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TRANSIENT RESPONSE OF ONE-DIMENSIONAL DISTRIBUTED SYSTEMS: A CLOSED FORM EIGENFUNCTION EXPANSION REALIZATION

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The exact and closed form transient response of general one-dimensional distributed dynamic systems subject to arbitrary external, initial and boundary disturbances is determined. Non-self-adjoint operators characterizing damping, gyroscopic and circulatory effects, and eigenvalue-dependent boundary conditions are considered. Through introduction of augmented operators, a closed form modal expansion of the displacement and internal forces of the distributed system is derived. The eigenfunction expansion is realized in a spatial state space formulation, which systematically yields exact eigensolutions, eigenfunction normalization coefficients and modal co-ordinates. The proposed method is illustrated on a cantilever beam with end mass, viscous damper and spring.

1. INTRODUCTION

Transient response analysis plays an important role in the design of a variety of distributed dynamic systems, such as bridges, buildings, automobiles, airplanes and machines with flexible components. This work is concerned with closed form eigenfunction expansion of the transient response of a class of non-self-adjoint distributed dynamic systems.

Eigenfunction expansion or modal expansion is a commonly used technique for transient analysis of distributed dynamic systems [1–5]. Conventional modal analysis of self-adjoint systems is facilitated by the orthogonality relations among system eigenfunctions, based on which an eigenfunction series representation of system response decouples the original equations of motion into a set of independent equations governing the unknown time-dependent coefficients of the series, or modal co-ordinates. Solution of those decoupled equations for the modal co-ordinates yields a closed form estimation of the system transient response. The convergence of the series solution is guaranteed by the completeness of system eigenfunctions.

Conventional modal expansion is not directly applicable to non-self-adjoint systems the eigenfunctions of which associated with the original equations of motion are non-orthogonal. For certain non-self-adjoint distributed systems, the closed form transient response can be expressed by a series of bi-orthogonal eigenfunctions in a state space formulation [6, 7]. In this generalized modal analysis, the completeness of the state space eigenfunctions is often assumed without justification.

The problem becomes more complicated if a non-self-adjoint system has eigenvalue-dependent boundary conditions, due to lumped masses and energy dissipative

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devices at its boundary. In this case, the state-space formulism fails to yield closed form solutions. In a recent study [8], a closed form series solution for a longitudinally vibrating bar with an end viscous damper is formulated by changing the spatial interval of the problem. Nonetheless, closed form eigenfunction expansion for general one-dimensional distributed systems is not available. Moreover, most previous studies assume homogeneous boundary conditions; the effects of external loads and specified motion at the boundary are usually neglected in a closed form solution.

Even if a closed form modal expansion is available, its realization in a specific problem is fully dependent on the estimation of system eigensolutions, eigenfunction normalization coefficients and modal co-ordinates, which has been a challenging task for non-self-adjoint distributed systems. Because of this and the problems discussed previously, transient analyses of non-self-adjoint systems often rely on approximate methods, such as the finite element method [9, 10]. Although the finite element method is capable of modelling complicated distributed systems, it is computationally intensive, especially for systems with high-frequency dynamics, and depends on particular cases of computation to understand the effects of model parameters on the system transient response. Closed form analytical solutions, if obtainable, are always desirable because of their numerical efficiency, and physical insight into the problem.

In this paper, an exact closed form solution method is proposed for transient analysis of general one-dimensional distributed systems which have non-self-adjoint operators and eigenvalue-dependent boundary conditions, and are subject to arbitrary external, initial and boundary disturbances. An eigenfunction series solution is derived through introduction of augmented spatial operators, and through application of a modal expansion theorem given in reference [11]. The convergence of the modal series is assured without the completeness assumption about system eigenfunctions. With a spatial state space formulation, exact eigensolutions, eigenfunction normalization coefficients and modal co-ordinates are systematically obtained, leading to a highly accurate closed form transient response prediction. Besides the displacement, the slope and internal forces of the distributed system are simultaneously determined, without differentiation of the displacement function. The proposed method is demonstrated on a cantilever beam with end mass, viscous damper and spring, and is compared with the finite element method.

2. STATEMENT OF PROBLEM

The displacement $w(x, t)$ of the distributed dynamic system is governed by the non-dimensional partial differential equation

$$(A\, \partial^2/\partial t^2 + B\, \partial/\partial t + C)w(x, t) = f(x, t), \qquad x \in (0, 1), \tag{1a}$$

with the initial conditions

$$w(x, 0) = u_0(x), \quad \partial w(x, t)/\partial t|_{t=0} = v_0(x), \qquad x \in (0, 1) \tag{1b}$$

and boundary conditions

$$(A_{L_j}\, \partial^2/\partial t^2 + B_{L_j}\, \partial/\partial t + C_{L_j})w(x, t)|_{x=0} = \vartheta_{L_j}(t),$$
$$(A_{R_j}\, \partial^2/\partial t^2 + B_{R_j}\, \partial/\partial t + C_{R_j})w(x, t)|_{x=1} = \vartheta_{R_j}(t), \tag{1c}$$

for $j=1,2,\ldots,N$, where $f(x,t)$, $u_0(x)$ and $v_0(x)$, and $\vartheta_{L_j}(t)$ and $\vartheta_{R_j}(t)$ are the external, initial and boundary disturbances, respectively, A, B and C are the spatial differential operators of order $2N$ given by

$$A=\sum_{k=0}^{2N} a_k \frac{\partial^k}{\partial x^k}, \qquad B=\sum_{k=0}^{2N} b_k \frac{\partial^k}{\partial x^k}, \qquad C=\sum_{k=0}^{2N} c_k \frac{\partial^k}{\partial x^k}, \tag{2}$$

with a_k, b_k and c_k being constant coefficients, and A_{L_j}, B_{L_j}, C_{L_j}, A_{R_j}, B_{R_j} and C_{R_j} are spatial differential operators of up to (N-1)th order, in a form similar to equation (2). Equations (1) describe a wide class of distributed parameter systems such as strings, bars, beams, beam-columns, flexible rotating shafts, axially moving materials and flexible robot arms. The operators A_{L_j}, B_{L_j}, A_{R_j} and B_{R_j} characterize lumped end masses and viscous dampers.

In this work, a closed form solution of equations (1) is sought via eigenfunction expansion. The eigenvalue problem associate with the original equations of motion (1) is defined by

$$(\lambda_k^2 A+\lambda_k B+C)u_k(x)=0, \qquad x\in(0,1), \quad k=\pm 1, \pm 2, \ldots; \tag{3a}$$

$$(\lambda_k^2 A_{L_j}+\lambda_k B_{L_j}+C_{L_j})u_k(x)|_{x=0}=0, \qquad j=1,2,\ldots,N;$$

$$(\lambda_k^2 A_{R_j}+\lambda_k B_{R_j}+C_{R_j})u_k(x)|_{x=1}=0, \qquad j=1,2,\ldots,N; \tag{3b}$$

where λ_k are the eigenvalues, and $u_k(x)$ are the corresponding eigenfunctions or mode shapes. The adjoint eigenvalue problem is

$$(\bar{\lambda}_k^2 A^*+\bar{\lambda}_k B^*+C^*)v_k(x)=0, \qquad x\in(0,1); \tag{4a}$$

$$(\bar{\lambda}_k^2 A_{L_j}^*+\bar{\lambda}_k B_{L_j}^*+C_{L_j}^*)v_k(x)|_{x=0}=0, \qquad j=1,2,\ldots,N;$$

$$(\bar{\lambda}_k^2 A_{R_j}^*+\bar{\lambda}_k B_{R_j}^*+C_{R_j}^*)v_k(x)|_{x=1}=0, \qquad j=1,2,\ldots,N; \tag{4b}$$

where A^* is the adjoint of A, and $A_{L_j}^*$ is the adjoint of A_{L_j}, etc., and the overbar denotes complex conjugation. Because the coefficients of all operators are real, $\lambda_{-k}=\bar{\lambda}_k$ for any k. The adjoint operators and the boundary conditions (4b) are obtained from the integral

$$\int_0^1 \{\bar{v}(x)[\lambda^2 A+\lambda B+C]u(x)-u(x)[\lambda^2 A^*+\lambda B^*+C^*]\bar{v}(x)\}\,\mathrm{d}x=0,$$

where λ is an arbitrary constant, and $u(x)$ and $v(x)$ are differentiable functions satisfying the boundary conditions (3b) and (4b), respectively, with λ_k replaced by λ. It is easy to show that

$$A^*=\sum_{k=0}^{2N}(-1)^k a_k \frac{\partial^k}{\partial x^k}, \qquad B^*=\sum_{k=0}^{2N}(-1)^k b_k \frac{\partial^k}{\partial x^k}, \qquad C^*=\sum_{k=0}^{2N}(-1)^k c_k \frac{\partial^k}{\partial x^k}.$$

Modal analysis of distributed dynamic systems depends on the establishment of orthogonality relations for system eigenfunctions, and normalization of system eigenfunctions. The distributed systems modeled by equations (1) in general is non-self-adjoint, mainly due to damping, gyroscopic and circulatory effects and due to eigenvalue-dependent boundary conditions (see equations (3b) and (4b)). Because of this, conventional modal analysis techniques are not directly applicable here; equations (1) are normally solved by numerical methods. Also, it should be noted that eigenfunction normalization has been a challenging task because calculation of normalization coefficients requires spatial integration of system eigenfunctions over the entire domain of the continuum, which is different for different boundary conditions. Additionally, the completeness of system eigenfunctions is often assumed to guarantee the convergence of the eigenfunction series—which, however, is difficult to verify for non-self-adjoint systems.

The objective of the current study is to develop a new modal analysis for obtaining an exact and closed form solution of equations (1), which has not been available in the literature. In the proposed method, an eigenfunction expansion is derived through introduction of equivalent augmented governing equations (section 3), and realized in a spatial state space formulation (section 4). As shall be seen, with the new method, the aforementioned difficulties can be overcome or avoided.

3. EIGENFUNCTION EXPANSION

As mentioned in the previous section, the boundary conditions of the distributed system are eigenvalue-dependent, which makes it difficult to obtain a closed form eigenfunction series solution for the original equations (1). In this section, equations (1) are cast into an equivalent augmented form, from which a closed form eigenfunction expansion results.

By defining the augmented operators

$$\hat{A} = \operatorname{diag}\{A \quad A_{L_1} \quad \cdots \quad A_{L_N} \quad A_{R_1} \quad \cdots \quad A_{R_N}\},$$
$$\hat{B} = \operatorname{diag}\{B \quad B_{L_1} \quad \cdots \quad B_{L_N} \quad B_{R_1} \quad \cdots \quad B_{R_N}\},$$
$$\hat{C} = \operatorname{diag}\{C \quad C_{L_1} \quad \cdots \quad C_{L_N} \quad C_{R_1} \quad \cdots \quad C_{R_N}\}, \tag{5}$$

equations (1) can be rewritten as

$$(\hat{A}\,\partial^2/\partial t^2 + \hat{B}\,\partial/\partial t + \hat{C})W(x, t) = F(x, t), \qquad x \in (0, 1); \tag{6a}$$

$$W(x, 0) = U_0(x), \qquad \partial w(x, t)/\partial t|_{t=0} = V_0(x), \tag{6b}$$

where the displacement and external force vectors are

$$W(x, t) = \begin{Bmatrix} w(x, t) \\ w(0, t)\cdot 1_N \\ w(1, t)\cdot 1_N \end{Bmatrix}, \qquad F(x, t) = \{f(x, t) \quad \vartheta_{L_1}(t) \quad \vartheta_{L_N}(t) \quad \vartheta_{R_1}(t) \quad \vartheta_{R_N}(t)\}^{\mathrm{T}}, \tag{7a}$$

with $1_N = \{1 \quad 1 \quad \cdots \quad 1\}^{\mathrm{T}} \in R^N$, and the initial displacement and velocity vectors are

$$U_0(x) = \begin{Bmatrix} u_0(x) \\ u_0(0)\cdot 1_N \\ u_0(1)\cdot 1_N \end{Bmatrix}, \qquad V_0(x) = \begin{Bmatrix} v_0(x) \\ v_0(0)\cdot 1_N \\ v_0(1)\cdot 1_N \end{Bmatrix}. \tag{7b}$$

Note that the boundary conditions (1c) have been absorbed in the augmented equation (6a).

The associate and adjoint eigenvalue problems of equation (6) are

$$\{\lambda_k^2 \hat{A} + \lambda_k \hat{B} + \hat{C}\}\phi_k = 0, \qquad \{\bar{\lambda}_k^2 \hat{A}^* + \bar{\lambda}_k \hat{B}^* + \hat{C}^*\}\psi_k = 0, \qquad k = \pm 1, \pm 2, \ldots, \tag{8a, b}$$

where the adjoint operator are given by

$$\begin{aligned} \hat{A}^* &= \operatorname{diag}\{A^* \quad A^*_{L_1} \quad \cdots \quad A^*_{L_N} \quad A^*_{R_1} \quad \cdots \quad A^*_{R_N}\}, \\ \hat{B}^* &= \operatorname{diag}\{B^* \quad B^*_{L_1} \quad \cdots \quad B^*_{L_N} \quad B^*_{R_1} \quad \cdots \quad B^*_{R_N}\}, \\ \hat{C}^* &= \operatorname{diag}\{C^* \quad C^*_{L_1} \quad \cdots \quad C^*_{L_N} \quad C^*_{R_1} \quad \cdots \quad C^*_{R_N}\}, \end{aligned} \tag{9}$$

The eigenfunctions of the original eigenvalue problems (equations (3) and (4)) and those of equations (8) are related by

$$\phi_k = \begin{Bmatrix} u_k(x) \\ u_0(0) \cdot 1_N \\ u_k(1) \cdot 1_N \end{Bmatrix}, \qquad \psi_k = \begin{Bmatrix} v_k(x) \\ v_k(0) \cdot 1_N \\ v_k(1) \cdot 1_N \end{Bmatrix}. \tag{10}$$

Through application of the modal expansion theorem in reference [11], the solution to equations (6) is expressed by the eigenfunction series

$$W(x, t) = \sum_{k=\pm 1}^{\pm\infty} \frac{1}{\lambda_k \rho_k} s_k(t) \phi_k(x), \tag{11}$$

where the eigenfunction normalization coefficients ρ_k and time-dependent modal co-ordinates $s_k(t)$ are given by

$$\rho_k = \langle \psi_k, (2\lambda_k \hat{A} + \hat{B})\phi_k \rangle, \tag{12a}$$

$$s_k(t) = \int_0^t e^{\lambda_k(t-\tau)} \langle \psi_k, F(\cdot, \tau) \rangle \, d\tau + e^{\lambda_k t} \langle \psi_k, \hat{B} U_0 + \hat{A} V_0 + \lambda_k \hat{A} U_0 \rangle. \tag{12b}$$

The inner product in equations (12) is defined by

$$\langle G, H \rangle = \int_0^1 \bar{g}(x) h(x) \, dx + \sum_{j=1}^{2N} \bar{g}_j h_j \tag{13}$$

for $G = (g(x) \ \ g_1 \ \cdots \ g_{2N})^T$ and $H = (h(x) \ \ h_1 \ \cdots \ h_{2N})^T$, where $g(x)$ and $h(x)$ are elements of a function space, and g_j and h_j are complex scalars. Thus, by equations (7a) and (10), the transient response of the distributed system subject to arbitrary external, initial and boundary disturbances is given in the exact and closed form

$$w(x, t) = \sum_{k=\pm 1}^{\pm\infty} \frac{1}{\lambda_k \rho_k} s_k(t) u_k(x). \tag{14}$$

According to reference [11], the convergence of the eigenfunction expansion, equation (14), is assured without the need to assume completeness of the basis formed by the eigenfunctions ϕ_k and ψ_k.

4. SPATIAL STATE SPACE REALIZATION

Given a specific system, the closed form eigenfunction expansion, equation (14), is only symbolic unless the eigensolutions (λ_k, ϕ_k, ψ_k), the eigenfunction normalization coefficients

ρ_k, and the modal co-ordinates $s_k(t)$ are known. However, determination of those quantities is not trivial. The non-self-adjoint operators and eigenvalue-dependent boundary conditions make it difficult to obtain exact eigensolutions. Estimation of the eigenfunction normalization coefficients involves spatial integrals of the eigenfunctions, the exact quadrature of which for general systems is not available. Also, calculation of the modal co-ordinates is conducted on a system-by-system basis, and often requires a totally different derivation if any system parameters, disturbance functions or boundary conditions are changed. Because of these difficulties, transient analyses in many cases eventually rely on approximate methods. In this section, through introduction of a spatial state space formulation, an exact and closed form realization of the eigenfunction expansion, equation (14), is developed.

4.1. EXACT EIGENSOLUTIONS

By equations (10), the augmented eigenfunctions (ϕ_k and ψ_k) are completely determined if the original eigenfunctions (u_k and v_k) are known. Thus, the solution for u_k and v_k is sufficient. In this study, a spatial state space formulation is adopted for systematic evaluation of system eigenvalues and eigenfunctions. Let s and $u(x)$ be an eigenpair satisfying equations (3). By defining the spatial state space vector

$$\eta(x) = \left\{ u(x) \quad \frac{\mathrm{d}}{\mathrm{d}x} u(x) \quad \cdots \quad \frac{\mathrm{d}^{2N-1}}{\mathrm{d}x^{2N-1}} u(x) \right\}^{\mathrm{T}}, \tag{15}$$

equations (3) are cast in the equivalent spatial state space form

$$\frac{\mathrm{d}}{\mathrm{d}x}\eta(x) = F(s)\eta(x), \quad x \in (0, 1); \qquad M_b(s)\eta(0) + N_b(s)\eta(1) = 0, \tag{16a, b}$$

where the $2N \times 2N$ state space matrix $F(s)$, and boundary matrices $M_b(s)$ and $N_b(s)$ are

$$F(s) = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ d_0(s) & d_1(s) & \cdots & d_{2N-1}(s) \end{bmatrix},$$

$$M_b(s) = \begin{bmatrix} m_b(s) \\ 0_{N\times 2N} \end{bmatrix}, \qquad N_b(s) = \begin{bmatrix} 0_{N\times 2N} \\ n_b(s) \end{bmatrix},$$

with $d_k(s) = -(a_k s^2 + b_k s + c_k)/(a_{2N} s^2 + b_{2N} s + c_{2N})$, $k = 0, 1, \ldots, 2N-1$, $0_{N\times 2N}$ being the $N \times 2N$ zero matrix, and $m_b(s)$ and $n_b(s)$ N × 2N complex matrices composed of the coefficients of the operators A_{L_j}, B_{L_j}, C_{L_j}, A_{R_j}, B_{R_j} and C_{R_j}. The above formulation can also be obtained from Laplace transform of equations (1) with vanishing disturbances [12].

The solution to equation (16a) is of the form

$$\eta(x) = \mathrm{e}^{F(s)x}\eta_0, \tag{17}$$

where η_0 is a constant vector to be determined. Substitution of equation (17) into equation (16b) gives

$$[M_b(s) + N_b(s)\,\mathrm{e}^{F(s)}]\eta_0 = 0. \tag{18}$$

The eigenvalues are the root of the characteristic equation

$$\det\,[M_b(s) + N_b(s)\,\mathrm{e}^{F(s)}] = 0; \tag{19}$$

that is, $s = \lambda_k$, $k = \pm 1, \pm 2, \ldots$ The eigenvector corresponding to λ_k is given by

$$\eta(x) = \mathrm{e}^{F(\lambda_k)x}\eta_0(\lambda_k), \tag{20}$$

where the vector $\eta_0(\lambda_k)$ is a non-trivial solution of equation (18) with $s = \lambda_k$.

Likewise, the adjoint eigenvalue problem, equations (4), is equivalent to the state space equations

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\theta(x) = F^*(s)\theta(x), \quad x \in (0, 1); \qquad M_b^*(s)\theta(0) + N_s^*(s)\theta(1) = 0, \tag{21a, b}$$

where the matrices $F^*(s)$, $M_b^*(s)$ and $N_b^*(s)$ consist of the coefficients of the adjoint operators in equations (4), and the adjoint state space vector

$$\theta(x) = \left\{ v(x) \quad \frac{\mathrm{d}}{\mathrm{d}x}\,v(x) \quad \cdots \quad \frac{\mathrm{d}^{2N-1}}{\mathrm{d}x^{2N-1}}\,v(x) \right\}^{\mathrm{T}},$$

with $v(x)$ being the adjoint eigenfunction. The adjoint eigenvector corresponding to λ_k is given by

$$\theta(x) = \mathrm{e}^{\mathrm{F}^*(\bar{\lambda}_k)x}\theta_0(\bar{\lambda}_k), \tag{22}$$

where $\theta_0(\bar{\lambda}_k)$ is a non-trivial solution of the homogeneous equation

$$[M_b^*(\bar{\lambda}_k) + N_b^*(\bar{\lambda}_k)\,\mathrm{e}^{F^*(\bar{\lambda}_k)}]\theta_0(\bar{\lambda}_k) = 0. \tag{23}$$

The eigenfunctions $u_k(x)$ and $v_k(x)$ are the first elements of $\eta(x)$ and $\theta(x)$ given in equations (20) and (22); namely,

$$u_k(x) = e_1^{\mathrm{T}}\,\mathrm{e}^{F(\lambda_k)x}\eta_0(\lambda_k), \qquad v_k(x) = e_1^{\mathrm{T}}\,\mathrm{e}^{F^*(\bar{\lambda}_k)x}\theta_0(\bar{\lambda}_k), \tag{24}$$

where $e_1 = (1 \;\; 0 \;\; \cdots \;\; 0)^{\mathrm{T}} \in R^{2N}$, and the superscript T denotes matrix/vector transposition.

In the above state space formulation, no approximation or discretization has been made. Exact eigensolutions can be obtained from equations (19) and (24). One advantage of the formulation is that different system types (strings, bars, beams, etc.), various non-self-adjoint effects (e.g., damping and gyroscopic forces) and arbitrary boundary conditions are systematically treated by easy formation of the state space and boundary matrices in equations (16) and (21). The algorithm for eigensolutions, however, remains the same in all cases.

4.2. EIGENFUNCTION NORMALIZATION COEFFICIENTS

The eigenfunction normalization coefficient, by equations (5), (10) and (12a), is

$$\rho_k = \int_0^1 \bar{v}_k(x)(2\lambda_k A + B)u_k(x)\,\mathrm{d}x + \bar{v}_k(0)\sum_{j=1}^{N}(2\lambda_k A_{L_j} + B_{L_j})u_k(x)|_{x=0}$$

$$+\bar{v}_k(1)\sum_{j=1}^{N}(2\lambda_k A_{R_j} + B_{R_j})u_k(x)|_{x=1}. \tag{25}$$

For exact evaluation of ρ_k, decompose the exponential matrices in equation (24) into

$$e^{F(\lambda_k)x} = Q\, e^{\Lambda x} Q^{-1}, \qquad e^{F^*(\bar{\lambda}_k)x} = R\, e^{\Lambda^* x} R^{-1}, \tag{26}$$

where

$$\Lambda = \text{diag}\,\{\mu_1 \quad \cdots \quad \mu_{2N}\}, \qquad \Lambda^* = \text{diag}\,\{\sigma_1 \quad \cdots \quad \sigma_{2N}\},$$
$$Q = [q_1 \quad \cdots \quad q_{2N}] \in C^{2N \times 2N}, \qquad R = [r_1 \quad \cdots \quad r_{2N}] \in C^{2N \times 2N}. \tag{27}$$

Here (μ_j, q_j) and (σ_j, r_j) are the eigensolutions of $F(\lambda_k)$ and $F^*(\bar{\lambda}_k)$; namely,

$$(\mu_j I - F(\lambda_k))q_j = 0, \quad (\sigma_j I - F^*(\bar{\lambda}_k))r_j = 0, \qquad j = 1, 2, \ldots, 2N. \tag{28}$$

Write

$$Q^{-1} = [\alpha_1 \quad \cdots \quad \alpha_{2N}]^{\mathrm{T}}, \qquad R^{-1} = [\beta_1 \quad \cdots \quad \beta_{2N}]^{\mathrm{T}}, \tag{29}$$

where $\alpha_j,\ \beta_j \in C^{2N}$. The exponential matrices in equation (26) become

$$e^{F(\lambda_k)x} = \sum_{j=1}^{2N} e^{\mu_j x} q_j \alpha_j^{\mathrm{T}}, \qquad e^{F^*(\bar{\lambda}_k)x} = \sum_{j=1}^{2N} e^{\sigma_j x} r_j \beta_j^{\mathrm{T}}. \tag{30}$$

Substitution of equations (24) and (30) into equation (25) yields

$$\rho_k = \bar{\theta}_0^T(\bar{\lambda}_k)[\Delta + \Delta_L + \Delta_R]\eta_0(\lambda_k), \tag{31}$$

where

$$\Delta = \sum_{j,l=1}^{2N} \bar{r}_{j,1} q_{l,1} \bar{\beta}_j \alpha_l^T \int_0^1 e^{\bar{\sigma}_j x}(2\lambda_k A + B)\, e^{\mu_l x}\, \mathrm{d}x,$$

$$\Delta_L = \sum_{j,l=1}^{2N} \bar{r}_{j,1} q_{l,1} \bar{\beta}_j \alpha_l^{\mathrm{T}} \sum_{i=0}^{N} [(2\lambda_k A_{L_i} + B_{L_i})\, e^{\mu_l x}]_{x=0}, \tag{32a, b}$$

$$\Delta_R = \sum_{j,l=1}^{2N} \bar{r}_{j,1} q_{l,1} \bar{\beta}_j \alpha_l^{\mathrm{T}}\, e^{\bar{\sigma}_j} \sum_{i=1}^{N} [(2\lambda_k A_{R_i} + B_{R_i})\, e^{\mu_l x}]_{x=1}, \tag{32c}$$

with $r_{j,1}$ and $q_{l,1}$ being the first elements of the vectors r_j and q_l, respectively. Exact quadrature for the integrals in equations (32a) can be easily obtained. Hence, the eigenfunction normalization coefficients can be precisely computed by equation (31).

In the above matrix decomposition, Λ and Λ^* are assumed to be diagonal; see equations (27). This condition can be lifted. If $F(\lambda_k)$ and $F^*(\bar{\lambda}_k)$ have less than $2N$ linearly independent eigenvectors, Λ and Λ^* become Jordan matrices. In this case, the exponential matrices have the form

$$e^{F(\lambda_k)x} = \sum_{j=1}^{2N} x^{m_j}\, e^{\mu_j x} D_j, \qquad e^{F^*(\bar{\lambda}_k)x} = \sum_{j=1}^{2N} x^{n_j}\, e^{\sigma_j x} E_j, \tag{33}$$

where m_j and n_j are non-negative integers that are less than $2N$, and D_j and E_j are $2N \times 2N$ constant matrices. An exact prediction of ρ_k similar to equation (31) can be derived.

4.3. MODAL CO-ORDINATES

By equations (7), (10) and (12a), the modal co-ordinates $s_k(t)$ are given by

$$s_k(t) = \int_0^t e^{\lambda_k(t-\tau)} \int_0^1 \bar{v}_k(x) f(x, \tau)\, dx\, d\tau + \int_0^t e^{\lambda_k(t-\tau)}$$
$$\times \sum_{i=1}^{N} [\bar{v}_k(0)\vartheta_{L_i}(\tau) + \bar{v}_k(1)\vartheta_{R_i}(\tau)]\, d\tau + e^{\lambda_k t} \int_0^1 \bar{v}_k(x)[A(\lambda_k u_0(x) + v_0(x)) + Bu_0(x)]\, dx$$
$$+ e^{\lambda_k t}\bar{v}_k(0) \sum_{i=1}^{N} [A_{L_i}(\lambda_k u_0(x) + v_0(x)) + B_{L_i} u_0(x)]_{x=0} + e^{\lambda_k t}\bar{v}_k(1) \sum_{i=1}^{N} [A_{R_i}(\lambda_k u_0(x)$$
$$+ v_0(x)) + B_{R_i} u_0(x)]_{x=1}. \tag{34}$$

It is easy to see that the computation of $s_k(t)$ involves the integrals

$$\int_0^t e^{\lambda_k(t-\tau)} \int_0^1 e^{\bar{\sigma}_j x} f(x, \tau)\, dx\, d\tau \qquad \text{and} \qquad \int_0^1 e^{\bar{\sigma}_j x}[A(\lambda_k u_0(x) + v_0(x)) + Bu_0(x)]\, dx,$$

the exact quadrature of which is obtainable for given disturbance functions. Following section 4.2, precise prediction of the modal co-ordinates is realized with the spatial space formulation.

4.4. SPATIAL DERIVATIVES

In a transient analysis, besides the displacement $w(x, t)$, the slope, internal forces or stresses of the distributed system are often determined, which calls for calculation of the spatial derivatives $(\partial^j/\partial x^j)w(x, t), j = 1, 2, \ldots, 2N-1$. One unique feature of the proposed state space realization is that these derivatives can be conveniently and accurately obtained without additional efforts. According to equations (15) and (20),

$$\frac{d^j}{dx^j} u_k(x) = e_{j+1}^T \eta(x) = e_{j+1}^T e^{F(\lambda_k)x} \eta_0(\lambda_k) \tag{35}$$

for $j = 0, 1, \ldots, 2N-1$, where e_{j+1} is a 2N-vector with the $(j+1)$th element being 1 and all others 0. It follows from equation (14) that

$$\frac{\partial^j}{\partial x^j} w(x, t) = e_{j+1}^T \sum_{k=\pm 1}^{\pm\infty} \frac{1}{\lambda_k \rho_k} s_k(t)\, e^{F(\lambda_k)x} \eta_0(\lambda_k). \tag{36}$$

Equation (36) reduces to equation (14) when $j = 0$. Thus, closed form slope and internal forces of the distributed system are determined without direct differentiation of the displacement function, which in many methods may lead to poor accuracy in computation.

In summary, the major steps in the state space realization are as follows: (1) solve equation (18) for the eigenvalues λ_k and eigenvectors $\eta_0(\lambda_k)$; (2) solve equation (23) for the adjoint eigenvectors $\theta_0(\bar{\lambda}_k)$; (3) evaluate the eigenfunction normalization coefficients ρ_k by equation (31); (4) for given external, initial and boundary disturbances, determine the modal co-ordinates $s_k(t)$ as described in section 4.3; (5) estimate the transient response $w(x, t)$ and its spatial derivates by equations (14) and (36).

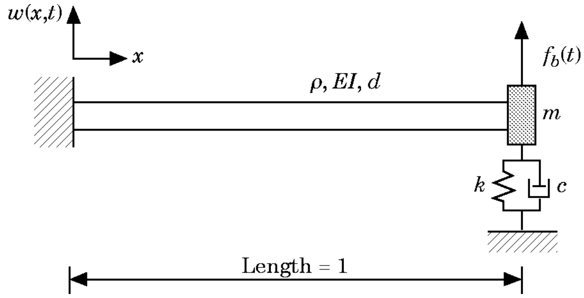

Figure 1. A cantilever beam with tip mass, damper and spring.

TABLE 1
The first ten eigenvalues of the beam ($j = \sqrt{-1}$)

k	λ_k
1	$-0{\cdot}2705 + j1{\cdot}1451$
2	$-0{\cdot}1357 + j4{\cdot}4930$
3	$-0{\cdot}0896 + j13{\cdot}2586$
4	$-0{\cdot}0727 + j26{\cdot}8877$
5	$-0{\cdot}0647 + j45{\cdot}4297$
6	$-0{\cdot}0602 + j68{\cdot}8941$
7	$-0{\cdot}0575 + j97{\cdot}2862$
8	$-0{\cdot}0558 + j130{\cdot}6088$
9	$-0{\cdot}0546 + j168{\cdot}8632$
10	$-0{\cdot}0537 + j212{\cdot}0505$

5. EXAMPLE

The proposed transient analysis is illustrated on a cantilever Euler–Bernoulli beam with end mass, damper and spring; see Figure 1. The lumped mass is subject to a transverse force $f_b(t)$. The transverse displacement $w(x, t)$ of the beam is governed by the differential equation

$$\rho \frac{\partial^2}{\partial t^2} w(x, t) + d \frac{\partial}{\partial t} w(x, t) + EI \frac{\partial^4}{\partial x^4} w(x, t) = 0, \qquad x \in (0, 1), \tag{37}$$

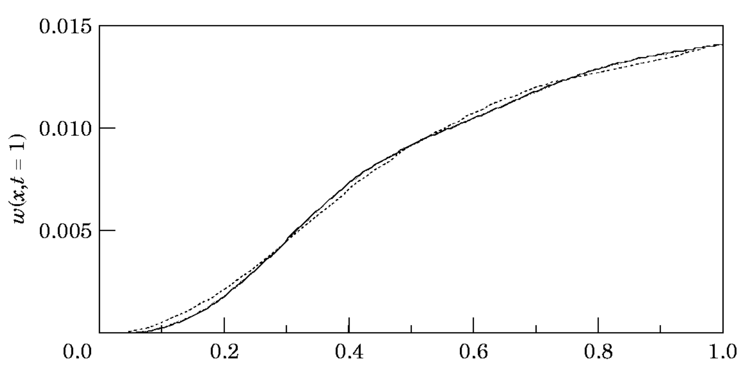

Figure 2. The spatial distribution of the beam response at $t = 1$: ····, FEM with eight elements; ——, the proposed method, and FEM with 16 elements.

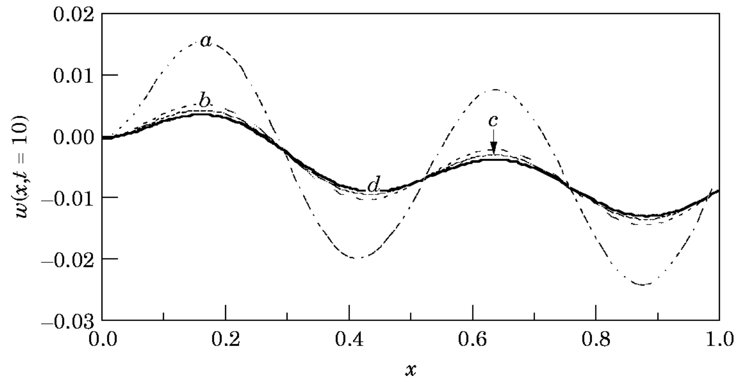

Figure 3. The spatial distribution of the beam response at $t = 10$: a, FEM with eight elements; b, FEM with 16 elements; c, FEM with 20 elements; d the proposed method.

with the boundary conditions

$$w(0, t) = 0, \qquad \frac{\partial}{\partial x} w(x, t)|_{x=0} = 0, \tag{38a, b}$$

$$\frac{\partial^2}{\partial x^2} w(x, t)|_{x=1} = 0, \qquad \left(m \frac{\partial^2}{\partial t^2} + c \frac{\partial}{\partial t} + k - EI \frac{\partial^2}{\partial x^3}\right) w(x, t)|_{x=1} = f_b(s) \tag{38c, d}$$

and zero initial disturbances. Here, EI, ρ and d are the bending stiffness, linear density and viscous damping of the beam, respectively, m the inertia of the lumped mass, c the

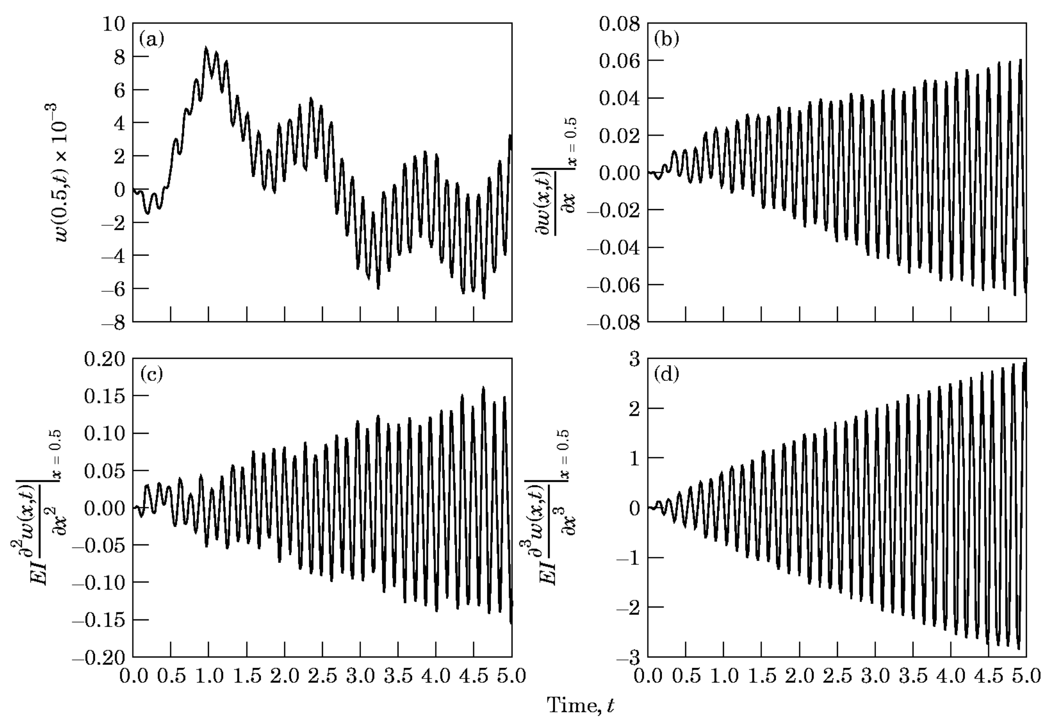

Figure 4. Transient response of the beam at $x = 0.5$: (a) displacement, (b) slope, (c) bending moment, (d) shear force.

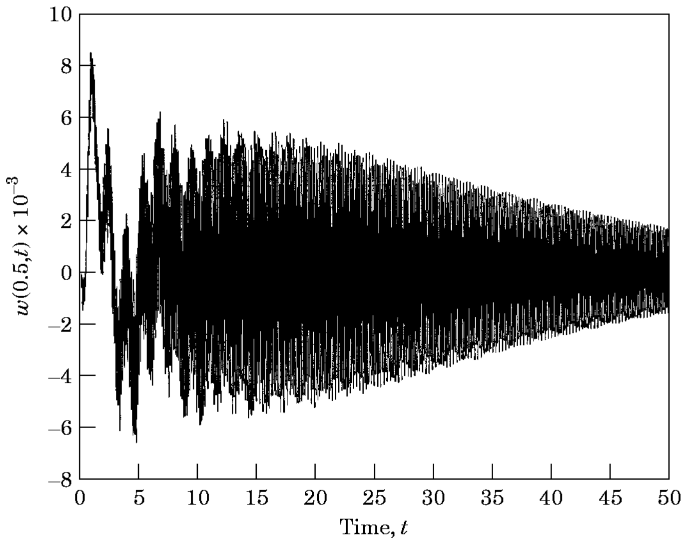

Figure 5. The transient displacement of the beam at $x = 0{\cdot}5$.

coefficient of the damper, and k the coefficient of the spring. The spatial differential operators defined in equations (1) are given by

$$A = \rho, \qquad B = d, \qquad C = EI\frac{\partial^4}{\partial x^4}, \tag{39a}$$

$$A_{L_1} = 0, \qquad B_{L_1} = 0, \qquad C_{L_1} = 1, \qquad A_{L_2} = 0, \qquad B_{L_2} = 0, \qquad C_{L_2} = \partial/\partial x, \tag{39b}$$

$$A_{R_1} = 0, \qquad B_{R_1} = 0, \qquad C_{R_1} = EI\frac{\partial^2}{\partial x^2}, \qquad A_{R_2} = m, \qquad B_{R_2} = c,$$

$$C_{R_2} = k - EI\frac{\partial^3}{\partial x^3}. \tag{39c}$$

Note that equation (38d) is an eigenvalue-dependent boundary condition. The state space matrices defined in equations (16) are of the form

$$F(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ d_0(s) & 0 & 0 & 0 \end{bmatrix}, \qquad M_b(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N_b(s) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha(s) & 0 & 0 & -EI \end{bmatrix}, \tag{40}$$

where $d_0(s) = -(\rho s^2 + ds)/EI$, and $\alpha(s) = ms^2 + cs + k$. It can be shown that the adjoint state space matrices defined in equations (21) are the same as those in equation (40). Accordingly, the system eigenfunctions in this particular case are related by

$$v_k(x) = u_{-k}(x) = \bar{u}_k(x), \qquad k = 1, 2, \ldots . \tag{41}$$

The transient response of the beam is given by equation (14), with the eigenfunction normalization coefficients and modal co-ordinates being

$$\rho_k = (2\lambda_k \rho + d) \int_0^1 u_k^2(x) dx + (2\lambda_k m + c) u_k^2(1), \qquad s_k(t) = u_k(1) \int_0^t e^{\lambda_k (t-\tau)} f_b(\tau) \, d\tau. \tag{42a, b}$$

The integrals in equations (42) are evaluated by exact quadrature, as discussed in sections 4.2 and 4.3. In the numerical simulation, the values of the system parameters are chosen as $\rho = 16$, $d = 1.6$, $EI = 1$, $m = 4$, $c = 4$ and $k = 8$. Here all the parameters are dimensionless. Listed in Table 1 are the first ten eigenvalues of the beam, which are the roots of the characteristic equation (18), and are computed by an iterative root-finding scheme with the corresponding undamped natural frequencies of the beam ($d = 0$, $c = 0$) as initial guesses.

Assume that the boundary disturbance is a decayed sinusoidal excitation

$$f_b(t) = 8 \, e^{-0.064t} \sin (45{\cdot}4t).$$

Note that the excitation frequency is close to the fifth damped frequency of the beam, $\mathrm{Im}(\lambda_5) = 45{\cdot}43$; see Table 1. Thus, it is expected that the fifth mode will be dominant in vibration. In the calculation of the beam transient response, the infinite series given in equation (14) has to be truncated. A convergence study shows that the maximum deviation between a seven-term prediction (i.e., the first seven terms taken from equation (14)) and a 30-term prediction is less than 10^{-10}. Thus, the seven-term model is accurate enough, and will be used in the computation.

Shown in Figures 2 and 3 are the spatial distributions of the beam displacement at times $t = 1{\cdot}0$ and $10{\cdot}0$, respectively, calculated by the proposed method, and the finite element method (FEM) with four-node beam elements. At $t = 1{\cdot}0$, the results obtained by both eight and 16 finite elements are in good agreement with that by the proposed method. However, as time passes, the deviation between the prediction by the finite element method and that by the proposed method grows. At $t = 10{\cdot}0$, eight finite elements lead to large numerical errors, and the 20-element prediction gets closer to that of the proposed seven-term modal expansion; see Figure 3. Further numerical simulation shows that at least 32 elements are needed in order for the finite element prediction to be in good agreement with that by the proposed method in a longer period of time; say, $0 \leqslant t \leqslant 100$. This implies that as the number of elements increases, the finite element prediction converges to the modal expansion prediction. Indeed, one advantage of the proposed method is to model the transient response of non-self-adjoint distributed systems with a few unknowns to be solved.

The displacement, slope, bending moment and shear force of the beam at the mid-span ($x = 0{\cdot}5$) are evaluated by equation (36), and plotted in Figures 4(a)–(d), for $0 \leqslant t \leqslant 5$. After a long time, the beam response eventually decays, as shown in Figure 5.

6. CONCLUSIONS

The transient response of general one-dimensional non-self-adjoint distributed systems subject to eigenvalue-dependent boundary conditions has been obtained in exact and closed form eigenfunction series. The proposed modal expansion differs from the existing modal analyses in three main aspects. First, the method systematically treats non-self-adjoint operators, eigenvalue-dependent boundary conditions, different model parameters and various disturbances in a compact spatial state space form. Second, the method obtains eigenfunction normalization coefficients and modal co-ordinates via exact and explicit quadrature. Third, the method simultaneously determines transient displacement and internal forces of the distributed system, without the need to differentiate the displacement function. Unlike many analytical approaches, the present method does not require specific derivations for specific problems, because its algorithm is the same for different systems.

The high accuracy and efficiency of the proposed method is justified in the numerical example. It is shown that if the spectra of external disturbances are narrow-banded, a few terms from the modal series are enough to deliver accurate results. Compared with the finite element method, the proposed method does not need to deal with large-scale matrices, which indicates much savings of computational efforts. This feature makes the proposed method potentially useful in the study of high frequency dynamic behaviors of distributed systems. This issue, among others, is beyond the scope of the current study, and will be deferred to a future investigation.

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