Bigg|_{k_1 = -\tilde{k}_1}\right.\right.$$

$$\left.+\frac{\exp\{ik_1\xi - (|y|/c_m)F(k_1)\}}{\frac{d}{dk_1}[(V^2 - c_s^2)k_1^2 - \gamma\sqrt{(c_m^2 - V^2)k_1^2 + \mu^2}]}\Bigg|_{k_1 = +\tilde{k}_1}\right)$$

$$-\int_\eta^\infty \frac{\exp\{k_1\xi\}}{(V^2 - c_s^2)^2k_1^4 + \gamma^2F^2(k_1)} \times \left\{(V^2 - c_s^2)k_1^2\right.$$

$$\left.\left.\times \sin\left(\frac{y}{c_m}F(k_1)\right) + \gamma F(k_1)\cos\left(\frac{y}{c_m}F(k_1)\right)\right\}\right\}, \qquad \text{for } \xi < 0, \quad (20)$$

where $F(k_1) = \sqrt{(c_m^2 - V^2)k_1^2 - \mu^2}$ and $\eta = \mu/\sqrt{(c_m^2 - V^2)}$. The results of the numerical calculations of equations (19) and (20) for $V/c_s = 1{\cdot}8$ and $c_m/c_s = 2{\cdot}5$ (the other parameters are taken as unity) are represented in Figure 7. The figure shows that the transcritically moving load ($c_s < V < c_m$) generates waves in the string. The oscillating string excites

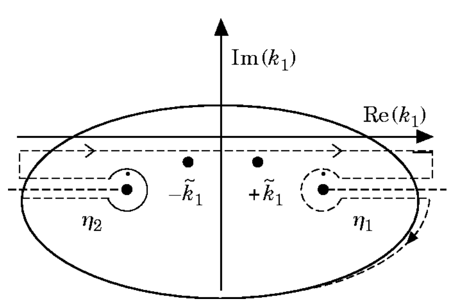

Figure 8. The branch points and cuts in the case $V > c_m$, c_s; contour of integration in the lower half-plane.

elastic waves into the membrane. The displacement of the string ahead of the moving load is not equal to zero, as it would be for a string on an elastic foundation.

3.3. SUPERCRITICAL CASE ($c_m, c_s < V$)

Now the velocity of the load is higher than the critical velocity in the membrane and in the string. In this case the moving load generates elastic waves both in the membrane and the string. The displacement of the membrane is described by the expression

$$U_\delta^m\,(\xi = x - Vt, y) = \frac{\tilde{P}}{2\pi}\int_{-\infty}^{\infty} \frac{\exp\{\mathrm{i}[k_1\,\xi + (|y|/c_m)\sqrt{k_1^2\,(V^2 - c_m^2) + 2\mathrm{i}\delta V k_1 - \mu^2}]\}}{(V^2 - c_s^2)k_1^2 + \mathrm{i}\gamma\sqrt{k_1^2\,(V^2 - c_m^2) + 2\mathrm{i}\delta V k_1 - \mu^2}}\;\mathrm{d}k_1, \tag{21}$$

with $\mathrm{Im}\sqrt{k_1^2\,(V^2 - c_m^2) + 2\mathrm{i}\delta V k_1 - \mu^2} > 0$ and $U^m(\xi, y) = \lim_{\delta\to+0} U_\delta^m\,(\xi, y)$. The integrand of the expression (21) has two branch points $\eta_{1,2}$, as a multiple-valued function and two poles $\pm\tilde{k}_1$ ($\delta\to 0$) in the lower half-plane (see Figure 8):

$$\eta_{1,2} = -\mathrm{i}\,\frac{(\delta\to 0)V}{(V^2 - c_m^2)} \pm \frac{\mu}{\sqrt{(V^2 - c_m^2)}}, \qquad \tilde{k}_1 = \pm\frac{\sqrt{\sqrt{\gamma^4\alpha^4 + 4\beta^4\mu^2\gamma^2} - \gamma^2\alpha^2}}{\sqrt{2}\beta^2},$$

where $\alpha = \sqrt{(V^2 - c_m^2)}$ and $\beta = \sqrt{(V^2 - c_s^2)}$. The branch cuts are chosen as shown in Figure 8. In this case the condition $\mathrm{Im}\sqrt{k_1^2\,(V^2 - c_m^2) + 2\mathrm{i}\delta V k_1 - \mu^2} > 0$ is satisfied in the whole complex k_1-plane (more precisely, on the sheet of the Riemann surface [6]). Note that the location of the poles of the integrand of equation (9) defines the string motion. The locations of the branch points in the complex plane define the membrane motion. Thus if poles have real parts it means that waves are excited in the string, and real parts of the branch points that waves are excited in the membrane. As one can see from Figure 8, the poles and branch points have real parts and they are located in the lower complex k_1-half-plane. Furthermore, there are no singular points in the upper half-plane. The locations of the poles and branch points correspond to wave processes in the string and the membrane, and the absence of singularities corresponds to the fact that outside the Mach cone, $\{\xi + |y|\sqrt{V^2/c_m^2 - 1}) > 0$, the system of string membrane is not excited.

After these considerations one may easily reduce the integral (21).

For $(\xi + |y|\sqrt{V^2/c_m^2 - 1}) > 0$ the contour of integration has to be closed in the upper half of the complex k_1-plane and since one has no singular points in this plane

$$U^m(\xi = x - Vt, y) = 0 \qquad \text{for } (\xi + |y|\sqrt{(V^2/c_m^2) - 1}) > 0, \tag{22}$$

and for $(\xi + |y|\sqrt{V^2/c_m^2 - 1}) < 0$ the contour of integration is closed in the lower complex k_1-plane, as shown in Figure 8. One obtains

$$U^m(\xi = x - Vt, y) = \frac{\tilde{P}}{2\pi}\left\{-2\pi\mathrm{i}\left(\frac{\exp\{\mathrm{i}[k_1\,\xi + (|y|/c_m)F(k_1)]\}}{\dfrac{\mathrm{d}}{\mathrm{d}k_1}[(V^2 - c_s^2)k_1^2 + \mathrm{i}\gamma\sqrt{(-V^2 - c_m^2)k_1^2 - \mu^2}]}\right|_{k_1 = +\tilde{k}_1}\right.$$

$$\left.\left. + \frac{\exp\{\mathrm{i}[k_1\,\xi + (|y|/c_m)F(k_1)]\}}{\dfrac{\mathrm{d}}{\mathrm{d}k_1}[(V^2 - c_s^2)k_1^2 + \mathrm{i}\gamma\sqrt{(V^2 - c_m^2)k_1^2 - \mu^2}]}\right|_{k_1 = -\tilde{k}_1}\right)$$

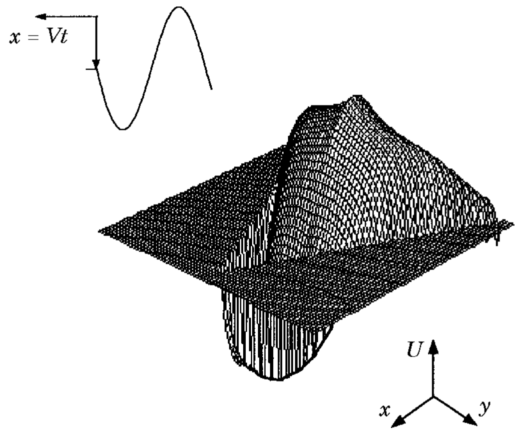

Figure 9. The membrane and string displacements for $V/c_s = 1{\cdot}8$, $c_m/c_s = 0{\cdot}5$.

$$-\int_{\eta}^{\infty} \frac{4\sin(k_1\xi)}{(V^2-c_s^2)^2k_1^4+\gamma^2F^2(k_1)} \times \{(V^2-c_s^2)k_1^2\sin(\arg y) + \gamma\sqrt{k_1^2(V^2-c_m^2)-\mu^2}\cos(\arg y)\}\,\mathrm{d}k_1\Bigg\}, \tag{23}$$

for $(\xi+|y|\sqrt{(V^2/c_m^2)-1})<0$, where $\arg y=(y/c_m)\sqrt{k_1^2(V^2-c_m^2)-\mu^2}$ and $\eta = \mu/\sqrt{(V^2-c_m^2)}$. The integral (23) has been calculated numerically for various ratios of the relevant velocities, and the qualitative results are shown in Figures 9 and 10.

As can be seen from the figures, the wave field is located inside the Mach cone, the angular slope of which depends on the relation $\sqrt{(V^2/c_m^2)-1}$. The displacement of the string is continuous on the cone border, but the first derivative of the displacement has a jump, as is to be expected for a string on an elastic foundation without the presence of the membrane. The membrane displacement as well as its first derivative with respect to ξ are not continuous on the cone border. Note that this jump in the displacement on the Mach cone is a property of a two dimensional system [7].

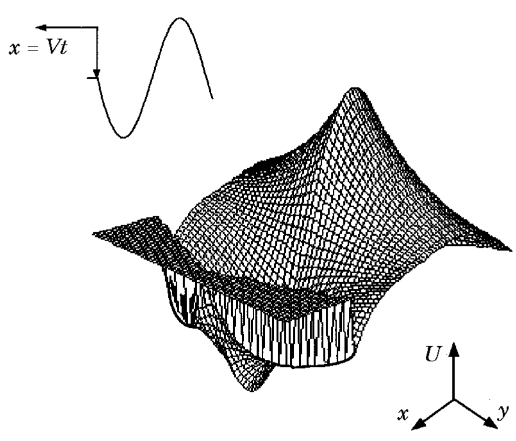

Figure 10. The membrane and string displacements for $V/c_s = 1{\cdot}9$, $c_m/c_s = 1{\cdot}55$.

4. DISCUSSION

This discussion is concerned with a qualitative comparison of the results obtained in this paper and results derived by more complex models in references [1–3]. In these references the steady state displacements due to a uniform moving load along an Euler–Bernoulli beam on an elastic half-space have been analyzed to model load–track–subsoil interaction. For a track–subsoil interaction the Rayleigh wave speed c_R of the subsoil will generally be smaller than the wave speed in the track. Therefore a comparison between the two models will be made for $c_m < c_s$.

It should be noted that the models are quite different in dispersion properties (string versus E–B-beam and Winkler-supported membrane versus half-space); however, it is of interest to give a qualitative comparison.

Comparing the results of references [1] and [3] to the result found in this paper shows the following.

1. The displacement under the load becomes infinite in [1] for a critical velocity V_{cr} below the Rayleigh wave speed c_R and at the speed c_R, while this occurs in the string–membrane problem for $V = c_s$; however, for $c_s < c_m$ only.
2. For $V < V_{cr}$ and $V < c_m$ one finds stationary eigenfields in both models.
3. For $V_{cr} < V < c_R$ waves are propagating only in the beam and not in the half-space while for $c_m < V < c_s$ the waves are radiating in the membrane.
4. For $c_R < V < c_t$ (c_t = shear wave speed in the half-space) surface waves radiate in the half-space, comparable to the range $c_m < V < c_s$.
5. For $V > c_t$ surface and bulk waves radiate in the half-space, while for $V > c_s$ only surface waves radiate in the supported membrane.

Comparing the equivalent stiffnesses of the half-space in reference [3] with that of the elastically supported membrane for the different velocity ranges show the following.

For the subcritical case the results are qualitatively similar. For small wave numbers and a Poisson ratio of $\nu = 0{\cdot}25$, the shear modulus μ of the half-space can be equated to $\sqrt{N}$ k, where N is the tension in the membrane and k is the stiffness per area of the elastic support.

For case 1 of the transcritical case, the equivalent stiffness is purely imaginary due to the absence of bulk waves in the support, differing from the results in reference [1] in which both a real and an imaginary part are found. Therefore, only the radiation part of the surface waves in the half-space can be modelled.

Case 2 cannot be compared to the results in references [1] and [3]. The equivalent stiffness of the supported membrane is now completely real.

For the supercritical case, the equivalent stiffness in our model is also purely imaginary, and again only the radiation part of the surface waves in the half-space can be modelled.

5. CONCLUSIONS

In this paper the steady state displacements due to a uniformly moving load along a string on a Winkler supported membrane have been derived, by using the concept of "equivalent stiffness". With this model the elastic wave fields in the system have been determined for subcritical, transcritical and supercritical load velocities and presented as graphs. The results show that the subcritically ($V < c_s, c_m$) moving load does not radiate any elastic waves but excites a localized eigenfield moving with the load. In the case of transcritical motion ($c_s < V < c_m$ or $c_m < V < c_s$), the load generates elastic waves in the string only or in the membrane only. When waves are generated in the string (case 2) then the membrane wave motion is localized near the string. When waves are generated in the membrane (case 1) then the string follows the membrane wave motion and the amplitude of this motion is decreasing with increasing distance from point source. The wave field is then located inside a Mach cone.

In the case of supercritical motion (c_s, $c_m < V$), the load generates elastic waves both in the string and in the membrane and the wave field is located inside a Mach cone.

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