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PROBABILITY DENSITY OF FINITE FOURIER SERIES WITH RANDOM PHASES

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Analysis has been made to determine the properties of a random process consisting of the sum of a series of sine waves with deterministic amplitudes and independent, random phase angles. The probability density of the series, its peaks and envelope have been found for an arbitrary number of sine waves in the series. These probability distributions are non-Gaussian.

1. INTRODUCTION

Engineering processes are often sums of a finite number of mutually independent modes,

$$Y = X_1(\phi_1, t) + X_2(\phi_2, t) + X_3(\phi_3, t) + \cdots + X_N(\phi_N, t). \tag{1}$$

The modes $X_n(\phi, t)$ are functions of time, t, and independent random variables such as phase, ϕ_n. We consider that each mode has a finite maximum value. The number of modes in the series is also finite so the maximum (peak) value of Y is finite. The central limit theorem does not apply to finite series, hence the probability distribution of Y is non-Gaussian.

When the modes in equation (1) are sine waves with equally spaced frequencies, equation (1) is known as a finite Fourier series. The Fourier series with random phases was first considered by Rayleigh [1]. Its properties are useful in modelling random multi-frequency processes. A search of the literature has revealed no solutions for probability distribution of peaks in finite Fourier series with random phases. In this paper, analysis is made of the sum of a series of sine waves with deterministic amplitudes and independent, random phase angles. The probability density of the series, its peaks and envelope have been found for 1, 2, 3 and an arbitrary number of sine waves in the series.

2. MAXIMUM AND RMS OF FINITE FOURIER SERIES

Consider the sum of N sine waves defined over the time interval, $0 \leqslant t < T$:

$$Y = \sum_{n=1}^{N} a_n \cos(\omega_n\ t + \phi_n), \qquad 0 \leqslant t_n \leqslant T, \quad a_n \geqslant 0, \quad 0 \leqslant \phi_n < 2\pi. \tag{2}$$

Equation (2) is a finite Fourier series. The amplitudes, a_n, and time t, are considered to be positive, deterministic numbers. The circular frequency of each sine wave, ω_n, is a positive, non-zero integer multiple of $2\pi/T$.

The phases ϕ_n are independent random variables. An ensemble of phases is developed by generating M sets of N randomly chosen phases ($\phi_{n,m}$ for $n = 1, N$ and $m = 1, M$) which

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are uniformly distributed over the interval $0 \leqslant \phi_{n,m} < 2\pi$. Substituting the M sets into the right side of equation (2) results in M values of the dependent random variable Y.

The sine waves on the right side of equation (2) are statistically independent of each other, since each sine wave is a function of an independent random variable. The maximum possible (peak) value of Y (equation (2)) is the sum of the amplitudes of the sine waves (recall that $a_n \geqslant 0$):

$$Y_{peak} = \sum_{n=1}^{N} a_n \tag{3a}$$

$$= Na, \qquad \text{for } a_n = a, \quad n = 1, 2, \ldots, N. \tag{3b}$$

The mean square of the sum of independent sine waves is the sum of the mean squares of each sine wave:

$$Y_{rms}^2 = \overline{Y^2} = \sum_{n=1}^{N} \frac{1}{2\pi} \int_0^{2\pi} [a_n \cos(2\pi t_n/T + \phi_n)]^2 d\phi_n = \tfrac{1}{2} \sum_{n=1}^{N} a_n^2 \tag{4a}$$

$$= \tfrac{1}{2} Na^2, \qquad \text{for } a_n = a, \quad n = 1, 2, \ldots, N. \tag{4b}$$

The ratio of peak to the root mean square of the sum of N mutually independent sine waves is

$$\frac{Y_{peak}}{Y_{rms}} = 2^{1/2} \sum_{n=1}^{N} a_n \Bigg/ \left(\sum_{n=1}^{N} a_n^2 \right)^{1/2}, \tag{5a}$$

$$= (2N)^{1/2}, \qquad \text{for } a_n = a, \quad n = 1, 2, \ldots, N. \tag{5b}$$

The peak-to-r.m.s. ratio for an equal amplitude series increases from $2^{1/2}$ for a single term ($N = 1$) and approaches infinity as the number of terms N approaches infinity, as shown in Figure 1. The probability of a values of Y being greater than the peak value is zero.

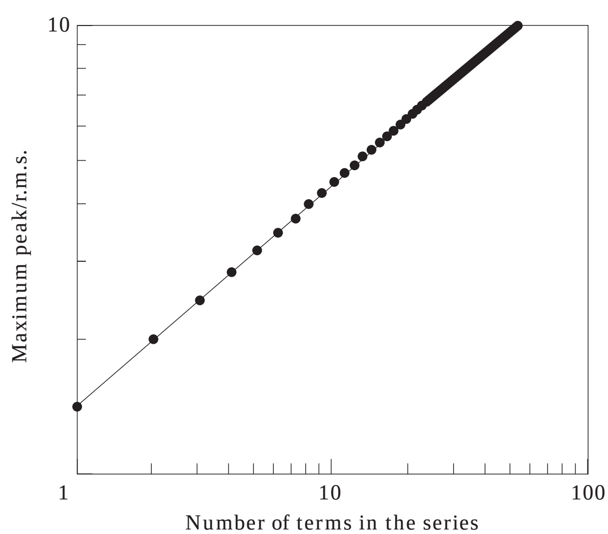

Figure 1. The ratio of the peak to the r.m.s. value of a Fourier series of N equal amplitude terms, equation (4b).

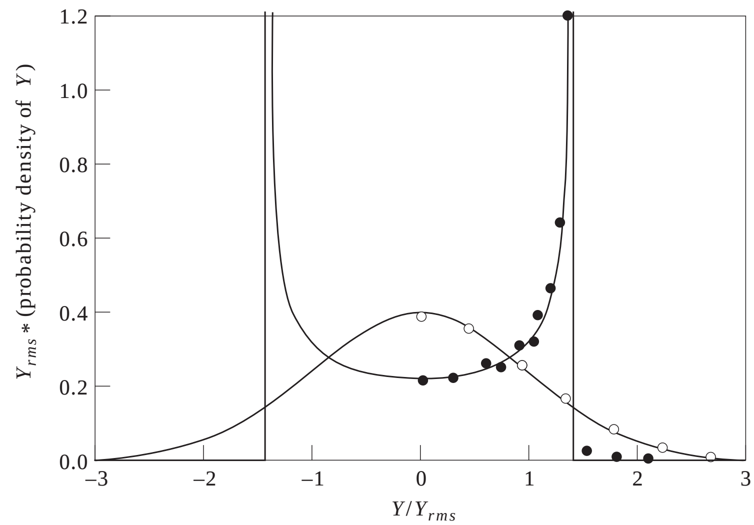

Figure 2. The normal probability density, equation (46), and the sine wave probability density, equation (8), in comparison with results of numerical integration of equation (40) for $N = 1$ and $N = 10$. ——, Normal distribution; ▬▬, sine wave distribution; ○; numerical integration, $N = 10$; ●, numerical integration, $N = 1$.

For example, there is no chance that the sum of any four ($N = 4$) independent sinusoidal terms will be greater than $8^{1/2} = 2{\cdot}828$ times the overall r.m.s. value.

These results can be generalized. The maximum value of a series, equation (1), cannot exceed the sum of the maximum values of its modes. The peak-to-r.m.s. ratio of the sum of independent, non-constant, modes increases with the number of modes. If all these modes have the same peak and r.m.s. values, then the peak-to-r.m.s. ratio of their sum increases with the square root of the number of modes.

3. SINGLE SINE WAVE

The probability density $p_Y(y)$ of the random variable Y is the probability that Y has values within the small range between y and $y + \mathrm{d}y$, divided by $\mathrm{d}y$. $p_Y(y)$ has the units of $1/Y$ [2, p. 48]. A sine wave of amplitude a_n, circular frequency ω_n,

$$Y = a_n \cos(\omega_n t + \phi_n), \tag{6}$$

and random phase ϕ_n which is uniformly distributed over the range $0 \leqslant \phi_n < 2\pi$,

$$p_{\phi_n}(\phi_n) = \begin{cases} 1/(2\pi), & \text{if } 0 \leqslant \phi_n < 2\pi, \\ 0, & \text{if } 0 > \phi_n \text{ or } \phi_n > 2\pi, \end{cases} \tag{7}$$

has a well known probability density [3, 4, art. 3.10]:

$$p_Y(y) = \begin{cases} \pi^{-1}(a_n^2 - y^2)^{-1/2}, & \text{if } -a_n < y < a_n, \\ 0, & \text{if } |y| \geqslant a_n. \end{cases} \tag{8}$$

The probability density of the sine wave is symmetric about zero, i.e., $p_Y(y) = p_Y(-y)$, it is singular at $y = \pm a_n$ and it falls to zero for $|y|$ greater than a_n, as shown in Figure 2.

The characteristic function of a random variable X is the expected value of $e^{j2\pi fX}$ which is the Fourier transform of its probability density function [2, 5, 6],

$$C(f) = \int_{-\infty}^{\infty} e^{j2\pi fx} p_X(x)\, dx. \tag{9}$$

$j = \sqrt{-1}$ is the imaginary constant. The characteristic function of the sine wave is found by substituting equation (8) into equation (9) and integrating over the range of non-zero probabilities (see Gradshteyn *et al*. [7, article 3.753]; Rice [4, article 3.16]):

$$C_n(f) = 2(\pi a_n)^{-1} \int_0^{a_n} \cos(2\pi f y)\, [1 - (y/a_n)^2]^{-1/2}\, dy = J_0(2\pi f a_n). \tag{10}$$

$J_0()$ is the Bessel function of the first kind and zero order. Equations (8) and (10) are starting points for determining the probability density of the Fourier series.

The joint probability density $p_{X_1 X_2}(x_1, x_2)$ of the two random variables X_1 and X_2 is the probability that X_1 falls in the range between x_1 and $x_1 + dx_1$ and X_2 falls in the range between x_2 and $x_2 + dx_2$, divided by $dx_1\, dx_2$. The joint probability density of a sine wave and its derivative is found by first noting that the derivative of the sine wave Y, equation (6), with respect to time can be expressed in terms of Y,

$$dY/dt = \dot{Y} = -a_n \omega_n \sin(\omega_n t + \phi_n) = \pm\omega_n \sqrt{a_n^2 - Y^2}, \qquad |Y| \leqslant a_n. \tag{11}$$

The probability density of $\dot{Y}$, given Y, is thus non-zero only for two possible values $\dot{Y} = \pm\omega_n \sqrt{a_n^2 - Y^2}$:

$$p_{\dot{Y}|Y}(\dot{y}|y) = \tfrac{1}{2}[\delta_{\dot{Y}}(\dot{y} + \omega_n \sqrt{a_n^2 - y^2}) + \delta_{\dot{Y}}(\dot{y} - \omega_n \sqrt{a_n^2 - y^2})]. \tag{12}$$

$\delta_{\dot{Y}}(\dot{y})$ is Dirac's delta function for the derivative $\dot{Y}$. Using conditional probability theory (Sveshnikov [2, p. 100]),

$$p_{Y\dot{Y}}(y, \dot{y}) = p_Y(y) p_{\dot{Y}|Y}(\dot{y}|y), \tag{13}$$

and substituting equations (8) and (12) into equation (13), the joint probability density of a sine wave and its derivative with respect to time is found:

$$p_{Y\dot{Y}}(y, \dot{y}) = (2\pi)^{-1}(a_n^2 - y^2)^{-1/2}[\delta_{\dot{Y}}(\dot{y} + \omega_n \sqrt{a_n^2 - y^2}) + \delta_{\dot{Y}}(\dot{y} - \omega_n \sqrt{a_n^2 - y^2})]. \tag{14}$$

This equation is valid for $|y| \leqslant a_n$ and $|\dot{y}| \leqslant \omega_n a_n$ and the joint probability is zero outside this range.

The joint characteristic function of Y and $\dot{Y}$ is the expected value of $e^{j2\pi(f_1 Y + f_2 \dot{Y})}$:

$$C_n(f_1, f_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j2\pi(f_1 y + f_2 \dot{y})} p_{Y\dot{Y}}(y, \dot{y})\, dy\, d\dot{y}, \tag{15}$$

which is the two-dimensional Fourier transform $p_{Y\dot{Y}}(y, \dot{y})$. Substituting equation (14) into equation (15) and integrating with respect to $\dot{y}$ gives

$$C_n(f_1, f_2) = \int_{-a_n}^{a_n} (\pi\sqrt{a_n^2 - x^2})^{-1/2} \cos(2\pi f_1 x) \cos(2\pi f_2 \omega_n \sqrt{a_n^2 - x^2})\, dx. \tag{16}$$

Now making the substitution $x = a_n \cos\theta$, a solution is found [7, articles 3·937·2 and 8·406·3]:

$$C_n(f_1, f_2) = (2\pi)^{-1} \int_0^{2\pi} \cos(2\pi f_1 a_n \cos\theta) \cos(2\pi f_2 a_n \omega_n \sin\theta)\, \mathrm{d}\theta$$

$$= \mathrm{J}_0\left(2\pi a_n \sqrt{f_1^2 + f_2^2 \omega_n^2}\right). \tag{17}$$

The resultant joint characteristic function of Y and $\dot{Y}$ is real; the imaginary part integrates to zero since $p_{Y\dot{Y}}(y, \dot{y})$ is symmetric about the origin.

The joint probability density of the sine wave, its first derivative and its second derivative is expressed using conditional probability theory, equation (13):

$$p_{Y\dot{Y}\ddot{Y}}(y, \dot{y}, \ddot{y}) = p_Y(y) p_{\dot{Y}|Y}(\dot{y}|y) p_{\ddot{Y}|Y\dot{Y}}(\ddot{y}|y, \dot{y}). \tag{18}$$

Since $\mathrm{d}^2Y/\mathrm{d}t^2 = -\omega_n^2 Y$ for a sine wave (equation (6)), $p_{\ddot{Y}|Y\dot{Y}}(\ddot{y}|y, \dot{y}) = \delta_{\ddot{Y}}(\ddot{y} + \omega_n^2 y)$. Substituting this equation, equation (8) and equation (12) into equation (18) gives the joint probability distribution of Y, $\dot{Y}$ and $\ddot{Y}$ for a single sine wave:

$$p_{Y\dot{Y}\ddot{Y}}(y, \dot{y}, \ddot{y}) = \frac{\delta_{\ddot{y}}(\ddot{y} + \omega_n^2 y)}{2\pi\sqrt{a_n^2 - y^2}} \left[\delta_{\dot{y}}\left(\dot{y} + \omega_n \sqrt{a_n^2 - y^2}\right) + \delta_{\dot{y}}\left(\dot{y} - \omega_n \sqrt{a_n^2 - y^2}\right)\right]. \tag{19}$$

The joint characteristic function of Y, $\dot{Y}$ and $\ddot{Y}$ is

$$C_n(f_1, f_2, f_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}2\pi(f_1 y + f_2 \dot{y} + f_3 \ddot{y})} p_{Y\dot{Y}\ddot{Y}}(y, \dot{y}, \ddot{y})\, \mathrm{d}y\, \mathrm{d}\dot{y}\, \mathrm{d}\ddot{y}. \tag{20}$$

Equation (19) is substituted into this equation. Integrating over the range of non-zero probability, $|y| \leqslant a_n$, gives the joint characteristic function of the sine wave and its first two derivatives:

$$C_n(f_1, f_2, f_3) = \mathrm{J}_0\left(2\pi a_n \sqrt{(f_1 - \omega_n^2 f_3)^2 + f_2^2 \omega_n^2}\right). \tag{21}$$

$C_n(f_1, f_2, f_3)$ is a real function. As a check we see that equation (21) with $f_3 = 0$ reduces to equation (17) and equation (21) with $f_2 = f_3 = 0$ reduces to equation (10).

4. SUM OF TWO SINE WAVES

The probability density of the sum of two mutually independent random variables X_1 and X_2,

$$Y = X_1 + X_2, \tag{22}$$

is the sum of the probabilities of all possible combinations of X_1 and X_2 such that $Y - \mathrm{d}Y - X_1 \leqslant X_2 < Y + \mathrm{d}Y - X_1$, divided by $\mathrm{d}Y$, and it is given by the convolution integral [2, p. 129]

$$p_Y(y) = \int_{-\infty}^{\infty} p_{X_1}(x) p_{X_2}(y - x)\, \mathrm{d}x. \tag{23}$$

$p_{X_1}(x)$ is the probability density of X_1 evaluated at x and $p_{X_2}(y - x)$ is the probability density of X_2 evaluated at $y - x$.

Consider the sum of two sine waves which are functions of the uniformly distributed, independent random variables ϕ_1 and ϕ_2, equation (8):

$$Y = a_1 \cos(\omega_1 t + \phi_1) + a_2 \cos(\omega_2 t + \phi_2). \tag{24}$$

Applying equation (23) to this sum with $X_1 = a_1 \cos(\omega_1 t_1 + \phi_1)$ and $X_2 = a_2 \cos(\omega_2 t_2 + \phi_2)$ and using equation (8) within the range $Y \geqslant 0$ and $a_2 \geqslant a_1$ gives

$$p_Y(y) = \begin{cases} \dfrac{1}{\pi^2}\displaystyle\int_{-a_1}^{a_1} [(a_1^2 - x^2)(y + a_2 - x)(-y + a_2 + x)]^{-1/2}\,\mathrm{d}x, & \text{if } a_2 - a_1 > y > 0, \\ \dfrac{1}{\pi^2}\displaystyle\int_{y - a_2}^{a_1} [(a_1^2 - x^2)(y + a_2 - x)(-y + a_2 + x)]^{-1/2}\,\mathrm{d}x, & \text{if } a_1 + a_2 > y > a_2 - a_1, \\ 0, & \text{if } y > a_1 + a_2. \end{cases} \tag{25}$$

There are three regions: (1) $a_2 - a_1 > y \geqslant 0$, where the sum is less than the difference between amplitudes; (2) $a_2 + a_1 > y \geqslant a_2 - a_1$, where the sum is between the sum and difference of the two amplitudes; and (3) $y > a_2 + a_1$, which has zero probability (Figure 3)

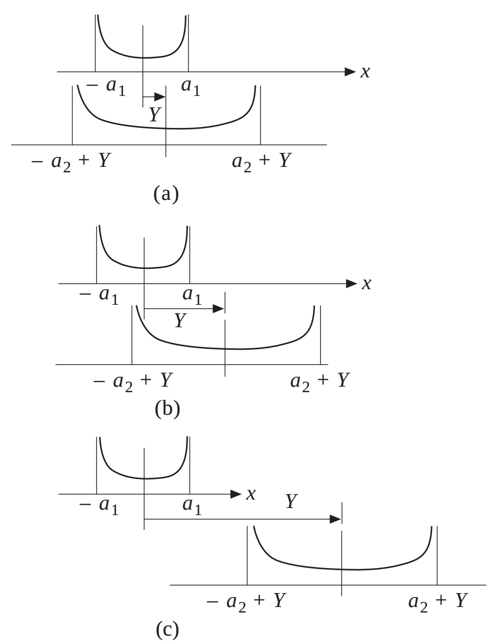

Figure 3. Convolution of the probability density of two sine waves: (a) $a_2 - a_1 > Y > 0$; (b) $a_1 + a_2 > Y > 0$; (c) $Y > a_1 + a_2$.

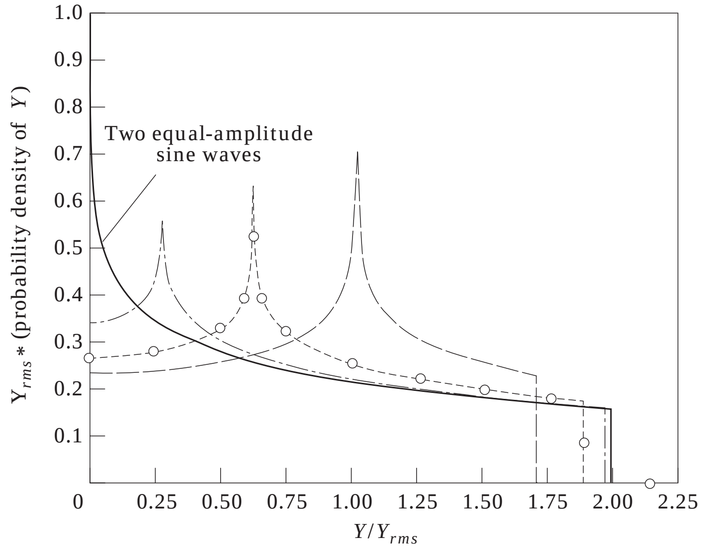

Figure 4. The probability density of the sum of two sine waves with various amplitudes, equation (26) ----, $a_1/a_2 = 0{\cdot}25$; . . . , 0·5; — · — · , 0·75; ——, 1·0; ○, numerical integration of equation (39) for $a_1/a_2=0.5$.

because y exceeds the sum of the amplitudes. Applying Gradshteyn and Ryzhik's [7, art. 3.14] integral solutions to the previous equations gives the probability density of the sum of two independent sine waves of amplitude a_1 and a_2.

$$p_Y(y) = \begin{cases} \dfrac{2}{\pi^2}\dfrac{1}{\sqrt{(a_1+a_2)^2-y^2}}K\left[\sqrt{\dfrac{4a_1\,a_1}{(a_1+a_2)^2-y^2}}\right], & \text{if } |a_2-a_1|>|y|>0; \\ \dfrac{1}{\pi^2}\dfrac{1}{\sqrt{a_1\,a_2}}K\left[\sqrt{\dfrac{(a_1+a_2)^2-y^2}{4a_1\,a_2}}\right], & \text{if } \ |a_2-a_1|<y<a_1+a_2; \\ 0, & \text{if } |y|>a_1+a_2. \end{cases} \tag{26}$$

$K(k)$ is the complete elliptic integral of the first kind. $K(k)$ is a single valued function that increases smoothly between the limits $K(0)=\pi/2$ and $K(1)=\infty$,

$$K(k) = \int_0^1 [(1-u^2)\,(1-k^2u^2)]^{-1/2}\,\mathrm{d}u, \qquad 0<k<1; \tag{27a}$$

$$\approx (\alpha_0+\alpha_1\,k_1^2+\alpha_2\,k_1^4) + (\beta_0+\beta_1\,k_1^2+\beta_2\,k_1^4)\ln(1/k_1^2). \tag{27b}$$

$k_1=(1-k^2)^{1/2}$ is the complementary modulus and $\alpha_0=1{\cdot}38629$, $\alpha_1=0{\cdot}11197$, $\alpha_2=0{\cdot}07252$, $\beta_0=0{\cdot}5$, $\beta_1=0{\cdot}12134$, $\beta_2=0{\cdot}02887$ [8].

The two-sine probability density is shown in Figure 4 for four values of the ratio a_1/a_2. As a_1 or a_2 approaches zero, the peak in the probability density shifts from the center to the edge and the two-sine probability distribution (equation (26)) approaches the single sine probability distribution (equation (8)). There are singularities at $Y=a_2-a_1$, where the argument of the complete elliptic integral is unity.

When the two sine wave amplitudes are equal, $a_1 = a_2 = a$, equation (26) simplifies to

$$p_Y(y) = \begin{cases} \dfrac{1}{\pi^2 a} K\left[\sqrt{1 - \dfrac{y^2}{4a^2}}\right], & \text{if } 2a > y > -2a; \\ 0, & \text{if } |y| > 2a. \end{cases} \tag{28}$$

5. SUM OF THREE SINE WAVES

Consider the sum of three sine waves which are functions of uniformly distributed independent random phases. Two of the sine waves have equal amplitudes ($a_1 = a_2 = a$):

$$Y = a\cos(\omega_1 t + \phi_1) + a\cos(\omega_2 t + \phi_2) + a_3\cos(\omega_3 t + \phi_3). \tag{29}$$

The probability density of Y is calculated by convolution (equation (23)) of the probability densities of one and two sine waves, equations (8) and (28):

$$p_Y(y) = \frac{1}{\pi^3 a a_3}\int_{limits}\left(1 - \frac{x^2}{a_3^2}\right)^{-1/2} K\left[\sqrt{1 - \left(\frac{y-x}{2a}\right)^2}\right]\mathrm{d}x. \tag{30}$$

This integral apparently does not possess a simple solution. The approximation for $K(k')$ given in equation (27b) is substituted and integrated term-by-term, but terms involving $\int[1/(Y-y)]\arcsin(y/a_3)\,\mathrm{d}y$ arise that have no closed form solution [7, article 2.83] and these are resolved by substituting the following series for arcsine [8, p. 81]:

$$\arcsin(x) \approx \pi/2 - (1-x)^{1/2}(\gamma_0 + \gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3), \tag{31}$$

where $\gamma_0 = 1{\cdot}5707288$, $\gamma_1 = -0{\cdot}2121144$, $\gamma_2 = 0{\cdot}0742610$ and $\gamma_3 = -0{\cdot}0187293$. The integrals were made piecewise around discontinuities.

The probability density of Y, equation (30), is zero for $Y > 2a + a_3$, $x^2/a_3^2 \geqslant 1$ and $(y-x)^2/a_3^2 \geqslant (2a/a_3)^2$. These conditions give rise to the limits of integration which fall into the four regions shown in Figure 5. The slope of the probability density is discontinuous across the regions. In region 1, $a_3 \leqslant 2a$, $0 \leqslant Y \leqslant 2a - a_3$, the integration is from $-a$ to

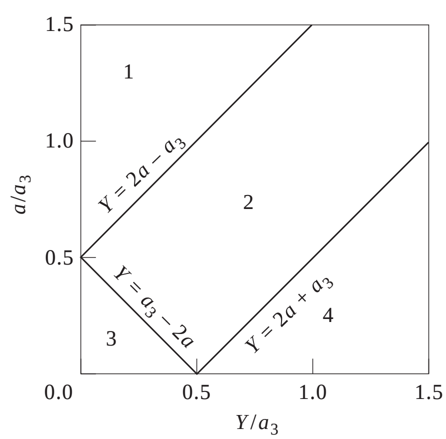

Figure 5. Regions for evaluations of integrals in probability density of the sum of three sine waves.

a_3, and the resultant three sine probability density carried though the α_1, β_0 and γ_1 terms in equations (27b) and (31) is

$$p_Y(y) = \frac{\alpha_0}{\pi^2 a} + \frac{\alpha_1}{2^2\pi^2 a}\left[\left(\frac{y}{a}\right)^2 + \frac{1}{2}\left(\frac{a_3}{a}\right)^2\right] + \frac{\beta_0}{\pi^2 a}\ln\left(\frac{2a}{y+a_3}\right)^2 + \frac{2^{5/2}\beta_0}{\pi^3 a}\left[\gamma_0 + \gamma_1\frac{y}{a_3} - \frac{2}{3}\gamma_1\right]$$

$$- \begin{cases} \dfrac{4\beta_0}{\pi^3 a}\sqrt{\dfrac{y}{a_3} - 1}\left(\gamma_0 + \gamma_1\dfrac{y}{a_3}\right)\arctan\dfrac{2^{1/2}}{\sqrt{y/a_3 - 1}}, & \text{if } a_3 \leqslant y; \\ \dfrac{2\beta_0}{\pi^3 a}\sqrt{1 - \dfrac{y}{a_3}}\left(\gamma_0 + \gamma_1\dfrac{y}{a_3}\right)\ln\dfrac{2^{1/2} + \sqrt{1 - y/a_3}}{2^{1/2} - \sqrt{1 - y/a_3}}, & \text{if } a_3 \geqslant y; \\ 0, & \text{if } a_3 = y. \end{cases} \tag{32}$$

In the limit as a_3 approaches zero, this equation approaches equation (28).

In region 2, $2a + a_3 \geqslant y \geqslant 2a - a_3$, $Y > a_3 - 2a$ and the limits on the integration of equation (30) are from $2a - y$ to a_3. The resultant three sine probability density carried though the γ_1, α_1 and β_0 terms, equations (27b) and (31), is

$$p_Y(y) = \frac{\alpha_0}{\pi^3 a}\left(\frac{\pi}{2} - \arcsin\frac{y-2a}{a_3}\right) + \frac{\alpha_1}{2^2\pi^3 a}\left[\left(\frac{y}{a}\right)^2 + \frac{1}{2}\left(\frac{a_3}{a}\right)^2\right]\left(\frac{\pi}{2} - \arcsin\frac{y-2a}{a_3}\right)$$

$$- \frac{\alpha_1 a_3}{2\pi^3 a^2}\left(\frac{3}{2}\frac{y}{a} + 1\right)\sqrt{1 - \left(\frac{y-2a}{a_3}\right)^2} + \frac{4\beta_0}{\pi^3 a}\left(\frac{2a + a_3 - y}{a_3}\right)^{1/2}\left(\gamma_0 + \gamma_1\frac{4y - 2a - a_3}{3a_3}\right)$$

$$- \begin{cases} \dfrac{4\beta_0}{\pi^3 a}\sqrt{\dfrac{y}{a_3} - 1}\left(\gamma_0 + \gamma_1\dfrac{y}{a_3}\right)\arctan\sqrt{\dfrac{2a + a_3 - y}{y - a_3}}, & \text{if } a_3 \leqslant y; \\ \dfrac{2\beta_0}{\pi^3 a}\sqrt{1 - \dfrac{y}{a_3}}\left(\gamma_0 + \gamma_1\dfrac{y}{a_3}\right)\ln\dfrac{\sqrt{(2a + a_3 - y)/a_3} + \sqrt{1 - y/a_3}}{\sqrt{(2a + a_3 - y)/a_3} - \sqrt{1 - y/a_3}}, & \text{if } a_3 \geqslant y; \\ 0, & \text{if } a_3 = y. \end{cases} \tag{33}$$

In region 3, $a_3 > 2a$ and $a_3 - 2a \geqslant y \geqslant 0$ and the probability density of three sine waves, to order β_0, α_0 and γ_0, is

$$p_Y(y) = \frac{\alpha_0}{\pi^3 a}\left(\arcsin\frac{y+2a}{a_3} - \arcsin\frac{y-2a}{a_3}\right) + \frac{4\beta_0\gamma_0}{\pi^3 a}\left[\left(1 - \frac{y}{a_3} + \frac{2a}{a_3}\right)^{1/2} - \left(1 - \frac{y}{a_3} - \frac{2a}{a_3}\right)^{1/2}\right]$$

$$+ \frac{2\beta_0\gamma_0}{\pi^3 a}\left(1 - \frac{y}{a_3}\right)^{1/2}\ln\frac{\sqrt{1 - y/a_3} + \sqrt{1 - y/a_3 - 2a/a_3}}{\sqrt{1 - y/a_3} - \sqrt{1 - y/a_3 - 2a/a_3}}$$

$$\times \frac{\sqrt{1 - y/a_3 + 2a/a_3} - \sqrt{1 - y/a_3}}{\sqrt{1 - y/a_3 + 2a/a_3} + \sqrt{1 - y/a_3}}. \tag{34}$$

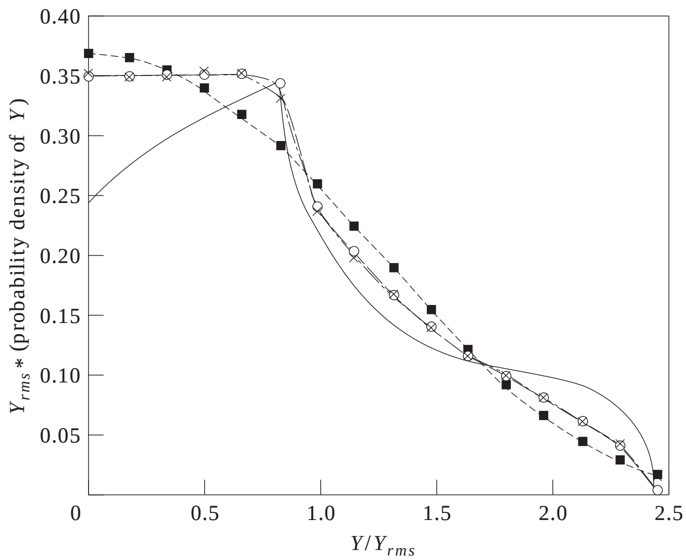

Figure 6. The probability density of the sum of three equal-amplitude sine waves computed by various techniques. ——, Asymptotic integration, equation (35); ■, power series, equation (45); ○, numerical integration, equation (40); ×, Fourier series, equation (42).

The probability density of the sum of three equal-amplitude sine waves ($a_1 = a_2 = a_3 = a$) is obtained from equations (33) and (34):

$$p_Y(y) = \frac{\alpha_0}{\pi^2 a} + \frac{\alpha_1}{2^2\pi^2 a}\left[\left(\frac{y}{a}\right)^2 + \frac{1}{2}\right] + \frac{\beta_0}{\pi^2 a}\ln\left(\frac{2a}{y+a}\right)^2 + \frac{2^{5/2}\beta_0}{\pi^3 a}\left[\gamma_0 + \gamma_1\left(\frac{y}{a} - \frac{2}{3}\right)\right]$$

$$- \frac{2\beta_0}{\pi^3 a}\sqrt{1 - \frac{y}{a}}\left(\gamma_0 + \gamma_1\frac{y}{a}\right)\ln\frac{2^{1/2} + \sqrt{1 - y/a}}{2^{1/2} - \sqrt{1 - y/a}}, \qquad \text{if } 0 \leqslant y \leqslant a;$$

$$p_Y(y) = \frac{\alpha_0}{\pi^3 a}\left(\frac{\pi}{2} - \arcsin\frac{y-2a}{a}\right) + \frac{\alpha_1}{2^2\pi^3 a}\left[\left(\frac{y}{a}\right)^2 + \frac{1}{2}\right]\left(\frac{\pi}{2} - \arcsin\frac{y-2a}{a}\right)$$

$$- \frac{\alpha_1}{2\pi^3 a}\left(1 + \frac{3}{2}\frac{y}{a}\right)\sqrt{1 - \left(\frac{y-2a}{a}\right)^2} + \frac{4\beta_0}{\pi^3 a}\left(3 - \frac{y}{a}\right)^{1/2}\left[\gamma_0 + \gamma_1\left(\frac{4y}{3a} - 1\right)\right]$$

$$- \frac{4\beta_0}{\pi^3 a}\sqrt{\frac{y}{a} - 1}\left(\gamma_0 + \gamma_1\frac{y}{a}\right)\arctan\sqrt{\frac{3a-y}{y-a}}, \qquad \text{if } a \leqslant y \leqslant 3a; \tag{35}$$

and $p_Y(y > 3a) = 0$. This equation is plotted in Figure 6. Note the slope discontinuity at $y = a$, i.e., at $y/Y_{rms} = \sqrt{2/3}$.

6. SUM OF N SINE WAVES

Rather than repeating convolution, it becomes preferable to generate an expression for the probability density of the finite Fourier series, equation (2), with the number of sine waves (N) as a parameter. This is done with characteristic functions. The characteristic function of the sum of N mutually independent random variables ($Y = X_1 + X_2 + \cdots + X_N$) is the product of their characteristic functions [9, p. 22], [2, pp. 124–129],

$$C(f) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}2\pi f(X_1 + X_2 + \cdots + X_N)} p(X_1) p(X_2) \cdots p(X_N) \, \mathrm{d}X_1 \, \mathrm{d}X_2 \cdots \mathrm{d}X_N,$$

$$= \prod_{n=1}^{N} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}2\pi f x_n} p_{X_n}(x_n) \, \mathrm{d}x_n = \prod_{n=1}^{N} C_n(f). \tag{36}$$

The symbol Π denotes a product of terms. The characteristic function for the sum of N independent sine waves is found from equations (10) and (36):

$$C(f) = \begin{cases} \prod_{n=1}^{N} J_0(2\pi f a_n), & \text{unequal } a_n; \\ [J_0(2\pi f a)]^N, & a_n = a, \quad n = 1, 2, \ldots, N. \end{cases} \tag{37}$$

The probability density of Y is the inverse Fourier transform of its characteristic function [2, p. 129]:

$$p_Y(y) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}2\pi f y} C(f) \, \mathrm{d}f. \tag{38}$$

By substituting equation (37) into equation (38) we obtain an integral equation for the probability density of an N-term finite Fourier series of independent sine waves [10]:

$$p_Y(y) = 2 \int_0^{\infty} \cos(2\pi y f) \left\{ \prod_{n=1}^{N} \mathrm{J}_0(2\pi f a_n) \right\} \mathrm{d}f, \qquad N = 1, 2, 3, \ldots. \tag{39}$$

If all N terms of the Fourier series have equal amplitudes $a_n = a$, $n = 1, 2, \ldots N$, then this simplifies to

$$p_Y(y) = 2 \int_0^{\infty} \cos(2\pi y f) \, [J_0(2\pi f a)]^N \, \mathrm{d}f, \qquad N = 1, 2, 3, \ldots. \tag{40}$$

These distributions are symmetric about $Y = 0$, as are all zero mean, sum-of-sine-wave distributions. Figures 2, 4 and 6 show results of numerically integrating equations (39) and (40) over interval $f = 0$ to $f = 15a$ using *Mathematica* [11].

Barakat [10], see also Weiss [9, p. 25] found a Fourier series solution to equation (39). The derivation is presented in Appendix A. The result is

$$p_Y(y) = \frac{1}{2L_Y} + \frac{1}{L_Y} \sum_{i=1}^{\infty} \left\{ \prod_{n=1}^{N} \mathrm{J}_0\left(\frac{i\pi a_n}{L_Y}\right) \right\} \cos\left(\frac{i\pi y}{L_Y}\right), \qquad |y| \leqslant L_Y. \tag{41}$$

For equal amplitudes, $a_n = a$, $n = 1, 2, \ldots, N$, $L_Y = Na$, and

$$p_Y(y) = \frac{1}{2Na} + \frac{1}{Na} \sum_{i=1}^{\infty} \left[\mathrm{J}_0\left(\frac{i\pi}{N}\right) \right]^N \cos\left(\frac{i\pi y}{Na}\right), \qquad |y| \leqslant Na. \tag{42}$$

In Figure 6 it is shown that the Fourier series solution, equation (42), carried to 20 terms to be virtually identical to numerical integration of equation (40), compares well with the approximate solution for $N = 3$ (equation (35)). Note that theory requires $p_Y(|y| > L_Y) = 0$.

A power series solution for equation (40) can be found with a technique used by Cramer [5] and Rice [4, art. 16]. The Bessel function term in equation (40) is expressed as an exponent of a logarithm which is then expanded in a power series (Appendix B):

$$[\mathrm{J}_0(2\pi fa)]^N = \exp[-N\pi^2a^2f^2]\left[1 - \tfrac{1}{4}N\pi^4a^4f^4 - \tfrac{1}{9}N\pi^6a^6f^6 - \left(\frac{11N}{192} - \frac{N^2}{32}\right)\pi^8a^8f^8 + \cdots\right]. \tag{43}$$

Substituting this expansion into equation (40) and rearranging gives a series of integrals,

$$p_Y(y) = 2\int_0^\infty \cos(2\pi yf)e^{-N\pi^2a^2f^2}\,\mathrm{d}f - (2N\pi^4a^4/4)\int_0^\infty f^4\cos(2\pi yf)e^{-N\pi^2a^2f^2}\,\mathrm{d}f$$
$$- (2N\pi^6a^6/9)\int_0^\infty f^6\cos(2\pi yf)\,\mathrm{e}^{-N\pi^2a^2f^2}\,\mathrm{d}f + O(N, N^2a^8f^8), \tag{44}$$

which are solved (Gradshteyn *et al*. [7, arts 3.896, 3.952]) to give a power series for the probability density of the equal-amplitude N-term Fourier series sum in terms of the number of sine waves (N) in the series:

$$p_Y(y) = \frac{\mathrm{e}^{-Y^2/(2Y_{rms}^2)}}{\sqrt{2\pi}Y_{rms}}\left(1 - \frac{\Gamma(5/2)}{4\pi^{1/2}N}\,{}_1F_1[-2, 1/2, y^2/(2Y_{rms}^2)]\right.$$
$$- \frac{\Gamma(7/2)}{9\pi^{1/2}N^2}\,{}_1F_1[-3, 1/2, y^2/(2Y_{rms}^2)]$$
$$\left. - \left(\frac{11}{192N^3} - \frac{1}{32N^2}\right)\frac{\Gamma(9/2)}{\pi^{1/2}}\,{}_1F_1[-4, 1/2, y^2/(2Y_{rms}^2)] + \cdots\right), \qquad |y| < Na. \tag{45}$$

$p_Y(|y| > Na) = 0$ and Y_{rms} is given by equation (4b). There are two special functions in equation (45), the gamma function Γ and the confluent hypergeometric function ${}_1F_1[n, \gamma, z]$. These are described in reference [7].

As N approaches infinity, the peak-to-r.m.s. ratio (equation (5b)) approaches infinity, and equation (45) approaches the normal distribution,

$$\lim_{N\to\infty} p_Y(y) = \frac{1}{\sqrt{2\pi}Y_{rms}}\,\mathrm{e}^{-y^2/(2Y_{rms}^2)}, \tag{46}$$

as predicted by the central limit theorem [5, 12].

7. ENVELOPE

If the time history can be expressed as a sine wave which is modulated by a slowly varying positive amplitude, with $A_e(t) \geqslant 0$, called the envelope of the time history, then

$$Y = A_e(t)\cos(\omega_0 t + \phi_0). \tag{47}$$

A single sine wave, equation (6) is in the form of equation (47) with $A_e(t) = a =$ constant and its envelope probability density is a delta function:

$$p_{A_e}(A) = \delta[A(t) - a]. \tag{48}$$

The sum of two sine waves, equation (24), can also be placed in the form of equation (47):

$$\begin{aligned} Y &= a_1 \cos(\omega_1 t + \phi_1) + a_2 \cos(\omega_2 t + \phi_2) \\ &= (a_1^2 + a_2^2 + 2a_1 a_2 \cos[(\omega_1 - \omega_2)t + \phi_1 - \phi_2])^{1/2} \cos[(\omega_1 + \omega_2)t/2 + (\phi_1 + \phi_2)/2]. \end{aligned} \tag{49}$$

The envelope amplitude,

$$A_e(t) = (a_1^2 + a_2^2 + 2a_1 a_2 \cos[(\omega_1 - \omega_2)t + \phi_1 - \phi_2])^{1/2}, \tag{50}$$

varies slowly in time if ω_1 is not much different from ω_2. Squaring both sides of this expression and moving the constant terms to the left side, we see that the right side is a sine wave:

$$A_e^2 - a_1^2 - a_2^2 = 2a_1 a_2 \cos[(\omega_1 - \omega_2)t + \phi_1 - \phi_2]. \tag{51}$$

This equation relates the envelope amplitude to the difference in the two independent random phases. The probability density of the difference in uniformly distributed phases, $\phi_1 - \phi_2$, has a triangular distribution but, as shown in Appendix C, the probability density of the sine of $\phi_1 - \phi_2$ is identical to the probability density of the sine of a single uniformly distributed phase, equations (7, 8). Thus, with the probability density of the right side of equation (51) the same as the probability density of a sine wave with amplitude $2a_1 a_2$,

$$p_{A_e^2 - a_1^2 - a_2^2}(x) = \begin{cases} \pi^{-1}[(2a_1 a_2)^2 - x^2]^{-1/2}, & -2a_1 a_2 < x < 2a_1 a_2; \\ 0, & |x| > 2a_1 a_2. \end{cases} \tag{52}$$

Since there is a one-to-one relationship between $A_e^2 - a_1^2 - a_2^2$ and A_e, the probability density of A_e can be found by integrating equation (52) to obtain its cumulative probability distribution, equation (110), and then differentiating this function with respect to A_e [12,

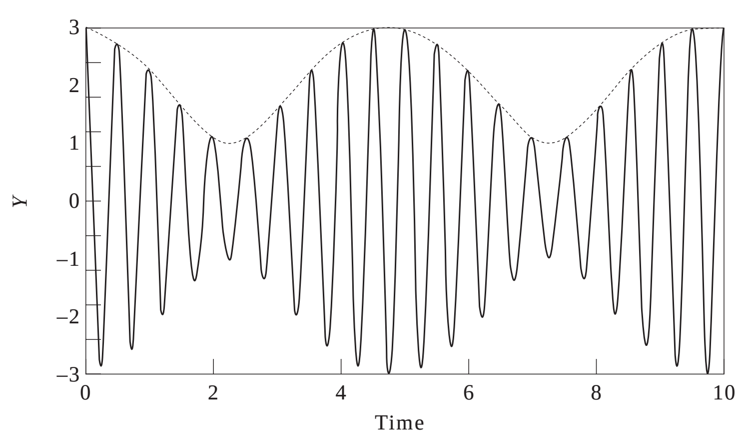

Figure 7. The time history (——), equation (49), and envelope (· · · ·), equation (50) for the sum of two sine waves; $a_1 = 2$, $a_2 = 1$, $\omega_1 = 4\pi$, $\omega_2 = 4{\cdot}4\pi$, $\phi_1 = 0$, $\phi_2 = 0{\cdot}3$ radian.

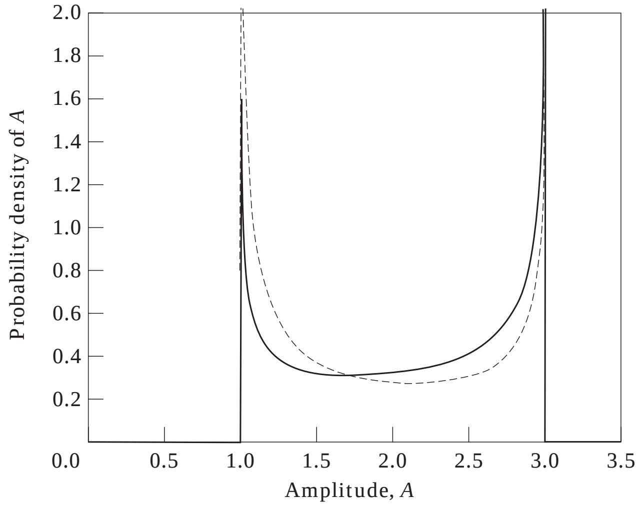

Figure 8. The probability density of the envelope (——), equation (54), and narrow-band peaks (----), equation (73), for the sum of two sine waves of unequal amplitude; $a_1 = 2$, $a_2 = 1$. See also Figure 7.

p. 22]; Gradshteyn *et al*. [7, art. 0.410, p. 23]. Sveshnikov [2, pp. 115–116] gives an equivalent method.

$$p_{A_e}(A) = \frac{\mathrm{d}}{\mathrm{d}A}\int_{-2a_1a_2}^{A^2-a_1^2-a_2^2} p_{A^2-a_1^2-a_2^2}(x)\,\mathrm{d}x = 2Ap_{A^2-a_1^2-a_2^2}(A^2-a_1^2-a_2^2). \tag{53}$$

The resultant solution for the probability density of the envelope of the sum of two independent sine waves is

$$p_{A_e}(A) = \begin{cases} \dfrac{2A}{\pi\sqrt{(2a_1a_2)^2-(A^2-a_1^2-a_2^2)^2}}, & |a_1-a_2| \leqslant A \leqslant a_1+a_2\,; \\ 0, & 0 < A < |a_1-a_2| \text{ or } A > a_1+a_2. \end{cases} \tag{54}$$

If two sine waves, equation (49), have equal amplitudes, $a_1 = a_2 = a$, then their sum can be expressed as the product of two sine waves, their envelope is sinusoidal, $A(t) = 2a\cos[\frac{1}{2}(\omega_1 - \omega_2)t + \frac{1}{2}(\phi_1 - \phi_2)]$, and the probability density of the positive envelope amplitude with $a_1 = a_2 = a$ is twice probability density of a sine wave, equation (8), of amplitude $2a$,

$$p_{A_e}(A)|_{a_1=a_2=a} = \frac{2}{\pi\sqrt{(2a)^2-A^2}}, \tag{55}$$

for $0 \leqslant A_e \leqslant 2a$ and 0 for $A_e > 2a$ or $A_e < 0$. Equations (54) and (55) are exact. A two-sine time history (equation (49)) and its envelope, equation (50), are plotted in Figure 7. The probability density of the envelope is plotted in Figure 8.

The envelope amplitude of the sum of N independent sine waves,

$$Y = \sum_{n=1}^{N} a_n \cos(\omega_n t + \phi_n), \tag{56}$$

is obtained by expanding each sine wave about a frequency ω_0,

$$\cos[\omega_0 t + (\omega_n - \omega_0)t + \phi_n] = \cos\omega_0 t \cos[(\omega_n - \omega_0)t + \phi_n - \sin\omega_0 t \sin[(\omega_n - \omega_0)t + \phi_n], \tag{57}$$

then summing terms proportional to $\sin\omega_0 t$ and $\cos\omega_0 t$,

$$Y = \left[\sum_{n=1}^{N} a_n \cos((\omega_n - \omega_0)t + \phi_n)\right]\cos\omega_0 t - \left[\sum_{n=1}^{N} a_n \sin((\omega_n - \omega_0)t + \phi_n)\right]\sin\omega_0 t. \tag{58}$$

Using equation (57), this equation is placed in the form of equation (47). The resultant phase and envelope amplitude are

$$\tan\phi_0 = -\sum_{n=1}^{N} a_n \sin[(\omega_n - \omega_0)t + \phi_n] \Bigg/ \sum_{n=1}^{N} a_n \cos[(\omega_n - \omega_0)t + \phi_n], \tag{59}$$

$$A_e^2(t) = \left[\sum_{n=1}^{N} a_n \cos((\omega_n - \omega_0)t + \phi_n)\right]^2 + \left[\sum_{n=1}^{N} a_n \sin((\omega_n - \omega_0)t + \phi_n)\right]^2. \tag{60}$$

As long as ω_n is not much different from ω_0, that is, the process is narrow band, then A_e, equation (60), is a slowly varying envelope of the high frequency portion of the time history.

Lord Rayleigh [13] noted that equation (60) has the same form as the equation for the radius to the end-point after N random-direction steps of length a_n. This random walk mathematical technology is described in Appendix D. It is used to determine the probability density of the envelope of the finite Fourier series.

The envelope of the sum of three sine waves, two of which have equal amplitude $a_1 = a_2 = a$, equation (29), is directly found from the exact solution of equation (D5) in Appendix D, and substituting a for l, a_3 for l_3 and A_e for r. If $2a \geqslant a_3$,

$$p_{A_e}(A) = \begin{cases} \dfrac{4A}{\pi^2}\dfrac{1}{\sqrt{[4a^2 - (A - a_3)^2](A + a_3)^2}} K\left(\sqrt{\dfrac{16Aa_3 a^2}{[4a^2 - (A - a_3)^2](A + a_3)^2}}\right), & 0 \leqslant A \leqslant -a_3 + 2a; \\[2ex] \dfrac{4A}{\pi^2}\dfrac{1}{\sqrt{16Aa_3 a^2}} K\left(\sqrt{\dfrac{[4a^2 - (A - a_3)^2](A + a_3)^2}{16Aa_3 a^2}}\right), & -a_3 + 2a \leqslant A \leqslant a_3 + 2a; \\[2ex] 0, & 0 > A, \quad \text{or} \quad A > a_3 + 2a. \end{cases} \tag{61}$$

If $2a < a_3$,

$$p_{A_e}(A) = \begin{cases} \dfrac{4A}{\pi^2}\dfrac{1}{\sqrt{16Aa_3\,a^2}}\,K\left(\sqrt{\dfrac{[4a^2-(A-a_3)^2]\,(A+a_3)^2}{16Aa_3\,a^2}}\right), & a_3-2a \leqslant A \leqslant a_3+2a; \\ 0, & A > a_3+2a, \text{ or } A < a_3-2a. \end{cases} \tag{62}$$

If $a_3 = a$ for equal amplitudes, then the probability density is

$$p_{A_e}(A) = \begin{cases} \dfrac{4A}{\pi^2}\dfrac{1}{\sqrt{[4a^2-(A-a)^2]\,(A+a)^2}}\,K\left(\sqrt{\dfrac{16Aa^3}{[4a^2-(A-a)^2]\,(A+a)^2}}\right), & 0 \leqslant A \leqslant a; \\ \dfrac{4A}{\pi^2}\dfrac{1}{\sqrt{16Aa^3}}\,K\left(\sqrt{\dfrac{[4a^2-(A-a)^2]\,(A+a)^2}{16Aa^3}}\right), & a \leqslant A \leqslant 3a; \\ 0, & A > 3a. \end{cases} \tag{63}$$

As a_3 approaches zero, equation (61) approaches equation (55). Equation (63) is also given by Pearson [14].

Kluyer [15] found a general integral solution for cumulative probability of the distance from the origin to the end of an N-step random walk. (See also Pearson [14], Rayleigh [16], Watson [17], Barber and Ninham [18] and Weiss [9].) Kluyer's solution is differentiated to obtain the probability density of the envelope of the sum of N mutually independent sine waves (equation (2)). For *unequal* amplitudes,

$$p_{A_e}(A) = A\int_0^\infty x\mathrm{J}_0(Ax)\left\{\prod_{n=1}^{N}\mathrm{J}_0(a_n x)\right\}\mathrm{d}x. \tag{64}$$

For *equal* amplitudes, $a_n = a$, $n = 1, 2, \ldots, N$, the envelope probability density becomes

$$p_{A_e}(A) = A\int_0^\infty x\mathrm{J}_0(Ax)\,[\mathrm{J}_0(ax)]^N\,\mathrm{d}x. \tag{65}$$

These equations can be numerically integrated, but the oscillatory integrand requires long computation times to achieve high accuracy, especially for small N and near the tail of the distribution [10, 19, 20].

Shmulei and Weiss [20] (see also Weiss [9, p. 280], and Barakat [10]) have developed series solutions to equation (64). The Shmulei and Weiss [20] series solution is

$$p_{A_e}(A) = \frac{2\pi A}{L_Y^2}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\alpha_{ik}\,\mathrm{J}_0\left(\frac{\pi A}{L_Y}(i^2+k^2)^{1/2}\right)\left\{\prod_{n=1}^{N}\mathrm{J}_0\left(\frac{\pi a_n}{L_Y}(i^2+k^2)^{1/2}\right)\right\}, \tag{66}$$

where $\alpha_{00} = 1/4$, $\alpha_{0k} = \alpha_{i0} = 1/2$ and $\alpha_{ik} = 1$ if $i > 0$ and $k > 0$. $L_Y = a_1 + a_2 + \cdots + a_N$. For equal amplitudes, $a_n = a$, $n = 1, 2, \ldots N$,

$$p_{A_e}(A) = \frac{2\pi A}{(Na)^2} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \alpha_{ik} \, \mathrm{J}_0\left(\frac{\pi A}{Na}(i^2+k^2)^{1/2}\right)\left[\mathrm{J}_0\left(\frac{\pi}{N}(i^2+k^2)^{1/2}\right)\right]^N. \tag{67}$$

These equations apply for $0 \leqslant A_e \leqslant L_y$ and $0 \leqslant A_e \leqslant Na$, respectively. $p_{A_e}(A) = 0$ outside of this range.

A power series expansion for equation (65) is found using the methodology of Appendix B. The Bessel function term in equation (65) is expressed as the exponent of a logarithm which is then expanded in a power series. The integral of each term in the power series has a closed form solution; reference [7, art. 6.631]. The series expansion for the probability density of the envelope of the N-term Fourier series sum of equal amplitude terms is

$$\begin{aligned} p_{A_e}(A) = \frac{A\,\mathrm{e}^{-A^2/(2Y_{rms}^2)}}{Y_{rms}^2}\Bigg(1 &- \frac{\Gamma(3)}{4N}\,{}_1F_1[-2, 1, A^2/(2Y_{rms}^2)] \\ &- \frac{\Gamma(4)}{9N^2}\,{}_1F_1[-3, 1, A^2/(2Y_{rms}^2)] \\ &- \left(\frac{11}{92} - \frac{N}{32}\right)\frac{\Gamma(5)}{N^3}\,{}_1F_1[-4, 1, A^2/(2Y_{rms}^2)] + \cdots\Bigg), \qquad 0 \leqslant A < Na. \end{aligned} \tag{68}$$

$p_{A_e}(A > Na) = 0$. The overall root mean square Y_{rms} is given by equation (4b). As N approaches infinity, Na approaches infinity, the peak-to-r.m.s. ratio approaches infinity, and the probability density of the envelope, equation (68), approaches the Rayleigh distribution [13, 16]:

$$\lim_{N\to\infty} p_{A_e}(A) = \frac{A}{Y_{rms}^2}\,\mathrm{e}^{-A^2/(2Y_{rms}^2)}. \tag{69}$$

8. PEAKS AND TROUGHS OF TWO SINE WAVES

Consider the sum of two sine waves, equation (24), which are functions of uniformly distributed independent random phase angles; equation (8). Peaks and troughs in their time history occur at points of zero slope, $\mathrm{d}Y(t)/\mathrm{d}t = 0$. Peaks occur at points of zero slope and negative curvature, while troughs occur at points of zeros slope and positive curvature. Setting the derivative with respect to time of the sum of two sine waves, equation (24), to zero gives

$$0 = -\omega_1 a_1 \sin(\omega_1 t + \phi_1) - \omega_2 a_2 \sin(\omega_2 t + \phi_2). \tag{70}$$

Equations (24) and (70) are two equations in two unknowns and they can be reduced to an equation describing $\cos(\omega_2 t + \phi_2)$ as a function of the peak amplitude Y such that $\mathrm{d}Y(t)/\mathrm{d}t = 0$:

$$\begin{aligned} \cos^2(\omega_2 t + \phi_2)\,[a_2^2 - a_1^2(\omega_2 a_2)^2/(\omega_1 a_1)^2] - 2Ya_2\cos(\omega_2 t + \phi_2) \\ + Y^2 - a_1^2 + a_1^2(\omega_2 a_2)^2/(\omega_1 a_1)^2 = 0. \end{aligned} \tag{71}$$

The nonmonotonic relationship between $\cos(\omega_2 t + \phi_2)$ and Y makes it difficult to concisely express the probability density of Y as a function of the probability density of $\cos(\omega_2 t + \phi_2)$. There are particular solutions as follows.

(1) *Single sine wave*. Consider $a_2 = 0$. Equation (71) reduces to $Y = \pm a_1$. A peak occurs at $Y = a_1$ and a trough occurs at $Y = -a_1$. The probability distribution of peaks is given by equation (48).

(2) *Two sine waves of equal frequency*. Setting $\omega_1 = \omega_2$ reduces equation (71) to

$$2a_2 \cos(\omega_2 t + \phi_2) = (Y^2 - a_1^2 + a_2^2)/Y. \tag{72}$$

Consider $a_1 > a_2$. For uniformly distributed ϕ_2, the probability density of the left side is the same as that of a sine wave, equation (8), with magnitude $2a_2$. However, for $a_1 \neq a_2$ there are two values of Y, one positive and one negative, for a given value of the left-side of this equation. Since the frequencies are equal, only peaks can occur for $Y > 0$ and troughs only occur for $Y < 0$. Considering each branch separately and using the methodology of equations (52)–(54), the probability density of peaks is

$$p_{A_p}(y) = \begin{cases} \dfrac{y^2 + a_1^2 - a_2^2}{\pi y \sqrt{(2a_1 a_2)^2 - (y^2 - a_1^2 - a_2^2)^2}}, & a_1 - a_2 \leqslant y \leqslant a_1 + a_2; \\ 0, & a_1 - a_2 > y \quad \text{or} \quad y > a_1 + a_2. \end{cases} \tag{73}$$

The probability density of troughs is a similar expression but it is non-zero in the range $-a_1 - a_2 < y < -a_1 + a_2$, and y on the right side of the above equation is replaced by $|y|$. When $a_1 = a_2 = a$ the two solutions coalesce and the probability density of peaks and troughs is given by

$$p_{A_{pt}}(y) = \begin{cases} \dfrac{1}{\pi \sqrt{(2a)^2 - y^2}}, & -2a \leqslant y \leqslant 2a; \\ 0, & |y| > 2a. \end{cases} \tag{74}$$

Equation (74) differs from the two-sine-wave envelope density, equation (54), by a factor of two, since equation (74) applies to the probability density of both peaks and troughs.

(3) *Sine waves with much different frequency*. In the limit, as the frequency ω_2 becomes much larger than ω_1, there is a peak followed by a trough, at time intervals of π/ω_2 which is at virtually every point along the time history path of the more slowly oscillating term. Equation (71) reduces to $\cos(\omega_2 t + \phi_2) = \pm 1$; hence equation (24) reduces to $Y = a_1 \cos(\omega_1 t + \phi_1) \pm a_2$ where $+$ is used for peak and $-$ for trough. The probability of a peak of at value Y is equal to the probability of the low frequency term achieving a value $Y - a_2$. The probability of peaks (A_p) and troughs (A_t) are adapted from equation (8):

$$p_{A_p}(y) = \begin{cases} \pi^{-1}((a_1^2 - (y - a_2)^2)^{-1/2}, & \text{if } -a_1 + a_2 < y < a_1 + a_2; \\ 0, & \text{if } y < -a_1 + a_2, \quad y > a_1 + a_2. \end{cases} \tag{75a}$$

$$p_{A_t}(y) = \begin{cases} \pi^{-1}((a_1^2 - (y + a_2)^2)^{-1/2}, & \text{if } -a_1 - a_2 < y < a_1 - a_2; \\ 0, & \text{if } y < -a_1 - a_2, \quad y > a_1 - a_2. \end{cases} \tag{75b}$$

The result directly extends to series of N terms. If one term has a frequency much greater than the remaining $N - 1$ terms, then the probability density of peaks of the sum of the N terms is obtained by offsetting the probability density of the more slowly varying $N - 1$ terms by the amplitude of the high frequency term.

9. PEAKS IN NARROW-BAND SERIES

If $Y(t)$ is *narrow band*, that is, each trajectory of $Y(t)$ which crosses zero has only a single peak, or trough, before crossing the axis again, then the number of peaks equals the number of times the time history crosses the axis with positive slope, and only positive peaks occur for $Y(t) > 0$ and they are located at points of zero slope, $\mathrm{d}Y(t)/\mathrm{d}t = 0$. Lin [12, p. 296, 304] gives expressions for the expected number of zero crossings with positive slope (peaks above the axis) per unit time for a general, not necessarily narrow-band, process:

$$\mathrm{E}[N_{0+}] = \int_0^\infty \dot{y} p_{Y\dot{Y}}(0, \dot{y})\,\mathrm{d}\dot{y}, \tag{76}$$

and the probability density of the peaks for a narrow-band process,

$$p_{A_p}(A) = -\frac{1}{\mathrm{E}[N_{0+}]}\frac{\mathrm{d}}{\mathrm{d}A}\int_0^\infty \dot{y} p_{Y\dot{Y}}(A, \dot{y})\,\mathrm{d}\dot{y}. \tag{77}$$

In order to apply these expressions, the joint probability distribution of Y and $\dot{Y}$ must be established. This is done using characteristic functions. The joint characteristic function of a sine wave and its first derivative is given by equation (17). The joint characteristic function of the sum and the derivative of the sum of N mutually independent sine waves is the product of the joint characteristic functions of the series terms (see equation (36)):

$$C(f_1, f_2) = \prod_{n=1}^{N} C_n(f_1, f_2) = \prod_{n=1}^{N} \mathrm{J}_0\left(2\pi a_n \sqrt{f_1^2 + f_2^2\,\omega_n^2}\right). \tag{78}$$

The joint probability density function of two random variables is the inverse Fourier transform of the joint characteristic function of the two random variables:

$$p_{Y\dot{Y}}(y, \dot{y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}2\pi(f_1 y + f_2 \dot{y})} C(f_1, f_2)\,\mathrm{d}f_1\,\mathrm{d}f_2. \tag{79}$$

The proof of equations (78) and (79) follows the same method used in equations (36)–(39). See also Chandrasekhar [21], Willie [22], Weiss and Shmueli [23], and Weiss [9, pp. 21–26].

Substituting equation (78) and (79) and expanding gives

$$p_{Y\dot{Y}}(y, \dot{y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\prod_{n=1}^{N} \mathrm{J}_0\left(2\pi a_n \sqrt{f_1^2 + f_2^2\,\omega_n^2}\right)\right\} \cos(2\pi f_1 y)\cos(2\pi f_2 \dot{y})\,\mathrm{d}f_1\,\mathrm{d}f_2. \tag{80}$$

Substituting this equation into equations (76) and (77) gives triple-integral expressions for the probability density of the number of peaks per unit time and their amplitude. These expressions can be integrated numerically, but their solution is difficult and it is desirable to obtain simpler solutions.

It is possible to expand the joint probability density of Y and $\dot{Y}$ in a double finite Fourier series as shown in Appendix A. The result is

$$p_{Y\dot{Y}}(y, \dot{y}) = \frac{1}{L_Y L_{\dot{Y}}} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \alpha_{ik} \left\{\prod_{n=1}^{N} \mathrm{J}_0\left(\pi a_n \sqrt{\left(\frac{i}{L_Y}\right)^2 + \left(\frac{k\omega_n}{L_{\dot{Y}}}\right)^2}\right)\right\} \cos(i\pi y/L_Y)\cos(k\pi \dot{y}/L_{\dot{Y}}). \tag{81}$$

The dimensionless coefficient α_{ik} is given by equation (A7). The expected number of peaks per unit time and the probability density of peaks of a narrow-band finite Fourier series is obtained by substituting this equation into equations (76) and (77) and integrating. The results are

$$\mathrm{E}[N_p] = \mathrm{E}[N_{0+}] = \frac{L_{\dot{Y}}}{L_Y} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \gamma_{ik} \left\{ \prod_{n=1}^{N} \mathrm{J}_0 \left(\pi a_n \sqrt{\left(\frac{i}{L_Y}\right)^2 + \left(\frac{k\omega_n}{L_{\dot{Y}}}\right)^2} \right) \right\}, \tag{82}$$

$$p_{A_p}(y) = \frac{\pi L_{\dot{Y}}}{L_Y^2\, \mathrm{E}[N_p]} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} i\gamma_{ik} \left\{ \prod_{n=1}^{N} \mathrm{J}_0 \left(\pi a_n \sqrt{\left(\frac{i}{L_Y}\right)^2 + \left(\frac{k\omega_n}{L_{\dot{Y}}}\right)^2} \right) \right\} \sin(i\pi y/L_Y), \tag{83}$$

where

$$\gamma_{ik} = \begin{cases} 1/8, & i = k = 0; \\ 1/4, & i > 0,\ k = 0; \\ (1/2)\,[(-1)^k - 1]/(k\pi)^2, & i = 0,\ k > 0; \\ [(-1)^k - 1]/(k\pi)^2, & i > 0,\ k > 0. \end{cases} \tag{84}$$

One advantage of the narrow-band analysis over envelope analysis is that narrow-band analysis gives a specific estimate for the number of peaks per unit time. If the frequencies are closely spaced so $\omega_n \approx \omega$ and hence $L_{\dot{Y}} \approx \omega L_Y$, then one positive peak is expected once per cycle,

$$\mathrm{E}[N_p] = \omega/(2\pi) \tag{85}$$

The probability density of narrow-band peaks becomes

$$p_{A_p}(y) = \frac{2\pi^2}{L_Y} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} i\gamma_{ik} \left\{ \prod_{n=1}^{N} \mathrm{J}_0 \left(\frac{\pi a_n}{L_Y} \sqrt{i^2 + k^2} \right) \right\} \sin\left(\frac{i\pi y}{L_Y}\right). \tag{86}$$

Equation (86) is similar, but not identical, to the expression (equation (66)) for the probability density of the envelope of the time history.

A power series solution exists for the probability density of the peaks. If the series terms have equal amplitude and frequency then the joint probability density of Y and $\dot{Y}$ is

$$p_{Y\dot{Y}}(y, \dot{y}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\mathrm{J}_0(2\pi a\sqrt{f_1^2 + \omega^2 f_2^2})]^N \cos(2\pi f_1 y) \cos(2\pi f_2 \dot{y})\, \mathrm{d}f_1\, \mathrm{d}f_2. \tag{87}$$

The Bessel function term is expanded in Cramer's series (Appendix B):

$$[\mathrm{J}_0(2\pi a\sqrt{f_1^2 + \omega^2 f_2^2})]^N = \mathrm{e}^{-Na^2\pi^2(f_1^2 + f_2^2\omega^2)}(1 - \tfrac{1}{4} N\pi^4 a^4 (f_1^2 + \omega^2 f_2^2) + \cdots), \tag{88}$$

which is substituted into equation (87) and integrated. The result is a power series for the joint probability density of Y and $\dot{Y}$,

$$p_{Y\dot{Y}}(y, \dot{y}) = \frac{\mathrm{e}^{-(1/2)(y^2/Y_{rms}^2 + \dot{y}^2/\dot{Y}_{rms}^2)}}{2\pi Y_{rms}\, \dot{Y}_{rms}} \left(1 - \frac{1}{4N} \left[1 - \left(\frac{y^2}{Y_{rms}^2} + \frac{\dot{y}^2}{\dot{Y}_{rms}^2} \right) + \frac{1}{8} \left(\frac{y^2}{Y_{rms}^2} + \frac{\dot{y}^2}{\dot{Y}_{rms}^2} \right)^2 \right] + \cdots. \right. \tag{89}$$

This equation is then substituted into equation (77) and integrated to give a power series for the probability density of peaks in a narrow-band series of equal amplitude and frequency sine waves,

$$p_{A_p}(y) = \frac{y\,\mathrm{e}^{-y^2/(2Y_{rms}^2)}}{Y_{rms}^2}\left(1 - \frac{1}{4N} + \frac{1}{4N}\frac{y^2}{Y_{rms}^2} - \frac{1}{32N}\frac{y^4}{Y_{rms}^4} + \cdots\right). \tag{90}$$

In the limit as N becomes infinite, the joint probability density of Y and $\dot{Y}$ has a Gaussian distribution,

$$\lim_{N\to\infty} p_{Y\dot{Y}}(y, \dot{y}) = \frac{1}{2\pi Y_{rms}\,\dot{Y}_{rms}}\,\mathrm{e}^{-(y^2/2Y_{rms}^2 + \dot{y}^2/2\dot{Y}_{rms}^2)}. \tag{91}$$

and the distribution of narrow-band process peaks approaches the Rayleigh distribution,

$$\lim_{N\to\infty} p_{A_p}(y) = \frac{1}{Y_{rms}^2}\,y\,\mathrm{e}^{-y^2/(2Y_{rms}^2)}. \tag{92}$$

Equations (91) and (92) are in agreement with expressions given by Crandall and Mark [20, p. 47, 51] for Gaussian random processes.

Equations (77), (83), (86) and (92) are conservative when applied to non-narrow-band processes in the sense that any troughs above the axis (points with $Y > 0$ and $\mathrm{d}Y/\mathrm{d}t = 0$ but $\mathrm{d}^2Y/\mathrm{d}t^2 > 0$) are counted as peaks [12, p. 304], [22], [23], [24].

10. PEAKS IN BROADBAND SERIES

A maximum, or local peak, in the continuous function $Y(t)$ occurs at a point at which $\mathrm{d}Y(t)/\mathrm{d}t$ is zero and $\mathrm{d}^2Y(t)/\mathrm{d}t^2$ is negative. This suggests that a general expression for the probability of peaks of $Y(t)$ can be obtain from the joint probability density of $Y(t)$, $\mathrm{d}Y(t)/\mathrm{d}t$, and $\mathrm{d}^2Y(t)/\mathrm{d}t^2$ [12, *p*. 299]. Expressions for the number of peaks per unit time and the probability density of those peaks have been developed by Rice [26] and adapted by Wirsching *et al*. [25, p. 155, 166] and Lin [12, pp. 300, 301]:

$$\mathrm{E}[N_p] = -\int_{-\infty}^{0} \ddot{y}\,p_{\dot{Y}\ddot{Y}}(0, \ddot{y})\,\mathrm{d}\ddot{y}, \tag{93}$$

$$p_{A_p}(y) = -\frac{1}{\mathrm{E}[N_p]}\int_{-\infty}^{0} \ddot{y}\,p_{Y\dot{Y}\ddot{Y}}(y, 0, \ddot{y})\,\mathrm{d}\ddot{y}. \tag{94}$$

In order to apply these expressions, we need to develop the joint probability densities $p_{\dot{Y}\ddot{Y}}(\dot{Y}, \ddot{Y})$ and $p_{Y\dot{Y}\ddot{Y}}(Y, \dot{Y}, \ddot{Y})$. These are found using characteristic functions. The joint characteristic function of the sine wave and its first two derivatives is given by equation (21). The corresponding joint characteristic function of the sum of N mutually independent sine waves is the product of the joint characteristic functions of the sine waves (see equations (36) and (78)),

$$C(f_1, f_2, f_3) = \prod_{n=1}^{N} \mathrm{J}_0\left(2\pi a_n\sqrt{(f_1 - \omega_n^2 f_3)^2 + f_2^2\,\omega_n^2}\right). \tag{95}$$

The joint probability density is the inverse Fourier transform of the characteristic function (see equations (38) and (79)),

$$p_{Y\dot{Y}\ddot{Y}}(y, \dot{y}, \ddot{y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}2\pi(f_1 y + f_2 \dot{y} + f_3 \ddot{y})}$$

$$\times \left\{\prod_{n=1}^{N} \mathrm{J}_0\left(2\pi a_n \sqrt{(f_1 - \omega_n^2 f_3)^2 + f_2^2\,\omega_n^2}\right)\right\} \mathrm{d}f_1\,\mathrm{d}f_2\,\mathrm{d}f_3. \tag{96}$$

The probability density $p_{\dot{Y}\ddot{Y}}(\dot{Y}, \ddot{Y})$ is obtained from this expression by setting $f_1 = 0$ and deleting the integration with respect to f_1,

$$p_{\dot{Y}\ddot{Y}}(\dot{Y}, \ddot{Y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}2\pi(f_2 \dot{y} + f_3 \ddot{y})}\left\{\prod_{n=1}^{N} \mathrm{J}_0\left(2\pi a_n \sqrt{(\omega_n^2 f_3)^2 + f_2^2\,\omega_n^2}\right)\right\} \mathrm{d}f_2\,\mathrm{d}f_3. \tag{97}$$

Substituting equation (97) into equation (93) and equation (96) into equation (94) gives integral expressions for the expected number of peaks per unit time and the probability density of those peaks in terms of a triple integral and a quadruple integral. It is desirable to obtain simpler solutions.

The joint probability density $p_{\dot{Y}\ddot{Y}}(\dot{y}, \ddot{y})$ can be expressed as a double Fourier series using the same process as was used in deriving equation (81); see Appendix A. The result is

$$p_{\dot{Y}\ddot{Y}}(\dot{y}, \ddot{y}) = \frac{1}{L_{\dot{Y}}\,L_{\ddot{Y}}} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \alpha_{km}\left\{\prod_{n=1}^{N} \mathrm{J}_0\left(\pi a_n \sqrt{\left(\frac{k\omega_n}{L_{\dot{Y}}}\right)^2 + \left(\frac{m\omega_n^2}{L_{\ddot{Y}}}\right)^2}\right)\right\}$$

$$\times \cos\left(\frac{k\pi\dot{y}}{L_{\dot{Y}}}\right)\cos\left(\frac{m\pi\ddot{y}}{L_{\ddot{Y}}}\right). \tag{98}$$

This equation is then substituted into equation (93) and integrated over $-L_{\ddot{Y}} \leqslant \ddot{y} \leqslant 0$ (recall that $p_{\dot{Y}\ddot{Y}}(\dot{y}, \ddot{y}) = 0$ for $\ddot{y} \leqslant -L_{\ddot{Y}}$) to give the expected number of positive peaks per unit time for a general (broadband) series sum of N mutually independent sine waves:

$$\mathrm{E}[N_p] = \frac{L_{\ddot{Y}}}{L_{\dot{Y}}} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \gamma_{km}\left\{\prod_{n=1}^{N} \mathrm{J}_0\left(\pi a_n \sqrt{\left(\frac{k\omega_n}{L_{\dot{Y}}}\right)^2 + \left(\frac{m\omega_n^2}{L_{\ddot{Y}}}\right)^2}\right)\right\}, \tag{99}$$

where $L_{\ddot{Y}} = \omega_1^2\,a_1 + \omega_2^2\,a_2 + \cdots + \omega_N^2\,a_N$, $L_{\dot{Y}} = \omega_1\,a_1 + \omega_2\,a_2 + \cdots + \omega_N\,a_N$ and $L_Y = a_1 + a_2 + \cdots + a_N$. If all the terms have a similar frequency, $\omega_n \approx \omega$, then equation (99) reduces to $\mathrm{E}[N_p] = \omega/(2\pi)$, the same result as for a narrow-band process, equation (85).

As discussed in Appendix *A*, equation (A15), the joint probability density of Y, $\dot{Y}$ and $\ddot{Y}$ can be expressed by a Fourier series,

$$p_{Y\dot{Y}\ddot{Y}}(y, \dot{y}, \ddot{y}) = \frac{1}{8L_Y\,L_{\dot{Y}}\,L_{\ddot{Y}}} \sum_{i=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \left\{\prod_{n=1}^{N} \mathrm{J}_0\left(\pi a_n \sqrt{\left(\frac{i}{L_Y} - \frac{\omega_n^2\,m}{L_{\ddot{Y}}}\right)^2 + \left(\frac{\omega_n\,k}{L_{\dot{Y}}}\right)^2}\right)\right\}$$

$$\times \cos\frac{k\pi\dot{y}}{L_{\dot{Y}}} \cos\left(\frac{i\pi y}{L_Y} + \frac{m\pi\ddot{y}}{L_{\ddot{Y}}}\right). \tag{100}$$

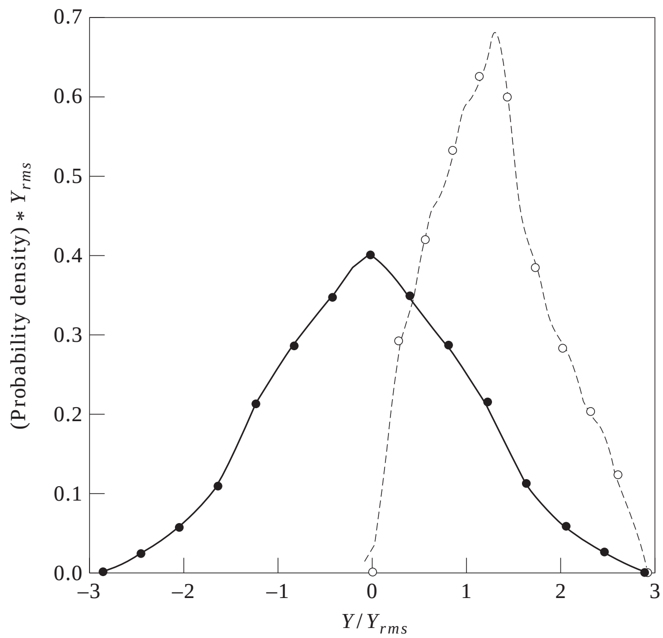

Figure 9. The probability density of peaks in a series of five sine waves with amplitudes $a_1 = a_2 = a_3 = a_4 = 1$ and $a_5 = 0{\cdot}1$ for both a narrow-band frequency distribution, $\omega_1 = \omega_2 = \omega_2 = \omega_4 = \omega_5 = 1$ and a broadband frequency distribution $\omega_1 = \omega_2 = \omega_2 = \omega_4 = 1$, $\omega_5 = 100$. ----, Broad band, with equal frequencies, equation (101); ○, narrow band, equation (80); ——, broadband with unequal frequencies, equation (101); ●, sum of five sine waves, equation (41).

Substituting this expression into equation (94) and integrating gives the probability density of peaks of a general (broadband) sum of N independent sine waves,

$$p_{A_p}(y) = -\frac{1}{\mathrm{E}[N_p]}\frac{L_{\ddot{Y}}}{8L_Y L_{\dot{Y}}}\sum_{i=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\prod_{n=1}^{N}\mathrm{J}_0\left(\pi a_n\sqrt{\left(\frac{i}{L_Y}-\frac{\omega_n^2 m}{L_{\ddot{Y}}}\right)^2+\left(\frac{\omega_n k}{L_{\dot{Y}}}\right)^2}\right)$$

$$\times\left(r_m\cos\frac{i\pi y}{L_Y}-s_m\sin\frac{i\pi y}{L_Y}\right),\qquad -L_Y\leqslant y\leqslant L_Y\,; \tag{101}$$

$$r_m=\begin{cases}-1/2, & m=0;\\ (1-(-1)^{|m|})/(m\pi)^2, & m=\pm 1,\pm 2,\pm 3,\ldots;\end{cases}$$

$$s_m=\begin{cases}0, & m=0;\\ -(-1)^{|m|}/(m\pi), & m=\pm 1,\pm 2,\pm 3,\ldots.\end{cases} \tag{102}$$

This equation is fairly general in that it applies to a series composed of an arbitrary number (N) of terms, with arbitrary amplitudes (a_n) and frequencies (ω_n) which are positive non-zero integer multiples of $2\pi/T$. Multiple peaks can occur between the time when Y crosses the axis and when it recrosses the axis and peaks may occur on both sides of the axis.

For numerical computation, each of the summations from minus infinity to plus infinity in equation (101) is made over a finite range; say, $-$NSUM to NSUM. This leads to $8 * \mathrm{NSUM}^3$ evaluations of the function within the summations. It is possible to reduce the number of evaluations by noting that the summations have the same value for positive and negative values of the index k so terms associated with $k = -1, -2, -3, \ldots$ can be included by doubling the value of the $k = 1, 2, 3, \ldots$ terms and summing over $k = 0, 1, 2, 3, \ldots$ NSUM.

The irregularity factor is defined as the ratio of the expected number of zero crossings with positive slope, equations (76) and (85), to the expected number of peaks, equations (93) and (98), [27, p. 155]:

$$\text{irregularity factor} = \mathrm{E}[N_{0+}]/\mathrm{E}[N_p] = F(N, \{a_n\}_{n=1,N}, \quad \{\omega_n\}_{n=1,N}). \tag{103}$$

The irregularity factor approaches unity as the process becomes increasingly narrow band and it decreases with increasing frequency differences. In Figure 9 is given the probability density of the peaks, equation (101), for a narrow-band random process which is the sum of five sine waves with amplitudes $a_1 = a_2 = a_3 = a_4 = 1$ and $a_5 = 0{\cdot}1$ and frequencies $\omega_1 = \omega_2 = \omega_3 = \omega_4 = 1$, which corresponds to an irregularity of 1·0, and for a broadband process which has the same amplitude and frequencies of ω_1–ω_4 but with a small high frequency oscillating term $\omega_5 = 100$, which gives an irregularity of 0·057. Equation (73) compares well with the narrow band result and results for the sum of four five sine waves (equation (41)) compare well with the broad band result, as predicted in the last paragraph of section 8.

11. POWER SERIES EXPANSION FOR BROADBAND SERIES PEAKS

It is possible to expand the probability density of peaks of a broadband finite Fourier series in a power series. The Bessel function of order zero has a maximum value of unity at the origin and decays with increasing values of its argument, so the maximum values of equation (95) occur near $f_1 = f_2 = f_3 = 0$ and as the number of terms in the series becomes large, non-trivial values are concentrated near the origin. Using Appendix B, the joint characteristic function of Y, $\dot{Y}$ and $\ddot{Y}$, equation (95), is expanded for small f_1, f_2 and f_3 about $f_1 = f_2 = f_3 = 0$:

$$C(f_1, f_2, f_2) = \exp\left[\ln\left(\prod_{n=1}^{N} \mathrm{J}_0\left(2\pi a_n \sqrt{(f_1 - \omega_n^2 f_3)^2 + f_2^2\,\omega_n^2}\right)\right)\right]$$

$$= \exp\left[-\pi^2 \sum_{n=1}^{N} a_n^2\,((f_1 - \omega_n^2 f_3)^2 + f_2^2\,\omega_n^2)\right] + O(f^4)$$

$$= \exp[-2\pi^2(Y_{rms}^2 f_1^2 - 2\dot{Y}_{rms}^2 f_1 f_3 + \ddot{Y}_{rms}^2 f_3^2 + \dot{Y}_{rms}^2 f_2^2)] + O(f^4). \tag{104}$$

Y_{rms} is given by equation (4b). The r.m.s. of the derivatives of Y follow similar laws,

$$\dot{Y}_{rms}^2 = \tfrac{1}{2}\sum_{n=1}^{N} \omega_n^2\,a_n^2, \qquad \ddot{Y}_{rms}^2 = \tfrac{1}{2}\sum_{n=1}^{N} \omega_n^4\,a_n^2. \tag{105}$$

Equation (104), to order f^2, is substituted into equation (96) and integrated (Gradshteyn *et al.* [7, articles 3·896, 3·897]) to produce the joint probability density of Y, $\dot{Y}$ and $\ddot{Y}$, for series with large numbers of terms:

$$p_{Y\dot{Y}\ddot{Y}}(y, \dot{y}, \ddot{y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\mathrm{j}2\pi(f_1 y + f_2 \dot{y} + f_3 \ddot{y})\right] \exp\left[-2\pi^2(Y_{rms}^2 f_1^2 - 2\dot{Y}_{rms}^2 f_1 f_3 + \ddot{Y}_{rms}^2 f_3^2 + \dot{Y}_{rms}^2 f_2^2)\right] \mathrm{d}f_1\, \mathrm{d}f_2\, \mathrm{d}f_2$$

$$= \frac{\exp[-(\dot{y}^2(Y_{rms}^2 \ddot{Y}_{rms}^2 - \dot{Y}_{rms}^4) + \ddot{y}^2 Y_{rms}^2 \dot{Y}_{rms}^2 + y^2 \dot{Y}_{rms}^2 \ddot{Y}_{rms}^2 + 2y\ddot{y}\dot{Y}^4)/(2(Y_{rms}^2 \dot{Y}_{rms}^2 \ddot{Y}_{rms}^2 - \dot{Y}_{rms}^6))]}{(2\pi)^{3/2}(Y_{rms}^2 \dot{Y}_{rms}^2 \ddot{Y}_{rms}^2 - \dot{Y}_{rms}^6)}. \tag{106}$$

Lin [12] derives an equivalent equation for the joint probability distribution of displacement, velocity and acceleration of a Gaussian process. The joint probability density of $\dot{Y}$ and $\ddot{Y}$ is found by substituting equation (104) with $f_1 = 0$ into equation (79) and integrating [7, art. 3.958]:

$$p_{\dot{Y}\ddot{Y}}(\dot{y}, \ddot{y}) = \frac{\mathrm{e}^{-\dot{y}^2/2\dot{Y}_{rms} - \ddot{y}^2/2\ddot{Y}_{rms}^2}}{2\pi \dot{Y}_{rms} \ddot{Y}_{rms}}. \tag{107}$$

The expected number of peaks per unit time is found by substituting this equation into equation (93). The integral has the simple solution [7, art. 3.958]

$$\mathrm{E}[N_p] = \ddot{Y}_{rms}/(2\pi \dot{Y}_{rms}). \tag{108}$$

Equation (108) agrees with the expression given by Lin [12, p. 302] for the expected number of peaks per unit time for a Gaussian process.

Equations (106) and (107) can be substituted into equation (94) to produce the probability density of peaks. The result is called Rice's distribution and it is reproduced here for completeness (Lin [12, p. 303], attributed to Huston and Skopinski [28], also see Rice [4, art. 3.6]; Broch [25]; Wirsching *et al*. [28]).

$$p_{A_p}(y) = (2\pi)^{-1/2}(1-\alpha^2)^{1/2} Y_{rms}^{-1} \exp[-y^2(2Y_{rms}^2(1-\alpha^2))^{-1}] + \alpha y(2Y_{rms}^2)^{-1}(1 + \mathrm{erf}[(y/Y_{rms})(2\alpha^{-2}-2)^{-1/2}]) \exp[-y^2/(2Y_{rms}^2)], \tag{109}$$

where $\alpha = \dot{Y}_{rms}^2/(Y_{rms}\ddot{Y}_{rms})$. For narrow-band processes, $\alpha = 1$, the first term on the right side of this equation is zero, and since $\mathrm{erf}(0) = 0$, the distribution of peaks reduces to the Rayleigh distribution, equation (92). For broadband series, α approaches zero, so only the first term on the right side of equation (109) survives and equation (109) becomes the normal distribution, equation (46). Thus we have shown that the probability distribution of peaks in a series of independent sine waves approaches classical Gaussian limits as the number of independent sine waves in the series becomes large.

12. CUMULATIVE PROBABILITY

The cumulative probability distribution of the random variable Y is the chance that the random variable has a realization in the range $-\infty < Y < y_*$. It is the integral of the probability density,

$$P_Y(y_*) = \int_{-\infty}^{y_*} p_Y(y)\, \mathrm{d}y. \tag{110}$$

$P_Y(y_*)$ is dimensionless. The well known cumulative probability density of a single sine wave is computed by substituting equation (8) into equation (110) and, integrating,

$$P_{Y-1-\text{sine}}(y_*) = \frac{1}{\pi}\int_{-a_n}^{y_*} \frac{1}{(a_n^2 - y^2)^{1/2}}\,\mathrm{d}y = \frac{1}{2} + \frac{1}{\pi}\arcsin\left(\frac{y_*}{a_n}\right), \qquad |y_*| \leqslant a_n. \tag{111}$$

The cumulative probability density of two sine waves of equal amplitude is obtained by substituting equation (28) into equation (110),

$$P_{Y-2-\text{sine}}(y_*) = (2/\pi^2)\int_{-2a}^{y_*} K(\sqrt{1 - x^2/(4a^2)})\,\mathrm{d}x. \tag{112}$$

The cumulative probability density of the sum of N independent sine waves is obtained by substituting equation (41) into equation (110) and integrating from $-L_Y$ to y_*,

$$P_{Y-N-\text{sine}}(y_*) = \frac{y_* + L_Y}{2L_Y} + \frac{1}{\pi}\sum_{i=1}^{\infty}\left(\frac{1}{i}\right)\left\{\prod_{n=1}^{N} \mathrm{J}_0\left(\frac{i\pi a_n}{L_Y}\right)\right\}\sin\left(\frac{i\pi y_*}{L_Y}\right), \qquad |y_*| \leqslant L_Y. \tag{113}$$

The cumulative probability of a narrow-band envelope and peaks is found by integrating their probability densities from 0 to the amplitude A_*,

$$P_A(A_*) = \int_0^{A_*} p(A)\,\mathrm{d}A, \qquad 0 \leqslant A_* < L_Y. \tag{114}$$

The cumulative probability distribution of the envelope of the sum of N independent sine waves is found by substituting equation (66) into equation (114) and integrating. The terms associated with the indices $i = 0$ and $k = 0$ are separated out:

$$P_A(A_*) = \frac{\pi}{4}\left(\frac{A_*}{L_Y}\right)^2 + \frac{2A_*}{L_Y}\sum_{i=1}^{\infty}\left(\frac{1}{i}\right)\left\{\prod_{n=1}^{N}\mathrm{J}_0\left(\frac{i\pi a_n}{L_Y}\right)\right\}\mathrm{J}_1\left(\frac{i\pi A_*}{L_Y}\right)$$
$$+ \frac{2A_*}{L_Y}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\sqrt{i^2+k^2}}\left\{\prod_{n=1}^{N}\mathrm{J}_0\left(\frac{\pi a_n}{L_Y}\sqrt{i^2+k^2}\right)\right\}\mathrm{J}_1\left(\frac{\pi A_*}{L_Y}\sqrt{i^2+k^2}\right). \tag{115}$$

The cumulative probability of narrow-band process peaks, under the assumption that all have nearly the same frequency, $\omega_n \approx \omega$, is found by substituting equation (86) into equation (114). The result is

$$P_{A_p}(A_*) = 2\pi\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\gamma_{ik}\left\{\prod_{n=1}^{N}\mathrm{J}_0\left(\frac{\pi a_n}{L_Y}\sqrt{i^2+k^2}\right)\right\}\left[1 - \cos\left(\frac{i\pi A_*}{L_Y}\right)\right]. \tag{116}$$

Theory requires $P_Y(-L_Y) = 0$ in equations (111)–(113) and $P_A(0) = 0$ in equations (114)–(116). These are confirmed by inspection. Theory requires $P_Y(L_Y) = 1$ and this is confirmed by inspection of equations (111) and (113) and by using an identity for equation (112) [7, art. 6.141]. While I have not been able to find an identity in the literature that proves $P_A(L_Y) = 1$ for equation (115), numerical evaluations confirm that its right side approaches unity at $A_* = L_Y$ regardless of the choice of a_n, provided that $a_n > 0$.

TABLE 1

Equation numbers of principal results

Quantity	Number of terms in series 1	2	3	N	∞
Probability density of Y	8	26, 28	32–35	39–42, 45	46
Probability density of envelope	48	54, 55	61, 62, 63	64–68	69
Probability density of narrow-band peaks	48	73		83, 86, 90	92, 109
Probability density of broadband peaks	48	75		101	109
Expected number of peaks per unit time	$\omega/2\pi$			82, 99, 108	109
Cumulative probability density of Y	111	112		113	
Cumulative probability density of A				115, 116	
Joint probability density of Y and $\dot{Y}$	14				
Joint probability density of $\dot{Y}$ and $\ddot{Y}$				98	107
Joint probability density of Y, $\dot{Y}$ and $\ddot{Y}$	19			100	106

Similarly, the expected number of peaks per unit time must approach $\omega/(2\pi)$ for series consisting of equal frequency components ($\omega_n = \omega, n = 1, 2, \ldots, N$) and from equation (82) or (99) this implies an identity,

$$1 = 2\pi \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \gamma_{ik} \left\{ \prod_{n=1}^{N} \mathrm{J}_0 \left(\frac{\pi a_n}{L_Y} \sqrt{i^2 + k^2} \right) \right\}, \tag{117}$$

which is equivalent to setting $P(L_Y) = 1$ in equation (116). Equation (117) apparently does not appear in the literature but it is confirmed by numerical evaluation. The cumulative probability distribution of peaks for broadband process is obtained by substituting equation (101) into equation (114). The result is similar to the right side of equation (101) but with $\cos(i\pi y/L_Y)$ replaced by $(L_Y/(i\pi)) \sin(i\pi A_*/L_Y)$ and $\sin(i\pi y/L_Y)$ replaced by $(L_Y/(i\pi))((-1)^{|i|} - \cos(i\pi A_*/L_Y))$.

13. DISCUSSION

1. The probability densities derived in sections 2–11 allow multiple terms in the series to possess the same frequency provided that the terms are statistically independent from each other and from the remaining terms. The probability density of a sine wave (equation (8)) is independent of its frequency. For example, the probability density of $\sin(2\pi t/T)$ is identical to the probability density of $\sin(10\pi t/T)$ for times uniformly distributed over the range $0 \leqslant t < T$.

2. The probability densities derived in the paper require each mode to have zero mean value when averaged over the ensemble. It is possible to extend the results to processes with a mean level by defining an equivalent zero mean process: $Y = Y_{actual} - \overline{Y}$, where the overbar denotes the mean level, and transforming the process to the equivalent zero mean process.

3. For a given N, the series that has the maximum peak-to-r.m.s. ratio is found by taking the derivative of equation (5a) with respect to an amplitude of a single term while holding the remaining amplitudes fixed. Solving the resulting equation shows that the

amplitude of an individual term that maximizes the peak-to-r.m.s. ratio of the series is equal to the sum of the squares of the remaining amplitudes divided by their sum,

$$a_m \big|_{max\text{-}peak\text{-}to\text{-}rms} = \sum_{n=1, n \neq m}^{N} a_n^2 \Bigg/ \sum_{n=1, n \neq m}^{N} a_n. \tag{118}$$

Applying this to each term implies that maximum peak-to-r.m.s. ratio series have equal amplitude terms, $a_1 = a_2 = \cdots = a_N = a$. That is, the peak-to-r.m.s. ratio of the series is maximized if all the amplitudes are equal. The Fourier series for the Dirac delta function has equal amplitude terms as does the finite Fourier series representation of band-limited white noise.

4. The principal results of this paper are indexed in Table 1. The probability density of a single sine wave and series with infinite numbers of independent terms are well established in the literature. The contribution of this paper is to provide the probability density of the sum of two and three sine waves, joint probability density of the series and its derivatives, and the probability densities of their narrow-band and broadband process peaks for an arbitrary number of independent sine waves. The envelope solutions (section 7) with the exception of the $N = 2$ and $N = 3$ solutions, equations (54) and (61)–(63), were adapted from the physical literature. The envelope power series expression, equation (68), is similar to a series given by Pearson [14], equations (39) and (41) are due to Barakat [10], and equation (109) is by Rice [4].

5. It is difficult to obtain high accuracy when numerically integrating equations (39), (40), (64) and (65) for probability density, especially *for* $N \leqslant 3$ and near the tails of the distribution, because of their oscillatory integrands that result in singularities in the probability density (see Figures 2, 4, 6 and 8 and Greenwood and Durand [19], Barakat [10]; Shmulei and Weiss [29]). The power series expansions converge slowly near the tail of the distributions and for small N [14, p. 9]. Fourier series solutions also converge slowly for small N. The closed form solutions for $N = 1, 2, 3$ (Table 1) could be used in preference to the more general series expansions for small N.

6. The principal assumption of this analysis is that the terms in the series are mutually independent. While this may seem to be restrictive, this same assumption is also made in the derivation of the normal and Rayleigh distributions and these distributions apply only in the limit that the number of terms become infinite [5, 20].

14. CONCLUSIONS

Analysis has been made to determine the probability density of a random process consisting of the sum of a series of sine waves with deterministic amplitudes and independent random phase angles. The joint probability density of the series and its first two derivatives is determined. The probability density of the series, its broadband and narrow-band peaks, envelope and cumulative distribution have been found for a series consisting of a finite number of statistically independent sine waves using convolution, characteristic functions and series expansions. The principal results are indexed in Table 1. The following conclusions are reached.

1. The peak-to-r.m.s. ratio of a series increases with the number of terms in the series. If all terms have the same peak and r.m.s. values, then the peak-to-r.m.s. ratio of the series sum increases with the square root of the number of terms in the series.

2. The probability of the series sum is zero beyond a maximum value, equal to the sum of the series amplitudes. Hence, the probability densities of the finite series, their peaks, and their envelopes are non-Gaussian.

3. The formulas allow the direct calculation of the probability density of the series, its envelope and its peaks from its power spectral density (PSD) under the assumption that each spectral component is statistically independent of the others.

4. It is shown that the probability density of the series, its peaks and its envelope approach classical Gaussian process results as the numbers of terms in the series become large.

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APPENDIX A: FOURIER SERIES EXPANSION OF PROBABILITY DENSITY

The probability density of a finite sum of finite terms, equation (1), is non-zero only in a limited range. As a result, it is well suited to expansion in a finite Fourier series over that range. Consider the expansion of the probability density of the sum of N sine waves, equation (2), in a Fourier series over the range $-L_Y$ to L_Y,

$$p_Y(y) = \sum_{i=0}^{\infty} b_i \cos(i\pi y/L_Y), \qquad -L_Y \leqslant y \leqslant L_Y. \tag{A1}$$

Only cosine terms are included because Y is symmetric about $Y = 0$, $p_Y(y) = p_Y(-y)$. The series coefficients are found using standard Fourier methods [30]:

$$b_0 = 1/(2L_Y), \qquad b_i = (1/L_Y)\int_{-L_Y}^{L_Y} p_Y(y)\cos(i\pi y/L_Y)\,\mathrm{d}y, \qquad i = 1, 2, 3.. \tag{A2}$$

Barakat [10] noted that since $p_Y(|y| > L_Y) = 0$, the equation for the characteristic function equation (9),

$$C(f) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}2\pi f x} p_Y(x)\,\mathrm{d}x = \int_{-L_Y}^{L_Y} \mathrm{e}^{\mathrm{j}2\pi f x} p_Y(x)\,\mathrm{d}x = \int_{-L_Y}^{L_Y} \cos(2\pi f x) p_Y(x)\,\mathrm{d}x, \tag{A3}$$

can be placed in the form of equation (A2) and the Fourier coefficients b_i can be expressed in terms of the characteristic functions,

$$b_i = (1/L_Y)C(f = i/2L_Y), \qquad i = 1, 2, 3, \ldots . \tag{A4}$$

Incorporating equations (10) and (A4) in equation (A1) gives a Fourier series expansion for the probability density of the sum of N mutually independent sine waves.

The joint probability density of Y and $\dot{Y}$ is expanded in a double Fourier series over the region $-L_y \leqslant Y \leqslant L_Y$, $-L_{\dot{Y}} \leqslant \dot{Y} \leqslant L_{\dot{Y}}$:

$$p_{Y\dot{Y}}(y, \dot{y}) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{ik} \cos(i\pi y/L_Y)\cos(k\pi \dot{y}/L_{\dot{Y}}), \tag{A5}$$

where $L_Y = a_1 + a_2 + \cdots + a_N$ and $L_{\dot{Y}} = \omega_1 a_1 + \omega_2 a_2 + \cdots + \omega_N a_N$. Only cosine terms are included because the joint probability density is symmetric about the origin, $p_{Y\dot{Y}}(-y, \dot{y}) = p_{Y\dot{Y}}(y, -\dot{y})$. The Fourier coefficients b_{ik} are found by standard techniques [30]:

$$b_{ik} = \frac{\alpha_{ik}}{L_Y L_{\dot{Y}}} \int_{-L_Y}^{L_{\dot{Y}}} \int_{-L_Y}^{L_Y} p_{Y\dot{Y}}(y, \dot{y}) \cos(i\pi y/L_Y) \cos(k\pi\dot{y}/L_{\dot{Y}})\, \mathrm{d}y\, \mathrm{d}\dot{y}, \tag{A6}$$

$$\alpha_{ik} = 1, \quad i, k > 0;\ 1/2, \quad i = 0 \qquad \text{or} \qquad k = 0;\ 1/4, \quad i = k = 0. \tag{A7}$$

The equation for the joint characteristic function of $p_{Y\dot{Y}}(y, \dot{y})$,

$$C(f_1, f_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{Y\dot{Y}}(y, \dot{y}) \cos(2\pi f_1 y) \cos(2\pi f_2 \dot{y})\, \mathrm{d}f_1\, \mathrm{d}f_2 \tag{A8}$$

is very similar to the equation for b_{ik} (equation (A6)). Since $p_{Y\dot{Y}}(y, \dot{y})$ is zero beyond $|y| > L_y$ or $|\dot{y}| > L_{\dot{Y}}$, the integration limits in equation (A6) can be replaced with finite limits. With the substitutions $f_1 = i/(2L_Y)$ and $f_2 = k/(2L_{\dot{Y}})$, the Fourier coefficients can be determined from the characteristic functions:

$$b_{ik} = (\alpha_{ik}/(L_Y L_{\dot{Y}}))C(2f_1 = i/L_Y, 2f_2 = k/L_{\dot{Y}}). \tag{A9}$$

Since each term in the series is a function of an independent random variable, the joint probability density of the sums $Y = X_1 + X_2 + \cdots + X_N$, $\dot{Y} = \dot{X}_1 + \dot{X}_2 + \cdots + \dot{X}_N$ is equal to the product of the joint probability density $p_{X_n \dot{X}_n}(x, \dot{x})$ of each term. This implies that the joint characteristic function of the sum is the product of the characteristic function of the terms

$$C(f_1, f_2) = \prod_{n=1}^{N} C_n(f_1, f_2), \tag{A10}$$

where $C_n(f_1, f_2)$ is given by equation (17). See also equation (37). Equation (17) is substituted into the right side of equation (A10) and this is substituted into equation (A9), which is substituted into equation (A5) to give the Fourier series solution for the joint probability density of Y and $\dot{Y}$:

$$p_{Y\dot{Y}}(y, \dot{y}) = \frac{1}{L_Y L_{\dot{Y}}} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \alpha_{ik} \left\{ \prod_{n=1}^{N} \mathrm{J}_0 \left(\pi a_n \sqrt{\left(\frac{i}{L_Y}\right)^2 + \left(\frac{k\omega_n}{L_{\dot{Y}}}\right)^2} \right) \right\}$$

$$\times \cos(i\pi y/L_Y) \cos(k\pi\dot{y}/L_{\dot{Y}}). \tag{A11}$$

The joint probability density of Y, $\dot{Y}$ and $\ddot{Y}$ is expanded in a triple complex Fourier series:

$$p_{Y\dot{Y}\ddot{Y}}(y, \dot{y}, \ddot{y}) = \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} b_{ikm}\, \mathrm{e}^{-\mathrm{j}\pi(iy/L_Y + k\dot{y}/L_{\dot{Y}} + m\ddot{y}/L_{\ddot{Y}})}. \tag{A12}$$

Note that the sums are over the double infinite range for the complex series [30, p. 177]. Multiplying both sides of this equation by $\exp[\mathrm{j}\pi(iy/L_Y + k\dot{y}/L_{\dot{Y}} + m\ddot{y}/L_{\ddot{Y}})]$ and integrating over the domain of non-zero probability, $-L_Y \leqslant y \leqslant L_Y$, $L_{\dot{Y}} \leqslant \dot{y} \leqslant L_{\dot{Y}}$, $L_{\ddot{Y}} \leqslant \ddot{y} \leqslant L_{\ddot{Y}}$, gives a triple integral expression for the Fourier coefficient:

$$b_{ikm} = \int_{-L_Y}^{L_Y} \int_{-L_{\dot{Y}}}^{L_{\dot{Y}}} \int_{-L_{\ddot{Y}}}^{L_{\ddot{Y}}} p_{Y\dot{Y}\ddot{Y}}(y, \dot{y}, \ddot{y}) \, e^{j\pi(iy/L_Y + k\dot{y}/L_{\dot{Y}} + m\ddot{y}/L_{\ddot{Y}})} \, dy \, d\dot{y} \, d\ddot{y}. \tag{A13}$$

Comparing this to the equation for the three-way joint characteristic function (equation (26)), we see that the two equations are very similar and that the coefficients may be determined from the characteristic functions:

$$b_{ikm} = \frac{1}{8 L_Y L_{\dot{Y}} L_{\ddot{Y}}} C\left(f_1 = \frac{i}{2L_Y}, \quad f_2 = \frac{k}{2L_{\dot{Y}}}, \quad f_3 = \frac{m}{2L_{\ddot{Y}}}\right). \tag{A14}$$

The characteristic functions of the sum of independent sine waves are the product of the joint characteristic function of a single sine wave (equation (21)). See equations (36) and (A10). Since $p_{Y\dot{Y}\ddot{Y}}(y, \dot{y}, \ddot{y})$ is real, only the real part of equation (A12) is non-zero. Furthermore, positive and negative values of $\dot{Y}$ are equally likely: $p_{Y\dot{Y}\ddot{Y}}(y, -\dot{y}, \ddot{y}) = p_{Y\dot{Y}\ddot{Y}}(y, y, \ddot{y})$. Hence,

$$p_{Y\dot{Y}\ddot{Y}}(y, \dot{y}, \ddot{y}) = \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} b_{ikm} \cos(\pi k \dot{y}/L_{\dot{Y}}) \cos(\pi i y/L_Y + \pi m \ddot{y}/L_{\ddot{Y}}). \tag{A15}$$

This equation applies over the range $-L_Y \leqslant y \leqslant L_Y$, $-L_{\dot{Y}} \leqslant \dot{y} \leqslant L_{\dot{Y}}$, $-L_{\ddot{Y}} \leqslant \ddot{y} \leqslant L_{\ddot{Y}}$. The probability is zero outside of this range.

APPENDIX B: SERIES EXPANSION OF PRODUCTS OF BESSEL FUNCTIONS

The properties of the asymptotic series obtained by expanding an exponential function were developed by Cramer [5, pp. 85–86] and used by Rice [4, article 1.6]. Here it is applied to expansions of the Bessel function of order zero.

Consider power series expansions for the Bessel function of order zero and the exponential function about zero and expansion of the natural logarithm about unity (Gradshteyn *et al*. [7, arts. 1.211, 1.511 and 8.441]):

$$J_0(x) - 1 = \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} = -\frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots, \tag{B1}$$

$$\ln(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(z-1)^k}{k} = (z-1) - \tfrac{1}{2}(z-1)^2 + \tfrac{1}{3}(z-1)^3 + \cdots, \tag{B2}$$

$$e^y = \exp(y) = \sum_{k=0}^{\infty} \frac{y^k}{k!} = 1 + y + \tfrac{1}{2}y^2 + \tfrac{1}{6}y^3 + \cdots. \tag{B3}$$

Setting $z = J_0(x)$ in equation (B1) and substituting into equation (B2), a series for the logarithm of the Bessel function of order zero is found:

$$\ln[J_0(x)] = \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} - \tfrac{1}{2}\left(\sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}\right)^2 + \cdots = -\frac{x^2}{4} - \frac{x^4}{64} - \frac{x^6}{576} + \cdots. \tag{B4}$$

There is an identity $y = \exp[\ln(y)]$; thus

$$J_0(x) = e^{\ln[J_0(x)]}. \tag{B5}$$

Now substituting equation (B4) into the right side of this expression and using equation (B3), an expansion for the Bessel function is found:

$$J_0(x) = \exp\left[-\frac{x^2}{4} - \frac{x^4}{64} - \frac{x^6}{576} + \cdots\right] = e^{-x^2/4}\left(1 - \frac{x^4}{64} - \frac{x^6}{576} - \frac{5x^8}{49152} + \cdots\right). \tag{B6}$$

This expansion has a useful form because the initial term in the exponential can be separated out. For example, the Bessel function of order zero raised to the power N is expanded as

$$[J_0(x)]^N = e^{N\ln[J_0(x)]} = e^{-Nx^2/4}\left(1 - \frac{Nx^4}{64} - \frac{Nx^6}{576} + \frac{1}{8}\left(\frac{N^2}{1024} - \frac{11N}{6144}\right)x^8 + \cdots\right). \tag{B7}$$

Equation (B7) is used in expansions of probability density for large values of N.

APPENDIX C: PROBABILITY DENSITY FOR PHASE DIFFERENCES

Consider two independent random phases ϕ_1 and ϕ_2 which are uniformly distributed over the range 0 to 2π and zero outside of this range:

$$p_{\phi_1}(\phi_1) = \begin{cases} 1/(2\pi), & 0 \leqslant \phi_1 < 2\pi, \\ 0, & \text{otherwise}, \end{cases} \tag{C1}$$

$$p_{\phi_2}(\phi_2) = \begin{cases} 1/(2\pi), & 0 \leqslant \phi_2 < 2\pi, \\ 0, & \text{otherwise}. \end{cases} \tag{C2}$$

Their difference is given by $\Delta\phi = \phi_1 - \phi_2$ and the probability of their difference is given by the convolution integral, equation (23):

$$p_{\Delta\phi}(\Delta\phi) = \int_{-\infty}^{\infty} p_{\phi_2}(\phi_1) p_{\phi_2}(\phi_1 - \Delta\phi)\,\mathrm{d}\phi_1. \tag{C3}$$

Substituting equations (C1) and (C2) into equation (C3) and integrating, taking into account the limits (see section 4 or Sveshnikov [2, p. 134] for a similar analysis), shows that the probability density of the difference between two uniformly distributed random variables,

$$p_{\Delta\phi}(\Delta\phi) = \begin{cases} (1/(2\pi)^2)(2\pi + \Delta\phi), & -2\pi \leqslant \Delta\phi < 0, \\ (1/(2\pi)^2)(2\pi - \Delta\phi), & 0 \leqslant \Delta\phi < 2\pi, \\ 0, & |\Delta\phi| > 2\pi, \end{cases} \tag{C4}$$

has a triangular distribution which is non-zero over the range between -2π and 2π and symmetric about $\Delta\phi = 0$.

Consider a sinusoidal function of the difference in phases,

$$Y = a\cos\Delta\phi, \qquad -2\pi \leqslant \phi \leqslant 2\pi. \tag{C5}$$

For a given value of Y which is less than a, there are four solutions for $\Delta\phi$, two which we call ϕ_1 and ϕ_2 in the range between -2π and 0 and two which we call ϕ_3 and ϕ_4 between 0 and 2π. The probability of Y having a given value within the range y and $y + \mathrm{d}y$ is equal to the sum of the probabilities of $\Delta\phi$ achieving a value between $\Delta\phi_i$ and $\Delta\phi + \mathrm{d}\Delta\phi_i$ about each of the four solutions [2, p. 115]. Thus the probability density of Y is

$$p_Y(y) = \sum_{i=1}^{4} p_{\Delta\phi}(\Delta\phi_i)|(\mathrm{d}\Delta\phi_i/\mathrm{d}Y)|. \tag{C6}$$

The derivative of the sine wave can be expressed interims of its value (equation (11)) $\mathrm{d}Y/\mathrm{d}\Delta\phi = \pm\sqrt{a^2 - Y^2}$. Substituting this equation and equations (C1) and (C2) into equation (C6) gives the probability density of Y,

$$p_Y(y) = \frac{1}{(2\pi)^2(a^2 - Y^2)^{1/2}}(8\pi + \Delta\phi_1 + \Delta\phi_2 - \Delta\phi_3 - \Delta\phi_4). \tag{C7}$$

Cyclic symmetry of the sine wave dictates $\cos(\Delta\phi) = \cos(\Delta\phi + 2\pi)$, thus the solutions $\Delta\phi_1$ and $\Delta\phi_2$ which fall in the range between -2π and 0 are related to the solutions between 0 and 2π by $\Delta\phi_3 = \Delta\phi_1 + 2\pi$, and $\Delta\phi_4 = \Delta\phi_2 + 2\pi$. Substituting these into equation (C6), we see that as a result the probability density of the sine of the difference in two uniformly distributed phases,

$$p_Y(y) = \frac{1}{\pi(a^2 - Y^2)^{1/2}}, \tag{C8}$$

is identical to the probability density of the sine of a single uniformly distributed phase, equation (8).

APPENDIX D: RANDOM WALKS

In 1905, Professor Karl Pearson [31] published a letter in *Nature*:

Can any of your readers refer me to a work wherein I should find a solution of the following problem, . . . A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r + \delta r$ from his starting point, O.

Consider a two-step random walk in the x–y plane, starting from the origin. The first step takes us to the co-ordinates x_1 and y_1 and the second step adds the values x_2 and y_2 to these co-ordinates. It is easily seen that the radius r from the origin to the end of the second step is

$$r^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2 = (l_1 \cos\theta_1 + l_2 \cos\theta_2)^2 + (l_1 \sin\theta_1 + l_2 \sin\theta_2)^2. \tag{D1}$$

Extending this to n steps, we see that if the random angles θ_i are identified with the random phase angles ϕ_i and the step lengths l_i are identified with the amplitudes a_i, then neglecting the phase offsets produced by the time terms, equation (D1) is identical to equation (60), as noted by Rayleigh [13, 16]. Pearson [14, 31] and Kluyver [15] developed solutions to the two-dimensional random walk. Reviews are given by Weiss [9], Barber and Ninham [18] and Watson [17]. Pearson's notation is used in this Appendix.

Pearson [14, p. 6] showed that the probability density, $\Phi_{n+1}(r^2)$, that r^2 is in a small area in the r–θ plane at a distance r from his starting point, O, on the $(n+1)$th step is related to the corresponding probability for the nth step by

$$\Phi_{n+1}(r^2) = \frac{1}{\pi}\int_0^{\pi} \Phi_n\,(r^2 + l^2 - 2rl\cos\theta)\,\mathrm{d}\theta. \tag{D2}$$

By integrating to obtain the cumulative distribution and then differentiating with respect to r, Rayleigh [16, p. 614] showed that the probability density of r is related to Pearson's $\Phi(r^2)$ by

$$p_r(r) = 2\pi r\Phi(r^2). \tag{D3}$$

Equations (D2) and (D3) allow computation of probability density of the envelope of the sum of three sine waves. Consider that the first two random steps are made with equal step lengths of l. The probability density after two steps is adapted from equation (55) with $a = l$ and $A = r$. The third step is of length l_3. (This is slightly more general than the equal length steps considered by Pearson [14].) Equations (D2) and (D3) imply that the probability density of the radius r from the origin to the end of the third step is

$$\Phi_3(r^2) = \frac{1}{\pi^3}\int_0^{\pi} (r^2 + l_3^2 - 2rl_3\cos\theta)^{-1/2}((2l)^2 - (r^2 + l_3^2 - 2rl_3\cos\theta))^{-1/2}\,\mathrm{d}\theta. \tag{D4}$$

The integrand is set to zero for $(2l)^2 < (r^2 + l_3^2 - 2rl_3\cos\theta)$ as this condition has zero probability after the first two steps. With the substitution $x = -2rl_3\cos\theta$, this equation becomes

$$\Phi_3(r^2) = \frac{1}{\pi^3}\int_{-2rl_3}^{U} [(4l^2 - r^2 - l_3^2 - x)\,(2rl_3 - x)\,(x + 2rl_3)\,(x + r^2 + l_3^2)]^{-1/2}\,\mathrm{d}x. \tag{D5}$$

The upper limit of integration U is the smaller of $2rl_3$, zero and $4l^2 - r^2 - l_3^2$. The integrand contains singularities at $x = 4l^2 - r^2 - l_3^2$, $2rl_3$, $-2rl_3$ and $-r^2 - l_3^2$, and their sequence depends on r and whether $2l > l_3$ or $2l < l_3$. There is an exact solution [7].

APPENDIX E: NOMENCLATURE

A	amplitude, peak, or envelope
A_e	envelope amplitude, $A_e \geqslant 0$
A_p	peak amplitude
A_t	trough amplitude
A_{pt}	amplitude of peaks and troughs
a_n	amplitude of the nth sine wave, $a_n \geqslant 0$
b_i	Fourier coefficient, Appendix A
$C(f)$	characteristic function with parameter f
$C(f_1, f_2)$	joint characteristic function with parameters f_1 and f_2
$E[N_p]$	expected number of positive peaks per unit time
$E[N_{0+}]$	expected number of zero crossing with positive slope per unit time
$\exp(x)$	e^x = exponential function
erf	error function
${}_1F_1$	confluent hypergeometric function [7]
f	parameter in Fourier transform
i	integer index

j	$\sqrt{-1}$ = imaginary constant
J_0	Bessel function of first kind and zero order
J_1	Bessel function of first kind and first order
k	integer index
K	complete elliptic integral of first kind [8], equation (27a)
L_Y	$a_1 + a_2 + \cdots + a_N$, sum of amplitudes
$L_{\dot{Y}}$	$\omega_1 a_1 + \omega_2 a_2 + \cdots + \omega_N a_N$, sum of velocity amplitudes
$L_{\ddot{Y}}$	$\omega_1^2 a_1 + \omega_2^2 a_2 + \cdots + \omega_N^2 a_N$, sum of acceleration amplitudes
m	integer index
N	number of terms in series
n	integer index, $n = 1, 2, \ldots, N$
$P_Y(y)$	cumulative probability, the integral of $p_Y(x)$ from $x = -\infty$ to y, equation (110)
$p_Y(x)$	probability density of random parameter Y evaluated at $Y = x$
$p_{XY}(x, y)$	joint probability density of X and Y evaluated at $Y = y$ and $X = x$
r_m	dimensionless coefficient, equation (102)
s_m	dimensionless coefficient, equation (102)
t	time, $0 \leqslant t < T$
T	length of time interval
X	a random variable
Y	sum of N modes or terms, $-L_Y \leqslant Y \leqslant L_Y$
$\dot{Y}$	first derivative with respect to time of Y, $-L_{\dot{Y}} \leqslant \dot{Y} \leqslant L_{\dot{Y}}$
$\ddot{Y}$	second derivative with respect to time of Y, $-L_{\ddot{Y}} \leqslant \ddot{Y} \leqslant L_{\ddot{Y}}$
α_{ij}	dimensionless coefficient, equation (A7)
Γ	gamma function [7], $\Gamma[(2n+1)/2] = \pi^{1/2}2^{-n}(2n-1)!!$
γ_{ij}	dimensionless coefficient, equation (84)
δ	Dirac's delta function
π	$3{\cdot}1415926\ldots$
$\Pi_{n=1}^{N} x_n$	$x_1 x_2 \ldots x_N$, product of terms
ϕ_n	phase angle of the nth sine wave, a uniformly distributed independent random variable
ω	circular frequency, a positive (non-zero) real number
ω_n	circular frequency of the nth term, a non-zero integer multiple of $2\pi/T$