Journal of Sound and Vibration (1997) **208**(4), 656–663

OPTIMAL TWO-PLANE BALANCE OF RIGID ROTORS

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(*Received* 23 *December* 1996, *and in final form* 6 *June* 1997)

1. INFLUENCE COEFFICIENT MODEL

A diagram of a rigid rotor is shown in Figure 1. All unbalance can be modelled as if it occurs in two distinct planes, called the balance planes. The vibration readings are taken in two distinct planes, called the measurement planes. We arbitrarily scribe two marks, one on each balance plane, to represent zero angle on that plane.

The objective is to spin the rotor multiple times at a constant speed (the speed at which we want to balance) and measure vibration magnitudes each time. The first spin, number 0, has no trial weights attached. For all other spins, we locate a single trial mass of constant known magnitude times radius and known angle on one of the balance planes. In subsequent spins, the previous trial weight must be removed. The only restriction on the trial weight angular placement is that for each plane they are uniformly spaced. For example, if we plan to locate three trial weights on balance plane 1 and four on balance plane 2, the weights on plane 1 must be 120 degrees apart and those on 2 must be 90 degrees apart.

An influence coefficient model for a rigid rotor relates measured vibration to unbalance and trial mass magnitude as

$$\vec{\vec{N}}(\vec{u} + \vec{w}_i) = \vec{V}_i, \tag{1}$$

where $\vec{\vec{N}}$ is a matrix of complex numbers (phasors) that depend on the system characteristics and running speed:

$$\vec{\vec{N}} = \begin{bmatrix} \tilde{n}_{1,1} & \tilde{n}_{1,2} \\ \tilde{n}_{2,1} & \tilde{n}_{2,2} \end{bmatrix}.$$

The individual terms ($\tilde{n}_{1,1}$ through $\tilde{n}_{2,2}$) are complex numbers that characterize the rotor and support system. If the balance and measurement planes are chosen properly the matrix $\vec{\vec{N}}$ will be non-singular, which is assumed. The vector $\vec{u}$ is

$$\vec{u} = \begin{bmatrix} \tilde{u}^{(1)} \\ \tilde{u}^{(2)} \end{bmatrix},$$

where $\tilde{u}^{(1)}$ and $\tilde{u}^{(2)}$ are complex numbers representing the magnitude and angle measured from the arbitrary zero angles on their respective balance planes. During the derivation, complex numbers may be written as a magnitude and a unit direction, such as $\tilde{u}^{(2)} = |\tilde{u}^{(2)}|\tilde{\lambda}_u^{(2)}$.

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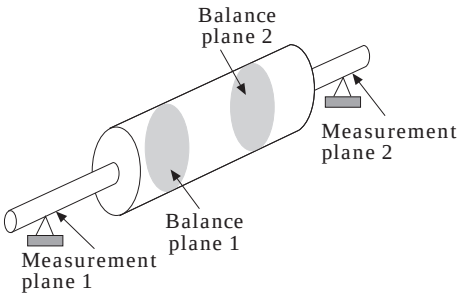

Figure 1. A schematic diagram of a rigid rotor.

2. THE OBJECTIVE FUNCTION

Since measurements will have errors in them, one should find an optimal estimate of the unbalance $\vec{u}$. To do this, we define the following objective function:

$$F = \sum_{i=0}^{m} \{[\vec{\vec{N}}(\vec{u} + \vec{w}_i) - \vec{V}_i]^*[\vec{\vec{N}}(\vec{u} + \vec{w}_i) - \vec{V}_i]\},$$

where $(\cdot)^*$ represents a complex conjugate transpose and m is the number of trial masses installed and spun. Now we determine (1) the angles (phase) of the vibration measurements, (2) the magnitudes and angles of the components of $\vec{\vec{N}}$, and (3) the magnitudes and angles of $\vec{u}$, that minimize the objective function, F. To do this, we take the partial derivatives with respect to the unknowns and set them to zero. Now, the problem with this direct approach is that there will be $m + 14$ unknown variables and that the equations have multiple solutions. In other words, F has a large number of local extrema; hence it is very difficult to determine the correct minimum point.

It is possible to determine the best estimate of $\vec{u}$ given all other variables. To find the best $\vec{u}$, we compute the partial of F with respect to $\vec{u}$ and set it to zero, as

$$\partial F/\partial \vec{u} = \sum_{i=0}^{m} \{[\vec{\vec{N}}(\vec{u} + \vec{w}_i) - \vec{V}_i]^*\vec{\vec{N}}\} = 0.$$

Solving for $\vec{\vec{N}}$ gives

$$\vec{\vec{N}}\vec{u} = \frac{1}{m+1}\left[\sum_{i=0}^{m} \vec{V}_i - \vec{\vec{N}}\left(\sum_{i=0}^{m} \vec{w}_i\right)\right].$$

Now, since the trial weights are equal magnitude and uniformly located on their respective balance planes, $\Sigma_{i=0}^{m}\, \vec{w}_i = 0$, and

$$\vec{\vec{N}}\vec{u} = \frac{1}{m+1}\sum_{i=0}^{m} \vec{V}_i = \vec{V}_a, \tag{2}$$

which defines the average vibration $\vec{V}_a$. In other words, the best estimate of $\vec{u}$ is the average vibration pre-multiplied by the inverse of $\vec{\vec{N}}$. The only problem that remains is to find best estimates of $\vec{\vec{N}}$ and the angles of $\vec{V}_i$ (the unit vector $\tilde{\lambda}_{Vi}$) which allows the vibration to be averaged and hence $\vec{V}_a$ computed. In the next section we discuss how this is done.

3. THE OPTIMAL $\vec{\vec{N}}$ AND $\tilde{\lambda}_{Vi}$

To begin, we use the best estimate of $\vec{\vec{N}}\vec{u}$ to eliminate it from the objective function. This gives

$$F = \sum_{i=0}^{m} \{[\vec{V}_a + \vec{\vec{N}}\vec{w}_i - \vec{V}_i]^*]\vec{V}_a + \vec{\vec{N}}\vec{w}_i - \vec{V}_i]\} = \sum_{i=0}^{m} (\tilde{f}_i^* \tilde{f}_i),$$

which defines $\vec{f}_i$. Since F is a sum of magnitudes, none of which can be negative, the minimum F occurs when all terms are minimum simultaneously. This is the strategy used in this paper. We consider each $\vec{f}_i$ individually.

We order the m spins into two sets. The first set contains m_1 members (at least three) of trial weights placed on plane 1. The second set contains $m - m_1$ members (at least three) of weights placed on plane 2. We consider the terms in the objective function which belong to set 1. They are

$$\tilde{f}_i = \begin{bmatrix} \tilde{V}_a^{(1)} \big| \tilde{\lambda}_{Va}^{(1)} + \vec{w}_i^{(1)} \tilde{n}_{1,1} - \tilde{V}_i^{(1)} \\ \tilde{V}_a^{(2)} \big| \tilde{\lambda}_{Va}^{(2)} + \vec{w}_i^{(1)} \tilde{n}_{2,1} - \tilde{V}_i^{(2)} \end{bmatrix}, \qquad \text{for } i = 1, \ldots, m_1.$$

The terms in set 2 are

$$\tilde{f}_k = \begin{bmatrix} \tilde{V}_a^{(1)} \big| \tilde{\lambda}_{Va}^{(1)} + \tilde{w}_k^{(2)} \tilde{n}_{1,2} - \tilde{V}_k^{(1)} \\ \tilde{V}_a^{(2)} \big| \tilde{\lambda}_{Va}^{(2)} + \tilde{w}_k^{(2)} \tilde{n}_{2,2} - \tilde{V}_k^{(2)} \end{bmatrix}, \qquad \text{for } k = m_1, \ldots, m.$$

The terms in which there is no trial weight are

$$\tilde{f}_0 = \begin{bmatrix} \tilde{V}_a^{(1)} \big| \tilde{\lambda}_{Va}^{(1)} - \tilde{V}_0^{(1)} \\ \tilde{V}_a^{(2)} \big| \tilde{\lambda}_{Va}^{(2)} - \tilde{V}_0^{(2)} \end{bmatrix}.$$

To find the minimum values of each term, we scale and rotate each one, as shown next:

$$\tilde{f}_i'^{(1)} = \frac{|\tilde{V}_a^{(1)}|}{\tilde{w}_i^{(1)}} + \frac{\tilde{n}_{1,1}}{\tilde{\lambda}_{Va}^{(1)}} - \frac{\tilde{V}_i^{(1)}}{\tilde{w}_i^{(1)} \tilde{\lambda}_{Va}^{(1)}}, \qquad \text{for } i = 1, \ldots, m_1; \tag{3}$$

$$\tilde{f}_i'^{(2)} = \frac{|\tilde{V}_a^{(2)}|}{\tilde{w}_i^{(1)}} + \frac{\tilde{n}_{2,1}}{\tilde{\lambda}_{Va}^{(2)}} - \frac{\tilde{V}_i^{(2)}}{\tilde{w}_i^{(1)} \tilde{\lambda}_{Va}^{(2)}}, \qquad \text{for } i = 1, \ldots, m_1; \tag{4}$$

$$\tilde{f}_i'^{(1)} = \frac{|\tilde{V}_a^{(1)}|}{\tilde{w}_k^{(2)}} + \frac{\tilde{n}_{1,2}}{\tilde{\lambda}_{Va}^{(1)}} - \frac{\tilde{V}_k^{(1)}}{\tilde{w}_k^{(2)} \tilde{\lambda}_{Va}^{(1)}}, \qquad \text{for } k = m_1 + 1, \ldots, m; \tag{5}$$

$$\tilde{f}_i'^{(2)} = \frac{|\tilde{V}_a^{(2)}|}{\tilde{w}_k^{(2)}} + \frac{\tilde{n}_{2,2}}{\tilde{\lambda}_{Va}^{(2)}} - \frac{\tilde{V}_k^{(2)}}{\tilde{w}_k^{(2)} \tilde{\lambda}_{Va}^{(2)}}, \qquad \text{for } k = m_1 + 1, \ldots, m; \tag{6}$$

$$\tilde{f}_0'^{(1)} = |\tilde{V}_a^{(1)}| - \frac{\tilde{V}_k^{(1)}}{\tilde{\lambda}_{Va}^{(1)}}, \qquad \tilde{f}_0'^{(2)} = |\tilde{V}_a^{(2)}| - \frac{\tilde{V}_k^{(2)}}{\tilde{\lambda}_{Va}^{(2)}}. \tag{7, 8}$$

Now, for the moment, we assume that $|\tilde{V}_a^{(1)}|$ and $|\tilde{V}_a^{(2)}|$ are known.

4. GRAPHICAL INTERPRETATION

Notice that equations (3)–(6) have the same form: hence we can interpret and solve one of them and the others will follow suit. Equations (7) and (8) will be discussed later.

The interpretation of equation (3) when $|\tilde{V}_a^{(1)}|$ is known is shown in Figure 2. For each value of i (each trial weight on one balance plane installed and spun), the equation

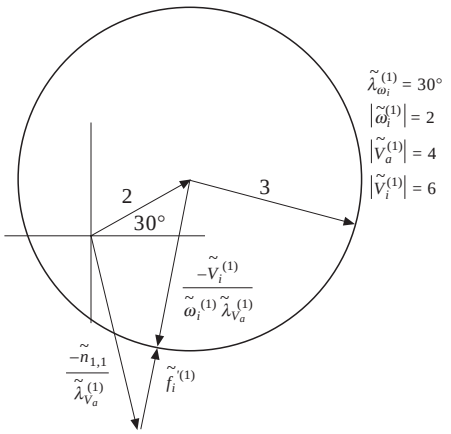

Figure 2. The interpretation of the vibration equation for one value of i and known $|\tilde{V}_a^{(1)}|$.

represents a known complex number minus a complex number of known magnitude ($-\tilde{V}_i^{(1)}/(\tilde{w}_i^{(1)}\tilde{\lambda}_a^{(1)})$ is represented by the circle of known radius) minus the objective function equaling the negative of the desired quantity $\tilde{n}_{1,1}/\tilde{\lambda}_{Va}^{(1)}$. The diagram also shows the minimum value of $\tilde{f}_i'^{(1)}$.

For every trial weight placed on plane 1, there will be two constructions similar to Figure 2, one for each measured vibration. Each construction is useful for determining part of the matrix $\vec{\vec{N}}$. If there is no measurement error, the objective functions have a minimum of zero and all the circles on each graphic intersect. The intersection is related to a component of $\vec{\vec{N}}$. To form a unique circle intersection, at least three circles are drawn, which means that at least three trial weights are used on each balance plane.

When there are measurement errors (in all real cases); the minimum $\tilde{f}_i'^{(1)}$ is determined by constructing m_1 (at least three) circles and finding the point that is minimum distance from all circles simultaneously. In Figure 2 is shown the minimum $\tilde{f}_i'^{(1)}$ for the one circle that is shown.

The effect of equations (7) and (8) is to determine the angle for the vibration without a trial mass being added. What the equations mean is that the angle of the vibration must be the same as the angle of the average vibration. Once the average vibration $\vec{V}_a$ is known, it is trivial to solve for the last two necessary angles.

5. THE EXISTENCE AND UNIQUENESS OF THE SOLUTION

An easy way to demonstrate the existence and uniqueness of the solution is to introduce an analog. Consider the situation in which there are m_1 linear springs, all of unit spring constant. Let spring i have a known free length of $|\vec{V}_i^{(1)}/\tilde{w}_i^{(1)}|$. All m_i springs are connected together at one of their ends. The second end of spring i is tied at point $|\vec{V}_a^{(1)}/\tilde{w}_i^{(1)}$. The $\tilde{f}_i'^{(1)}$ represents the deformation in spring i.

The solution for minimum $\tilde{f}_i'^{(1)}$ is equivalent to the solution that provides the minimum potential energy in all m_i springs—in other words, the static equilibrium point. With this interpretation, it is a simple matter to prove that the equations are convex and therefore have a single unique solution.

6. SOLUTION FOR UNKNOWN $|\tilde{V}_a^{(1)}|$

An iterative technique is used to find the solution when $|\tilde{V}_a^{(1)}|$ and $|\tilde{V}_a^{(2)}|$ are unknown. To start the process, we use the estimate that $\vec{\vec{N}}\vec{u} = \vec{V}_0$. With this estimate we find the best

intersection of the circles defined by equations (3) through (6). With these solutions, determine estimates of the vibration angles and hence compute an estimate of $|\tilde{V}_a^{(1)}|$ and $|\tilde{V}_a^{(2)}|$ by simple calculation. To help achieve faster convergence, an acceleration factor can be used to estimate the values of $|\tilde{V}_a^{(1)}|$ and $|\tilde{V}_a^{(2)}|$. For example, let $|\tilde{V}_a^{(1)}|_{p-1}$ and $|\tilde{V}_a^{(2)}|_{p-1}$ be estimated values in the $p-1$ iteration. With these estimates, circles are drawn from which vibration angles are computed and new estimates for $|\tilde{V}_a^{(1)}|_{p'}$ and $|\tilde{V}_a^{(2)}|_{p'}$ are obtained. For the p iteration, however, we use the following estimate:

$$|\tilde{V}_a^{(1)}|_p = |\tilde{V}_a^{(1)}|_{p-1} + K(|\tilde{V}_a^{(1)}|_{p'} - |\tilde{V}_a^{(1)}|_{p-1}), \qquad |\tilde{V}_a^{(2)}|_p = |\tilde{V}_a^{(2)}|_{p-1} + K(|\tilde{V}_a^{(2)}|_{p'} - |\tilde{V}_a^{(2)}|_{p-1}).$$

Convergence occurs within a few iterations with an acceleration (K) of anywhere between 0·05 and 1. Experience has shown that $K = 0{\cdot}5$ for a maximum of ten unattended iterations works well. If K is manually adjusted, faster convergence can be achieved.

7. SIMULATION RESULTS

This section gives simulation results for the optimal and non-optimal [8] balance of several rotors. In the simulation there was a distinction made between "perceived" values and "real" values. Perceived values are what one would think or measure them to be, while real values are what they really are. The goal was to estimate the real value of $\check{u}$ given the perceived values of trial unbalance and vibration magnitude.

After estimating the unbalance $\check{u}$, trial weights equal and opposite to $\check{u}$ are "placed" on the rotor and the resulting vibration is calculated. These residual vibration magnitudes are denoted as

$$\begin{bmatrix} |V|_{rn}^{(1)} \\ |V|_{rn}^{(2)} \end{bmatrix}$$

for the non-optimal solution and

$$\begin{bmatrix} |V|_{ro}^{(1)} \\ |V|_{ro}^{(1)} \end{bmatrix}$$

for the optimal technique. Results are given as a fraction of the original vibration magnitudes, where R_n, the non-optimal "result", is computed as

$$R_n = \frac{|V|_{rn}^{(1)} + |V|_{rn}^{(2)}}{|V|_0^{(1)} + |V|_0^{(2)}};$$

the optimal "result" is computed as

$$R_o = \frac{|V|_{ro}^{(1)} + |V|_{ro}^{(2)}}{|V|_0^{(1)} + |V|_0^{(2)}}.$$

All results are expressed as percentages. Obviously, if the perceived estimate of the unbalance is exactly correct, the "result" is zero.

One example of the data used in the simulation study given is in Table 1. Real trial weight location errors were normally distributed with zero mean and a standard deviation of ± 5 degrees from where they were "perceived" to be located. The real trial unbalance magnitude was uniformly distributed between $-0{\cdot}1$ and 0·1 from perceived values. Real vibration is simulated using the values chosen for the system and the "real" trial unbalance. The perceived or measured vibration is computed by adding a normally distributed (zero mean, $\sigma = 5$) percent error to the magnitude of the "real" vibration phasors.

Using the data in Table 1, a non-optimal solution for the unbalance was determined using the method in reference [8]. Applying a correction weight based on the non-optimal solution, the "result" vibration for the data in Table 1 is 37·2%.

Using the data in Table 1, an optimal solution for the unbalance was determined using the method discussed in this paper. An example of the circle constructions for this data and the optimal intersection points are shown in Figure 3. Note that, in general, not only are the intersection points slightly different than the non-optimal solution, but even the circles are different. This is due to the method used to approximate $\vec{\vec{N}}\vec{u}$. In the non-optimal method, $\vec{\vec{N}}\vec{u} \approx \vec{V}_0$, but this estimate is inaccurate and biased when there is a measurement error in $\vec{V}_0$. Using the optimal constructions to estimate $\vec{\vec{N}}$ determines the unbalance. Applying a correction weight based on this estimated unbalance, the "result" is 13.6%.

Depending on the system, and on errors in the vibration measurements, it is possible that the non-optimal solution produces a smaller residual vibration than the optimal solution. Of course, one cannot tell *a priori* whether the optimal solution method will be better than the non-optimal one. What one should do is use whichever method has the highest probability of producing the smallest residual vibration. To determine which method is better, 30 simulations were performed, each simulation having randomly chosen systems and measurement errors. For each system, a non-optimal and an optimal solution were computed. For the optimal solutions, the algorithm iterated a maximum of ten times. The "result" vibration was computed for each solution.

For the 30 uniformly distributed random systems, the non-optimal result had a mean and standard deviation of 19.7% and 12.0%, whereas the optimal had a mean and standard deviation of 15.8% and 10.2%. Using these statistics and a cumulative normal distribution chart, one can conclude with 96% confidence that the optimal method of balancing produces a better residual measure than does the non-optimal technique.

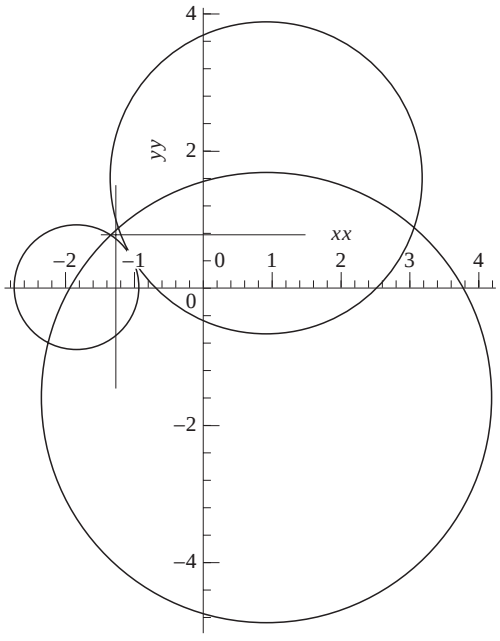

Figure 3. Circle constructions for the optimal solution for vibration on plane 1, and trial unbalance on plane 1.

TABLE 1

Sample values for one simulation

Reference	Description	Value
1	Randomly chosen real $\vec{\vec{N}}$	$\vec{\vec{N}} = \begin{bmatrix} 1{\cdot}30 + 0{\cdot}523I & -3{\cdot}95 - 3{\cdot}29I \\ 2{\cdot}98 - 7{\cdot}82I & -0{\cdot}619 - 5{\cdot}13I \end{bmatrix}$
2	Randomly chosen real $\vec{u}$	$\vec{u} = \begin{bmatrix} -4{\cdot}25 + 0{\cdot}957I \\ 1{\cdot}18 + 1{\cdot}04I \end{bmatrix}$
3	Magnitude of perceived trial unbalance	6
4	Perceived location of plane 1 weights (degrees)	$0, 120, -120$
5	Perceived phasors of plane 1 weights	$6, -3 + 5{\cdot}20I, -3 - 5{\cdot}20I$
6	Real phasors of plane 1 weights	$5{\cdot}91 + 0{\cdot}413I, -2{\cdot}23 + 5{\cdot}49I, -3{\cdot}29 - 4{\cdot}93I$
7	Perceived location of plane 2 weights (degrees)	$90, 210, -30$
8	Perceived phasors of plane 2 weights	$6I, -5{\cdot}20 - 3I, 5{\cdot}20 - 3I$
9	Real phasors of plane 2 weights	$0{\cdot}076 + 5{\cdot}93I, -5{\cdot}27 - 2{\cdot}70I, 4{\cdot}92 - 3{\cdot}31I$
10	Real vibration phasors, system in 1 and 2 together with unbalance in 6 and 9; plane 1 and 2 vibration	$\vec{V}_0 = \begin{bmatrix} \vec{V}_0^{(1)} \\ \vec{V}_0^{(2)} \end{bmatrix} = \begin{bmatrix} -7{\cdot}27 - 8{\cdot}97I \\ -0{\cdot}572 + 29{\cdot}4I \end{bmatrix}$ $\vec{V}_1 = \begin{bmatrix} 0{\cdot}219 - 5{\cdot}34I \\ 20{\cdot}3 - 15{\cdot}6I \end{bmatrix}$, $\vec{V}_2 = \begin{bmatrix} -13{\cdot}0 - 2{\cdot}98I \\ 35{\cdot}7 + 63{\cdot}2I \end{bmatrix}$ $\vec{V}_3 = \begin{bmatrix} -8{\cdot}99 - 17{\cdot}1I \\ -48{\cdot}9 + 40{\cdot}4I \end{bmatrix}$, $\vec{V}_4 = \begin{bmatrix} 11{\cdot}9 - 32{\cdot}6I \\ 29{\cdot}8 + 25{\cdot}3I \end{bmatrix}$ $\vec{V}_5 = \begin{bmatrix} 4{\cdot}67 + 19{\cdot}0I \\ -11{\cdot}2 + 58{\cdot}1I \end{bmatrix}$, $\vec{V}_6 = \begin{bmatrix} -37{\cdot}6 - 12{\cdot}1I \\ -20{\cdot}6 + 6{\cdot}22I \end{bmatrix}$
11	Perceived vibration random error added to magnitude of vibration in line 10	$\lvert\vec{V}_0\rvert = \begin{bmatrix} 11{\cdot}4 \\ 30{\cdot}1 \end{bmatrix}$, $\lvert\vec{V}_1\rvert = \begin{bmatrix} 5{\cdot}42 \\ 25{\cdot}6 \end{bmatrix}$, $\lvert\vec{V}_2\rvert = \begin{bmatrix} 13{\cdot}6 \\ 67{\cdot}8 \end{bmatrix}$, $\lvert\vec{V}_3\rvert = \begin{bmatrix} 19{\cdot}6 \\ 57{\cdot}6 \end{bmatrix}$, $\lvert\vec{V}_4\rvert = \begin{bmatrix} 33{\cdot}2 \\ 40{\cdot}3 \end{bmatrix}$, $\lvert\vec{V}_5\rvert = \begin{bmatrix} 18{\cdot}9 \\ 60{\cdot}3 \end{bmatrix}$, $\lvert\vec{V}_6\rvert = \begin{bmatrix} 38{\cdot}3 \\ 22{\cdot}2 \end{bmatrix}$

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