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A NOTE ON A FINITE ELEMENT FOR VIBRATING THIN, ORTHOTROPIC RECTANGULAR PLATES

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1. INTRODUCTION

A reasonable amount of finite element models has been developed for dealing with thin, orthotropic plates. Among them, the one developed by Tsay and Reddy [1] is very convenient, especially when dealing with every-day design-type problems.

On the other hand, and when dealing with vibrating, thin rectangular isotropic plates, the element developed by Bogner *et al*. [2] appears to be one of the most accurate ones, ideal for scientific, academically oriented investigations.

The present study deals with an extension of the rectangular, thin plate element developed by Bogner *et al*. in the 1960s [2] for static, elastic stability and vibration problems of thin, rectangular orthotropic plates. The essential details of the analysis are given in this note, as well as some examples which show the convenience and accuracy of the approach.

2. THE FINITE ELEMENT ORTHOTROPIC MODEL

The rectangular element referred to the local co-ordinate system x, y and the adopted local numbering system of the nodes is shown in Figure 1.

The transverse displacement w and its derivatives $\partial w/\partial x$, $\partial w/\partial y$ and $\partial^2 w/\partial x\,\partial y$ are the degrees of freedom corresponding to each node. The vector of the nodal displacements is expressed as

$$\{w_e\}^t = [w_1 \quad (\partial w/\partial x)_1 \quad (\partial w/\partial y)_1 \quad (\partial^2 w/\partial x\,\partial y)_1 \quad \cdots \quad (\partial^2 w/\partial x\,\partial y)_4]. \tag{1}$$

Now introducing the dimensionless variables $\xi = x/a$, $\eta = y/b$ and using the interpolation polynomials used in reference [2] one obtains the following shape functions:

$$\begin{aligned}
&N_1(\xi,\eta) = (2\xi^3 - 3\xi^2 + 1)(2\eta^3 - 3\eta^2 + 1), && N_9(\xi,\eta) = \xi^2\eta^2(3 - 2\xi)(3 - 2\eta),\\
&N_2(\xi,\eta) = a\xi(\xi^2 - 2\xi + 1)(2\eta^3 - 3\eta^2 + 1), && N_{10}(\xi,\eta) = a\xi^2\eta^2(\xi - 1)(3 - 2\eta),\\
&N_3(\xi,\eta) = b\eta(2\xi^3 - 3\xi^2 + 1)(\eta^2 - 2\eta + 1), && N_{11}(\xi,\eta) = b\xi^2\eta^2(3 - 2\xi)(\eta - 1),\\
&N_4(\xi,\eta) = ab\xi\eta(\xi^2 - 2\xi + 1)(\eta^2 - 2\eta + 1), && N_{12}(\xi,\eta) = ab\xi^2\eta^2(\xi - 1)(\eta - 1),\\
&N_5(\xi,\eta) = \eta^2(2\xi^3 - 3\xi^2 + 1)(3 - 2\eta), && N_{13}(\xi,\eta) = \xi^2(3 - 2\xi)(2\eta^3 - 3\eta^2 + 1),\\
&N_6(\xi,\eta) = a\xi\eta^2(\xi^2 - 2\xi + 1)(3 - 2\eta), && N_{14}(\xi,\eta) = a\xi^2(\xi - 1)(2\eta^3 - 3\eta^2 + 1),\\
&N_7(\xi,\eta) = b\eta^2(2\xi^3 - 3\xi^2 + 1)(\eta - 1), && N_{15}(\xi,\eta) = b\xi^2\eta(3 - 2\xi)(\eta^2 - 2\eta + 1),\\
&N_8(\xi,\eta) = ab\xi\eta^2(\xi^2 - 2\xi + 1)(\eta - 1), && N_{16}(\xi,\eta) = ab\xi^2\eta(\xi - 1)(\eta^2 - 2\eta + 1).
\end{aligned} \tag{2}$$

The displacement at an arbitrary point of the element is now given by

$$w(\xi,\eta) = [N]\{w_e\}. \tag{3}$$

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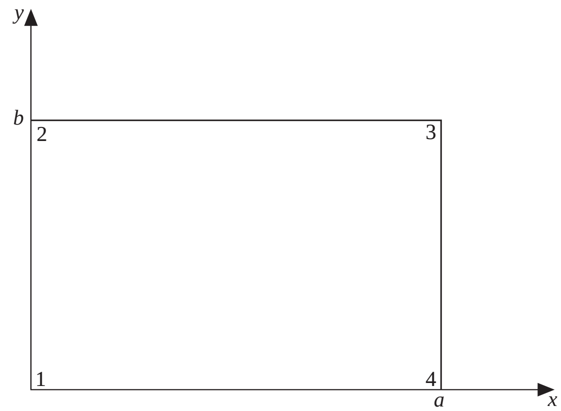

Figure 1. The finite element and the local numbering of its nodes.

Considering the case in which the directions of orthotropy coincide with the co-ordinate axes, one expresses the strain energy of the plate by [3]

$$U = \tfrac{1}{2} \iint \left\{ D_1 \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2\nu_2 D_1 \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + D_2 \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 4D_k \left(\frac{\partial^2 w}{\partial x \, \partial y} \right) \right\} \mathrm{d}x \, \mathrm{d}y, \qquad (4)$$

where

$$D_1 = \frac{E_1 h^3}{12(1 - \nu_1 \nu_2)}, \qquad D_2 = \frac{E_2 h^3}{12(1 - \nu_1 \nu_2)}, \qquad D_k = \frac{Gh^3}{12} \qquad (5)$$

and where h is the plate thickness; E_1 and E_2 are the orthotropic elasticity moduli; ν_1 and ν_2 are the orthotropic Poisson moduli ($E_1 \nu_2 = E_2 \nu_1$); and G is the transverse elasticity coefficient.

Substituting equation (3) in equation (4) and integrating over the rectangular element subdomain one obtains

$$U = \tfrac{1}{2} \{w_e\}^t \frac{D_1}{ab} [\alpha^2 [k^{(1)}] + \delta \alpha^{-2} [k^{(2)}] + \varphi [k^{(3)}] + \nu_2 [k^{(4)}]] \{w_e\}, \qquad (6)$$

where

$$\alpha = b/a, \qquad \delta = D_2/D_1, \qquad \varphi = 2D_k / D_1 \qquad (7)$$

and

$$[k^{(1)}] = \int_0^1 \int_0^1 [N_{\xi\xi}]^t [N_{\xi\xi}] \, \mathrm{d}\xi \, \mathrm{d}\eta, \qquad [k^{(2)}] = \int_0^1 \int_0^1 [N_{\eta\eta}]^t [N_{\eta\eta}] \, \mathrm{d}\xi \, \mathrm{d}\eta,$$

$$[k^{(3)}] = 2 \int_0^1 \int_0^1 [N_{\xi\eta}]^t [N_{\xi\eta}] \, \mathrm{d}\xi \, \mathrm{d}\eta, \qquad [k^{(4)}] = \int_0^1 \int_0^1 \{[N_{\xi\xi}]^t [N_{\eta\eta}] + [N_{\eta\eta}]^t [N_{\xi\xi}]\} \, \mathrm{d}\xi \, \mathrm{d}\eta. \quad (8)$$

The subscripts denote the derivatives with respect to the dimensionless spatial variables.

In the case of an isotropic plate one has

$$\nu_1 = \nu_2 = \nu, \qquad \delta = 1, \qquad \varphi = 1 - \nu, \tag{9}$$

and, accordingly, the stiffness matrix of the isotropic element is

$$[k] = \frac{D}{ab}\{\alpha^2[k^{(1)}] + \alpha^{-2}[k^{(2)}] + [k^{(3)}] + \nu[k^{(4)}] - \nu[k^{(3)}]\}, \tag{10}$$

where $D = Eh^3/12(1-\nu^2)$.

Bogner *et al*. [2] give the following expression for the generic component of the same matrix:

$$\tilde{q}_{ij} = \frac{D}{ab}[\alpha^2\tilde{\gamma}_{ij}^{(1)} + \alpha^{-2}\tilde{\gamma}_{ij}^{(2)} + \tilde{\gamma}_{ij}^{(3)} + \nu\tilde{\gamma}_{ij}^{(4)}]a^{\tilde{\lambda}_{ij}}b^{\tilde{\mu}_{ij}}. \tag{11}$$

Also, Table 6 of reference [2] contains the numerical values of $\tilde{\gamma}_{ij}^{(1)}$, $\tilde{\gamma}_{ij}^{(2)}$, $\tilde{\gamma}_{ij}^{(3)}$, $\tilde{\gamma}_{ij}^{(4)}$, $\tilde{\lambda}_{ij}$ and $\tilde{\mu}_{ij}$ for $i = 1, \ldots, 16$ and $j = 1, \ldots, i$.

Comparing equations (10) and (11) one immediately concludes that

$$k_{ij}^{(1)} = \tilde{\gamma}_{ij}^{(1)}a^{\tilde{\lambda}_{ij}}b^{\tilde{\mu}_{ij}}, \qquad k_{ij}^{(2)} = \tilde{\gamma}_{ij}^{(2)}a^{\tilde{\lambda}_{ij}}b^{\tilde{\mu}_{ij}},$$
$$k_{ij}^{(3)} = \tilde{\gamma}_{ij}^{(3)}a^{\tilde{\lambda}_{ij}}b^{\tilde{\mu}_{ij}}, \qquad k_{ij}^{(4)} = (\tilde{\gamma}_{ij}^{(3)} + \tilde{\gamma}_{ij}^{(4)})a^{\tilde{\lambda}_{ij}}b^{\tilde{\mu}_{ij}}. \tag{12}$$

Accordingly, the numerical values given in reference [2] allow for the straightforward transcription of the stiffness matrix of the orthotropic plate element. Regarding the inertia matrix, its generic component is [2]

$$m_{ij} = \rho\frac{abh}{1225}\tilde{\gamma}_{ij}^{(5)}a^{\tilde{\lambda}_{ij}}b^{\tilde{\mu}_{ij}}, \tag{13}$$

where ρ is the mass density and the values of $\tilde{\gamma}_{ij}^{(5)}$ being given in Table 6 of reference [2].

3. NUMERICAL RESULTS

In order to investigate the advantages and accuracy of the orthotropic element developed in this study, several problems were solved in cases in which exact or very

TABLE 1

The frequency coefficients of a simply supported, square, isotropic plate

Number of elements	Degrees of freedom	Ω_1	$\Omega_2 = \Omega_3$	Ω_4	$\Omega_5 = \Omega_6$
25	100	19·7403	49·4014	79·0265	99·3402
100	400	19·7393	49·3514	78·9611	98·7390
225	900	19·73922	49·3587	78·9577	98·7046
400	1600	19·739213	49·3482	78·9571	98·6988
625	2500	19·739211	49·3481	78·9569	98·6972
225*	900	19·739210	49·34806	78·9569	98·6966
400*	1600	19·739209	49·34804	78·95685	98·6962
625*	2500	19·7392089	49·348027	78·956842	98·69611
2500†	7500	19·7400	49·3513	78·9698	98·7034
Exact solution		19·7392088	49·348022	78·956835	98·69604

* Results obtained using the present element considering 1/4 of the plate.
† Results obtained using ALGOR, considering also 1/4 of the plate.

TABLE 2

The frequency coefficients of a simply supported, square, orthotropic plate ($D_2/D_1 = 0{\cdot}5$, $D_3/D_1 = 0{\cdot}5$, $\nu_2 = 0{\cdot}3$)

Number of elements	Degrees of freedom	Ω_1	Ω_2	Ω_3	Ω_4
25	100	15·6062	35·6226	44·7449	62·4856
100	400	15·6053	35·5877	44·6903	62·4249
225	900	15·60523	35·5858	44·6873	62·4217
400	1600	15·60522	35·5855	44·6868	62·4211
625	2500	15·605216	35·58543	44·6866	62·42096
225*	900	15·6052155	35·58539	44·68658	62·42091
400*	1600	15·6052150	35·585374	44·68655	62·42088
625*	2500	15·6052148	35·585369	44·686541	62·420865
2500†	7500	15·6059	35·5882	44·6888	62·4311
Exact solution		16·6052147	35·585365	44·686534	62·420859

* Results obtained using the present element considering 1/4 of the plate.
† Results obtained using ALGOR, considering also 1/4 of the plate.

accurate results were known. Three of those problems are reported herein. The eigenvector and corresponding eigenvalues were determined by the method of inverse iteration [4].

In Table 1 are depicted the lower eigenvalues $\Omega_i = \omega_i a^2\sqrt{\rho h/D}$ in the case of a simply supported square isotropic plate. The frequency coefficients have been evaluated using (1) the newly developed orthotropic plate element degenerated into the isotropic case and (2) the ALGOR system [5]. Excellent agreement with the exact eigenvalues is achieved.

In Table 2 is shown a comparison of natural frequency coefficients, $\Omega_i = \omega_i a^2\sqrt{\rho h/D_1}$, in the case of a square simply supported, orthotropic plate. The exact eigenvalues have been computed using the well known expression for the rectangular simply supported, thin orthotropic plate:

$$\Omega_{nm} = a^2\sqrt{\rho h/D_1}\omega_{nm} = \pi^2[n^4 + 2n^2m^2(a/b)^2D_3/D_1 + m^4(a/b)^4D_2/D_1]^{1/2}, \qquad (14)$$

TABLE 3

The frequency coefficients of a clamped, square, orthotropic plate ($D_2/D_1 = 0{\cdot}5$, $D_3/D_1 = 0{\cdot}5$, $\nu_2 = 0{\cdot}3$)

Number of elements	Degrees of freedom	Ω_1	Ω_2	Ω_3	Ω_4
25	100	30·0006	54·5135	68·0546	88·5513
100	400	29·9797	54·3484	67·8148	88·1860
225	900	29·9795	54·3390	67·8011	88·1647
400	1600	29·9793	54·3374	67·7988	88·1609
625	2500	29·9792	54·3370	67·7981	88·1598
225*	900	29·97919	54·3368	67·7979	88·1595
400*	1600	29·97917	54·3367	67·7977	88·1592
625*	2500	29·979169	54·33668	67·79768	88·15914
2500†	7500	29·9813	54·3434	67·8030	88·1801
Reference [6]		29·979167	54·336663	67·797655	88·159097

* Results obtained using the present element considering 1/4 of the plate.
† Results obtained using ALGOR, considering also 1/4 of the plate.

where

$$D_3 = \nu_2 D_1 + 2D_k. \tag{15}$$

Finally, in Table 3 are shown results for the case of a vibrating, thin, clamped orthotropic square plate.

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