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ANALYSIS OF AN ENGINE–MOUNT SYSTEM WITH TIME-DEPENDENT MASS AND VELOCITY MATRICES

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In this paper a formulation of a dynamic model of a flexibly supported engine is presented, in which the contribution of rotating and reciprocating parts are taken into account with accuracy. This dynamic model contains a mass matrix and a velocity matrix as periodic functions of time. The derivation of these equations of motion on the basis of Lagrange's equations is presented.

Using the formulated dynamic model, an analysis of parametric resonance phenomena for engine–mount systems is conducted. Numerical examples demonstrate the possibility of dynamic instability in some cases.

As a particular case, flexibly supported rotor systems can be treated as well using the developed dynamic model.

1. INTRODUCTION

The problem of self-excited vibrations (or dynamic instability) exists in many engineering applications. In the mathematical sense, this problem is usually associated with linear differential systems having periodic coefficients. Mechanical systems which are described by such type of equations are called linear parametrically excited systems. A significant portion of bibliography devoted to such systems can be found; see, e.g., reference [11].

In regards to rotating shaft systems possessing unequal rigidities, the problem of self-excited vibrations is widely considered; see, e.g., references [6, 4]. Here the term "unequal rigidity" means unequal flexural rigidity of the shaft.

However, such a factor as asymmetric mass distribution in a rigid shaft appears to be less studied. This factor may also induce self-excited vibrations. Here, by "asymmetric mass distribution" is implied the unequal principal mass moments of inertia in the plane of the shaft's cross-section. This factor provides a time varying contribution to the system mass matrix. An example of such shaft systems which can exhibit self-excited vibrations will be given in this study.

For such mechanical systems as piston engines, there will be an additional time-varying factor due to the piston reciprocating motion. An analysis of instability for some examples of engine–mount systems will also be given in this paper.

The application of vibration mounts (isolators) for piston engines is quite a common practice [12, 5]. Models of engine–mount systems described in the literature traditionally have constant matrices; i.e., mass M, stiffness K and viscous damping matrix C. For example, a model considered in reference [4], represented the engine as a rigid body with

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six degrees of freedom which is mounted on elastic isolators. The six natural frequencies and modes of this system were analyzed.

In references [14, 3] the authors described a technique which is aimed at the minimization of forces transmitted through engine mounts, in which the engine was also modelled as a rigid body subjected to periodic loadings. Mounts were assumed to be viscously damped; i.e., a dashpot element was present in parallel to a linear elastic element. The formulated equation of motion contained three constant (6×6) matrices M, K and C. In other words, a coupled system of six differential equations with constant coefficients was considered.

The application of the finite element procedure (see, e.g., reference [2]) to engine–mount systems leads to the formulation of the equation of motion which contains constant mass and stiffness matrices, because rotational and moving parts traditionally contribute only to the right side of equations of motion; i.e., to the loading functions.

Some finite element packages (see, e.g., reference [1]) have features that allow the treatment of engine–mount systems; however, the effects of rotating and reciprocating parts on the inertia characteristics of the system are not included.

Free and/or forced vibration responses of an engine–mount model are used to evaluate the performance of the engine–mount system, and for further consideration of optimization problems, where the response of the system is utilized as an input information. The accurate prediction of the free and/or forced responses is therefore necessary to provide reliable input information for the optimization problems.

There appear to be no attempts reported in the literature which accurately treat an internal combustion engine–mount system as a system consisting of several rotating and reciprocating rigid bodies.

Note that a system with constant matrices M, K and C in the equation of motion leads to the eigenvalue problem which yields the eigenvalues with the negative real parts (or zero parts for elastic mounts): hence the problem of instability does not arise. The system is stable, because the homogeneous solutions are decaying in the case of a damped system, or bounded for an undamped system. However when a system of differential equations with time-dependent, periodic coefficients is considered such phenomena as parametric resonance [7] (i.e., dynamic instability) can occur. In this case, homogeneous solutions can be unbounded. This is the essential difference between the model with constant matrices and the model when the time-variations of the inertia properties are included.

An analysis of steady state responses $X(t)$ of an engine–mount system described by the equation

$$M(t)\ddot{X}(t) + D(t)\dot{X}(t) + KX(t) = F(t) \tag{1}$$

is presented in reference [10]. Here matrices $M(t)$, $D(t)$ are T-periodic (6×6) mass and velocity matrices, the forcing vector function $F(t)$ is also T-periodic and the stiffness matrix K is assumed to be constant. A comparison of two models is given in reference [10] in terms of steady state responses. The second model is obtained from equation (1) by dropping the time-dependent components in matrices $M(t)$ and $D(t)$. It was obtained that in many cases when the time-dependent components are small in comparison with the constant components in matrices M and D, the difference in steady state responses betwen these two models is negligible; i.e., one can use an M, D-constant model. However, in the case of crankshafts with asymmetric mass distribution (in terms of moments of inertia), or significant contribution of the reciprocating parts (pistons), the use of the M, D-constant model may yield a different response [10]. This is the case in which it is important to include time-varying components generated by crankshaft rotation and by reciprocating pistons.

In this paper, a formulation of a dynamic model of an engine–mount system will be presented. In this dynamic model the contribution of rotating and reciprocating parts will be taken into account which leads to the formulation of a mass matrix and a velocity matrix (i.e., the matrix coefficient at the velocity vector) as periodic functions of time. Based on the developed dynamic model, an investigation of the problem of dynamic instability for an engine–mount system will be conducted.

It should be noted that the models of an engine–mount system used in references [3, 4, 14] are non-parametrically excited models, and they can be deduced from the model which is developed in this paper by neglecting certain components in the mass and velocity matrices.

2. DERIVATION OF EQUATIONS OF MOTION

The system which is under consideration consists of the following rigid bodies: (1) body A, the engine framework (including all the non-moving parts of the engine) with mass m_A and tensor of moments of inertia $\mathbf{J}_A$ taken at the mass centre (point A in Figure 1); (2) body B, the crankshaft with the mass m_B and tensor of moments of inertia $\mathbf{J}_B$ taken at the mass centre of the crankshaft (point B); (3) the pistons, each with the mass m_{pi}, $i = 1, N$. The connecting rods are not included, because their mass is assumed to be distributed between the crankshaft and the pistons. It is a customary assumption [12] to distribute the mass of the connecting rod m_{cr} into two concentrated masses m_1, m_2 using the relations

$$m_1 L_1 = m_2 L_2, \qquad m_1 + m_2 = m_{cr},$$

where L_1 is the distance from the axis of the crank joint to the centre of mass of the connecting rod, and L_2 is the distance from the axis of the piston joint to the centre of mass of the connecting rod.

One can introduce the following generalized co-ordinates for the system: displacements u_{1A}, u_{2A} and u_{3A} of the body A mass centre in a ground-based system of co-ordinates $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ (Figure 1) and angular rotations ϕ_1, ϕ_2, ϕ_3 about axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ respectively. The angular rotations are presumed to be very small, so that they can play the role of generalized co-ordinates [8].

The angular speed of the crankshaft in rotation about axis b–b is presumed to be constant (ω) with respect to the ground-based system of co-ordinates $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

One can begin [10] with a derivation of the kinetic energy of the system. For the body A, it will be

$$T_A = \tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} \omega_A \cdot \mathbf{J}_A \cdot \omega_A, \tag{2}$$

where the second term is the kinetic energy of rotational motion about the mass centre (point A), and is calculated as a double scalar product of the vector of angular velocity

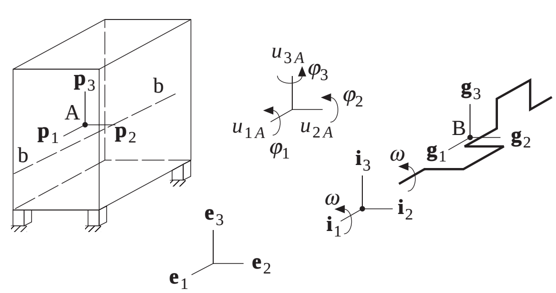

Figure 1. A schematic view of the engine framework and crankshaft.

and tensor of moments of inertia. The vector of angular velocity is expressed in terms of the generalized co-ordinates as

$$\omega_A = \dot{\phi}_1 \mathbf{e}_1 + \dot{\phi}_2 \mathbf{e}_2 + \dot{\phi}_3 \mathbf{e}_3.$$

Tensor $\mathbf{J}_A$ will also be expressed in terms of projections on to the axes parallel to $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and therefore will be time-dependent. However, assuming that the angles ϕ_1, ϕ_2, ϕ_3 are very small, it will be assumed that $\mathbf{J}_A$ is constant.

Expression (2) can be rewritten as

$$T_A = \tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2}[\dot{\phi}_1, \dot{\phi}_2, \dot{\phi}_3] J_A [\dot{\phi}_1, \dot{\phi}_2, \dot{\phi}_3]^{\mathrm{T}},$$

where J_A is the matrix of components of tensor $\mathbf{J}_A$.

The kinetic energy of body B (the crankshaft) will be

$$T_B = \tfrac{1}{2} m_B v_B^2 + \tfrac{1}{2} \omega_B \cdot \mathbf{J}_B \cdot \omega_B, \tag{3}$$

where the second term is the kinetic energy of rotational motion about the mass centre of body B. The vector of angular velocity is expressed in terms of generalized co-ordinates as

$$\omega_B = \omega \mathbf{e}_1 + \dot{\phi}_2 \mathbf{e}_2 + \dot{\phi}_3 \mathbf{e}_3.$$

The tensor $\mathbf{J}_B$ is time-dependent when expressed in terms of ground-basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Consider a set of basis vectors $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$, rigidly connected with the body B, in which the tensor of moments of inertia $\mathbf{J}_B$ has constant components

$$\mathbf{J}_B = J_{ij}^{B_0} \mathbf{g}_i \mathbf{g}_j, \tag{4}$$

where summation on repeated indices is assumed. Also introduce an auxiliary rotating set of basis vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, which has the angular velocity vector $\omega \mathbf{e}_1$. These basis vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ can be expressed in terms of the ground basis vectors as

$$\begin{bmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix},$$

where $\gamma = \omega t + \phi_0$ (angle of crankshaft rotation).

Given the angular displacements ϕ_1, ϕ_2, ϕ_3 at a given moment of time t, the basis vectors $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ rigidly connected with the body B can be expressed in terms of the ground-basis vectors as

$$\begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{bmatrix} + \begin{bmatrix} 0 & \phi_3 & -\phi_2 \\ \phi_2 \sin\gamma - \phi_3 \cos\gamma & -\phi_1 \sin\gamma & \phi_1 \cos\gamma \\ \phi_3 \sin\gamma + \phi_2 \cos\gamma & -\phi_1 \cos\gamma & -\phi_1 \sin\gamma \end{bmatrix} \right\} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

or, introducing notation T for the first matrix term and Φ for the second, one obtains

$$\mathbf{g}_i = (T_{ik} + \Phi_{ik}) \mathbf{e}_k = R_{ik} \mathbf{e}_k, \qquad i, k = 1, 2, 3, \tag{5}$$

where summation on repeated indices is assumed. Matrix R will be called the transformation matrix.

Note that components of matrix T in equation (5) are functions of γ only, whereas components Φ_{ik} are functions of angular co-ordinates ϕ_1, ϕ_2, ϕ_3 as well as of γ. Thus the basis $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ is related to the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ through matrix R:

$$R = \begin{bmatrix} 1 & \phi_3 & -\phi_2 \\ \phi_2 \sin\gamma - \phi_3 \cos\gamma & \cos\gamma - \phi_1 \sin\gamma & \sin\gamma + \phi_1 \cos\gamma \\ \phi_3 \sin\gamma + \phi_2 \cos\gamma & -\sin\gamma - \phi_1 \cos\gamma & \cos\gamma - \phi_1 \sin\gamma \end{bmatrix}.$$

The details of the derivation of the matrix R are presented in Appendix A. The contribution of components containing ϕ_1 will be neglected, because the angular displacements are assumed to be very small ($|\phi_i| \ll 1$, $i = 1, 2, 3$). Note that comparing the C-norms [9] of these functions, one obtains

$$\|\phi_1 \cos\gamma\|_c = |\phi_1 \sin\gamma\|_c = |\phi_1| \ll \|\cos\gamma\|_c = \|\sin\gamma\|_c = 1, \qquad \gamma \subset [0, 2\pi].$$

Thus the final form of the transformation matrix R will be further assumed as

$$R = \begin{bmatrix} 1 & \phi_3 & -\phi_2 \\ \phi_2 \sin\gamma - \phi_3 \cos\gamma & \cos\gamma & \sin\gamma \\ \phi_3 \sin\gamma + \phi_2 \cos\gamma & -\sin\gamma & \cos\gamma \end{bmatrix}.$$

Substituting equation (5) into equation (4), one obtains

$$\mathbf{J}_B = J_{ij}^{B_0} R_{ik} \mathbf{e}_k R_{jm} \mathbf{e}_m = R_{ik} J_{ij}^{B_0} R_{jm} \mathbf{e}_k \mathbf{e}_m. \tag{6}$$

Using the matrix form, expression (6) will correspond to

$$J_B = R^{\mathrm{T}} J^{B_0} R. \tag{7}$$

Note that matrix J_B is a function of time and, strictly speaking, of angular co-ordinates ϕ_1, ϕ_2, ϕ_3, because matrix R is a function of these variables.

Now one can rewrite the expression for the kinetic energy of the body B in matrix form. Namely,

$$T_B = \tfrac{1}{2} m_B v_B^2 + \tfrac{1}{2} [\omega, \dot{\phi}_2, \dot{\phi}_3] J_B(t) [\omega, \dot{\phi}_2, \dot{\phi}_3]^{\mathrm{T}}. \tag{8}$$

It is left to express the velocity $\mathbf{v}_B$ in terms of the generalized co-ordinates. The following vector relation holds (Figure 2):

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{AD} + \mathbf{r}_{DB}, \tag{9}$$

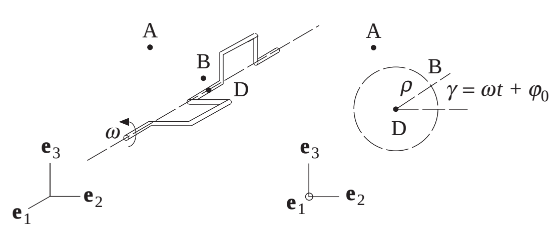

Figure 2. The determination of the velocity vector $\mathbf{v}_B$.

where

$$\mathbf{r}_A = u_{1A}\mathbf{e}_1 + u_{2A}\mathbf{e}_2 + u_{3A}\mathbf{e}_3.$$

Vectors $\mathbf{r}_{AD}$ and $\mathbf{r}_{DB}$ are represented in the co-ordinate system $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ (which is rigidly connected to body A; Figure 1) as

$$\mathbf{r}_{AD} = s_1\mathbf{p}_1 + s_2\mathbf{p}_2 + s_3\mathbf{p}_3, \qquad \mathbf{r}_{DB} = \rho\cos\gamma\,\mathbf{p}_2 + \rho\sin\gamma\,\mathbf{p}_3,$$

where point D is the projection of mass centre B on the axis of shaft rotation, and quantities s_1, s_2, s_3 are constant and assumed to be known.

Differentiating equation (9), one obtains

$$\mathbf{v}_B = \mathbf{v}_A + \dot{\mathbf{r}}_{AD} + \dot{\mathbf{r}}_{DB}. \tag{10}$$

Differentiation of $\mathbf{r}_{AD}$ yields

$$\dot{\mathbf{r}}_{AD} = \omega_A \times \mathbf{r}_{AD} = (\dot{\phi}_2 s_3 - \dot{\phi}_3 s_2)\mathbf{e}_1 + (\dot{\phi}_3 s_1 - \dot{\phi}_1 s_3)\mathbf{e}_2 + (-\dot{\phi}_2 s_1 + \dot{\phi}_1 s_2)\mathbf{e}_3,$$

where vector $\mathbf{r}_{AD}$ in the above multiplication was represented by the components s_1, s_2, s_3 in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ instead of basis $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ (the angles ϕ_1, ϕ_2, ϕ_3 are assumed to be small).

Differentiation of $\mathbf{r}_{DB}$ yields

$$\dot{\mathbf{r}}_{DB} = \omega_A \times \mathbf{r}_{DB} - \rho\omega\sin\gamma\mathbf{p}_2 + \rho\omega\cos\gamma\mathbf{p}_3.$$

Neglecting the first term and replacing basis vectors $\mathbf{p}_2$, $\mathbf{p}_3$ by $\mathbf{e}_2$, $\mathbf{e}_3$ (angles ϕ_1, ϕ_2, ϕ_3 are assumed to be small) one obtains

$$\dot{\mathbf{r}}_{DB} = -\rho\omega\sin\gamma\mathbf{e}_2 + \rho\omega\cos\gamma\mathbf{e}_3.$$

Thus from equation (10) the velocity of the mass centre of body B will be

$$\mathbf{v}_B = (\dot{u}_{1A} + \dot{\phi}_2 s_3 - \dot{\phi}_3 s_2)\mathbf{e}_1 + (\dot{u}_{2A} + \dot{\phi}_3 s_1 - \dot{\phi}_1 s_3 - \rho\omega\sin\gamma)\mathbf{e}_2 + (\dot{u}_{3A} - \dot{\phi}_2 s_1 + \dot{\phi}_1 s_2 + \rho\omega\cos\gamma)\mathbf{e}_3$$

and the kinetic energy of linear motion of body B will be

$$T_{Bl} = \tfrac{1}{2}m_B([\dot{u}_{1A} + \dot{\phi}_2 s_3 - \dot{\phi}_3 s_2]^2 + [\dot{u}_{2A} + \dot{\phi}_3 s_1 - \dot{\phi}_1 s_3 - \rho\omega\sin\gamma]^2 + [\dot{u}_{3A} - \dot{\phi}_2 s_1 + \dot{\phi}_1 s_2 + \rho\omega\cos\gamma]^2).$$

Now consider the determination of the kinetic energy of the pistons (Figure 3). It will be presumed that pistons are moving in the direction of $\mathbf{p}_3$ (Figure 1); i.e., the engine cylinders are parallel to each other. The derivation will be shown for one cylinder and then generalized for the case of several cylinders.

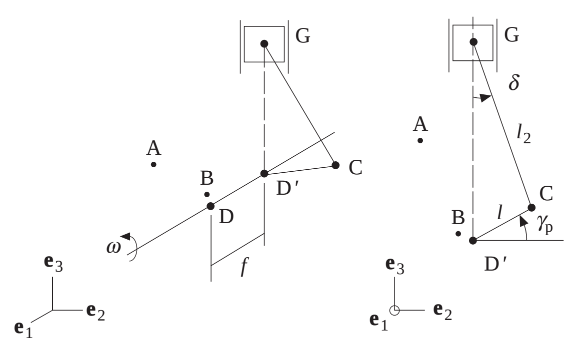

Figure 3. The determination of the kinetic energy of the pistons.

The velocity vector of point G (mass centre of the piston) can be calculated as

$$\mathbf{v}_G = \mathbf{v}_A + \omega_A \times \mathbf{r}_{AG} + v_{Gr}\mathbf{p}_3, \tag{11}$$

where v_{Gr} is the velocity of linear motion of the piston with respect to the cylinder along the vertical direction. One can find that this velocity v_{Gr} will satisfy the relation

$$v_{Gr} = l\omega \sin\gamma_p \tan\delta + l\omega \cos\gamma_p, \tag{12}$$

where $\gamma_p = \gamma + \gamma_{p0}$, γ_{p0} is the initial phase of the piston, and

$$\sin\delta = \left(\frac{l}{l_2}\right)\cos\gamma_p.$$

The radius vector $\mathbf{r}_{AG}$ can be represented as

$$\mathbf{r}_{AG} = \mathbf{r}_{AD'} + \mathbf{r}_{D'G} = (s_1 + f)\mathbf{p}_1 + s_2\mathbf{p}_2 + (s_3 + l\sin\gamma_p + l_2\cos\delta)\mathbf{p}_3, \tag{13}$$

where f is the offset of the cylinder line (point D′) with respect to point D. We introduce the notation $s_1 + f = a_1$, which will be used below.

Upon the calculation of $\omega_A \times \mathbf{r}_{AG}$ in equation (11), vector $\mathbf{r}_{AG}$ is represented by the same components as in equation (13), but in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ instead of basis $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ (the assumption about the smallness of angles ϕ_1, ϕ_2, ϕ_3 is used) and analogously the term $v_{Gr}\mathbf{p}_3$ is replaced by $v_{Gr}\mathbf{e}_3$.

Substituting relations (13) and (12) in equation (11) and taking into account the above-mentioned replacements, one obtains

$$\begin{aligned}\mathbf{v}_G = {} & (\dot{x}_A + \dot{\phi}_2(s_3 + l\sin\gamma_p + l_2\cos\delta) - s_2\dot{\phi}_3)\mathbf{e}_1 \\ & + (\dot{y}_A - \dot{\phi}_1(s_3 + l\sin\gamma_p + l_2\cos\delta) + a_1\dot{\phi}_3)\mathbf{e}_2 \\ & + (\dot{z}_A + l\omega\sin\gamma_p\tan\delta + l\omega\cos\gamma_p + s_2\dot{\phi}_1 - a_1\dot{\phi}_2)\mathbf{e}_3.\end{aligned}$$

Now, considering the case of N cylinders, we introduce the following auxiliary notations which will be used later:

$$M_p = \sum_{i=1}^{N} m_{pi}, \qquad h_i = s_3 + l\sin\gamma_{pi} + l_2\cos\delta_i, \qquad d_i = l\cos\gamma_{pi}\omega - l_2\sin\delta_i\dot{\delta}_i,$$

$$A = \sum_{i=1}^{N} m_{pi}h_i, \qquad B = \sum_{i=1}^{N} m_{pi}d_i, \qquad P = \sum_{i=1}^{N} m_{pi}h_i^2, \qquad E = \sum_{i=1}^{N} m_{pi}h_i d_i,$$

$$S = \sum_{i=1}^{N} m_{pi}a_{1i}, \qquad U = \sum_{i=1}^{N} m_{pi}a_{1i}^2, \qquad W = \sum_{i=1}^{N} m_{pi}a_{1i}h_i, \qquad L = \sum_{i=1}^{N} m_{pi}a_{1i}d_i, \tag{14}$$

where m_{pi} is the mass of the ith cylinder. The kinetic energy of all pistons will be

$$\begin{aligned}T_p = {} & \tfrac{1}{2}\sum_{i=1}^{N} m_{pi}[(\dot{x}_A + \dot{\phi}_2 h_i - s_2\dot{\phi}_3)^2 + (\dot{y}_A - \dot{\phi}_1 h_i + a_{1i}\dot{\phi}_3)^2 \\ & + (\dot{z}_A + s_2\dot{\phi}_1 - a_{1i}\dot{\phi}_2 + l\omega\sin\gamma_{pi}\tan\delta_i + l\omega\cos\gamma_{pi})^2],\end{aligned} \tag{15}$$

where the h_i were defined in equation (14).

The total kinetic energy of the system will be

$$T = T_A + T_B + T_p,$$

where T_A, T_B and T_p are given by equations (2), (3) and (15) respectively. We introduce a new notation for the generalized co-ordinates; namely,

$$q_1 = u_{1A}, \quad q_2 = u_{2A}, \quad q_3 = u_{3A}, \quad q_4 = \phi_1, \quad q_5 = \phi_2, \quad q_6 = \phi_3.$$

Using Lagrange's equations [8]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j, \qquad j = 1, 6, \tag{16}$$

one obtains the following equation in matrix form:

$$M(t)\ddot{q} + D(t)\dot{q} = Q(t), \tag{17}$$

where the inertia matrix $M(t)$ is symmetric, time-dependent and has components M_{ij} defined by

$$M_{11} = m_A + m_B + M_p, \quad M_{12} = 0, \quad M_{13} = 0,$$

$$M_{14} = 0, \quad M_{15} = m_B s_3 + A, \quad M_{16} = -m_B s_2 - M_p s_2,$$

$$M_{22} = m_A + m_B + M_p, \quad M_{23} = 0,$$

$$M_{24} = -m_B s_3 - A, \quad M_{25} = 0, \quad M_{26} = m_B s_1 + S, \quad M_{33} = m_A + m_B + M_p,$$

$$M_{34} = m_B s_2 + M_p s_2, \quad M_{35} = -m_B s_1 - S, \quad M_{36} = 0,$$

$$M_{44} = J_{11}^A + m_B(s_3^2 + s_2^2) + P + M_p s_2^2,$$

$$M_{45} = J_{12}^A - m_B s_1 s_2 - S s_2, \quad M_{46} = J_{13}^A - m_B s_1 s_3 - W,$$

$$M_{55} = J_{22}^A + m_B(s_3^2 + s_1^2) + J_{22}^B + P + U, \quad M_{56} = J_{23}^A - m_B s_2 s_3 + J_{23}^B - A s_2,$$

$$M_{66} = J_{33}^A + m_B(s_1^2 + s_2^2) + J_{33}^B + M_p s_2^2 + U, \tag{18}$$

where J_{ij}^A are the components of the tensor $\mathbf{J}_A$, and $J_{ij}^B(t)$ are the components of the tensor $\mathbf{J}_B(t)$. The expressions for J_{22}^B, J_{23}^B, J_{33}^B are

$$J_{22}^B(t) = J_{22}^{B_0}\cos^2\gamma + J_{33}^{B_0}\sin^2\gamma - J_{23}^{B_0}\sin 2\gamma,$$

$$J_{23}^B(t) = \tfrac{1}{2}(J_{22}^{B_0} - J_{33}^{B_0})\sin 2\gamma + J_{23}^{B_0}\cos 2\gamma,$$

$$J_{33}^B(t) = J_{22}^{B_0}\sin^2\gamma + J_{33}^{B_0}\cos^2\gamma + J_{23}^{B_0}\sin 2\gamma.$$

The quantities M_p, A, P, S, W and U used in equation (18) were defined in equation (14).

The matrix coefficient $D(t)$ associated with the velocity vector will be

$$D(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -B & 0 & 2E & 0 & -L \\ B & 0 & 0 & 0 & \dot{J}_{22}^B + 2E & \dot{J}_{23}^B - s_2 B \\ 0 & 0 & 0 & -L & \dot{J}_{32}^B - s_2 B & \dot{J}_{33}^B \end{bmatrix} + G(t) + C, \tag{19}$$

where the additional term $G(t)$ arises from $\partial T_B/\partial q_j$ ($j = 5, 6$) and will be presented below. The quantities B, E and L used in equation (19) were defined in equation (14). Matrix C (assumed to be constant) reflects the contribution of the viscous damping in the mounts.

Below the assumptions which were made in the process of the derivation of term $G(t)$ are presented. These assumptions allow the obtained matrices M and D to be independent of generalized co-ordinates.

In the calculation of

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_B}{\partial \dot{q}_j}\right) \qquad (j = 5, 6),$$

the assumption that the angular displacements ϕ_1, ϕ_2 and ϕ_3 are small is used, so the second term (components Φ_{ik}) in equation (5) are neglected in comparison with components T_{ik}; i.e., the transformation matrix R is assumed to be a function of γ only. However, in the calculation $\partial T_B/\partial q_j$, ($j = 5, 6$) this term (matrix Φ) is taken into account; i.e., using equation (8),

$$\frac{\partial T_B}{\partial q_j} = \tfrac{1}{2}[\omega \quad \dot{q}_5 \quad \dot{q}_6]\,\frac{\partial J_B}{\partial q_j}\,[\omega \quad \dot{q}_5 \quad \dot{q}_6]^{\mathrm{T}}, \qquad j = 5, 6,$$

where (using equation (7))

$$\frac{\partial J_B}{\partial q_j} = \frac{\partial R^{\mathrm{T}}}{\partial q_j} J^{B_0} R + R^{\mathrm{T}} J^{B_0} \frac{\partial R}{\partial q_j}, \qquad j = 5, 6,$$

and where matrix R (when it is not under differentiation) is assumed to depend only on γ. Therefore the calculation of the term $\partial T_B/\partial q_5$ yields

$$\frac{\partial T_B}{\partial q_5} = \tfrac{1}{2}(G_{11}^{(5)}\omega^2 + 2\omega G_{12}^{(5)}\dot{q}_5 + 2\omega G_{13}^{(5)}\dot{q}_6),$$

where the terms containing $\dot{q}_5\dot{q}_5$, $\dot{q}_5\dot{q}_6$ and $\dot{q}_6\dot{q}_6$ have been neglected in comparison with the terms containing $\dot{q}_5\omega$, $\dot{q}_6\omega$ and ω^2. Analogously,

$$\frac{\partial T_B}{\partial q_6} = \tfrac{1}{2}(G_{11}^{(6)}\omega^2 + 2\omega G_{12}^{(6)}\dot{q}_5 + 2\omega G_{13}^{(6)}\dot{q}_6),$$

where

$$G_{12}^{(5)} = \tfrac{1}{2}\sin 2\gamma(J_{22}^{B_0} - J_{33}^{B_0}) + J_{32}^{B_0}\cos 2\gamma, \qquad G_{13}^{(6)} = \tfrac{1}{2}\sin 2\gamma(J_{33}^{B_0} - J_{22}^{B_0}) - J_{32}^{B_0}\cos 2\gamma,$$

$$G_{13}^{(5)} = -J_{11}^{B_0} + J_{22}^{B_0}\sin^2\gamma + J_{33}^{B_0}\cos^2\gamma + J_{32}^{B_0}\sin 2\gamma,$$

$$G_{12}^{(6)} = J_{11}^{B_0} - J_{22}^{B_0}\cos^2\gamma - J_{33}^{B_0}\sin^2\gamma + J_{32}^{B_0}\sin 2\gamma.$$

Therefore, the additional matrix G, which can be called a gyroscopic matrix, will be expressed as

$$G(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega G_{12}^{(5)} & -\omega G_{13}^{(5)} \\ 0 & 0 & 0 & 0 & -\omega G_{12}^{(6)} & -\omega G_{13}^{(6)} \end{bmatrix}.$$

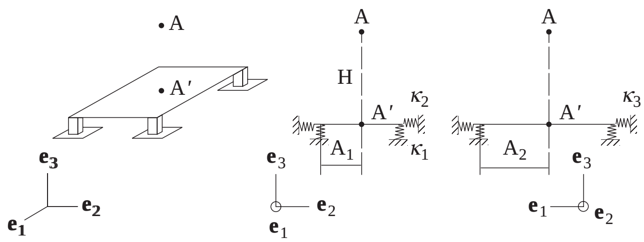

Figure 4. The assumed location of the mounts.

One can see that with an axisymmetric body rotating about the axis of symmetry, the terms $G^{(5)}_{12} = G^{(6)}_{13} = 0$ and $G^{(5)}_{13} = -G^{(6)}_{12}$, which yields a skew-symmetric addition to the velocity matrix D, but in the general case when the body is not axisymmetric this addition to the velocity matrix will not be skew-symmetric.

Now consider the generalized forces Q_j in equation (16), or vector $Q(t) = [Q_1, \ldots, Q_6]^T$ in equation (17). One contribution to these generalized forces are the forces arising from interaction of body A with the mounts (Figure 1). The location of the mounts is assumed as shown in Figure 4; i.e., with a two-plane symmetry. Point A is the mass centre of the engine framework, and k_3, k_2, k_1 are elastic springs parallel to axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ respectively. Note that some dashpot elements are in general assumed to be present as well. However, the derivation of the viscous damping matrix C is analogous to the derivation of the stiffness matrix; i.e., to obtain matrix C in the final form of the stiffness matrix, coefficients k_3, k_2, k_1 should be replaced by the corresponding dashpot coefficients.

Each of the mounts will be modelled as a combination of three elastic uniaxial elements. The relation between the generalized co-ordinates q_i and generalized forces arising from forces and moments (we denote them as vector Q^M) transmitted through the mounts to the engine framework can be written as

$$Q^M = -Kq, \tag{20}$$

where matrix K will be called the stiffness matrix. To derive this matrix, the definition of generalized forces is used. According to this definition, for a system with N points at which the external forces are applied, the generalized force will be [8]

$$Q_j = \sum_{k=1}^{N} \frac{\partial r_k}{\partial q_j} \cdot F_k, \tag{21}$$

where r_k is the radius vector of the point k, and F_k is the external force vector acting at the point k.

Given generalized co-ordinates $q = [q_1, \ldots, q_6]$, the following resulting forces and moments (with respect to the axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) are transmitted from the mounts to the engine framework:

$$F_x = -4k_3 q_1 + 4k_3 H q_5, \qquad F_y = -4k_2 q_2 - 4k_2 H q_4, \qquad F_z = -4k_1 q_3,$$

$$M_{xA} = -4k_2 H q_2 - 4(k_2 H^2 + k_1 A_1^2) q_4, \qquad M_{yA} = -4(k_1 A_2^2 + k_3 H^2) q_5 + 4k_3 H q_1,$$

$$M_{zA} = (-4k_2 A_2^2 - 4k_3 A_1^2) q_6,$$

where the index A means that the moments are calculated with respect to the axes passing through the point A. Now one can calculate the generalized forces contributed from the mounts using equation (21); namely, for this case the components of the vector Q^M will be

$$Q_1^M = F_x, \qquad Q_2^M = F_y, \qquad Q_3^M = F_z,$$

$$Q_4^M = M_{xA}, \qquad Q_5^M = M_{yA}, \qquad Q_6^M = M_{zA}.$$

From these expressions, the stiffness matrix K mentioned in equation (20) can be obtained. It will be symmetric with the components

$$K_{11} = 4k_3, \qquad K_{12} = K_{13} = K_{14} = K_{16} = 0, \qquad K_{15} = -4k_3 H,$$

$$K_{22} = 4k_2, \qquad K_{23} = K_{25} = K_{26} = 0, \qquad K_{24} = 4k_2 H,$$

$$K_{33} = 4k_1, \qquad K_{34} = K_{35} = K_{36} = 0,$$

$$K_{44} = 4(k_2 H^2 + k_1 A_1^2), \qquad K_{45} = K_{46} = 0,$$

$$K_{55} = 4(k_1 A_2^2 + k_3 H^2), \qquad K_{56} = 0, \qquad K_{66} = 4k_2 A_2^2 + 4k_3 A_1^2. \tag{22}$$

Note that the generalized force contribution from gravity does not depend upon the generalized co-ordinates. The same situation exists regarding the generalized force contributions from the gas pressure in the cylinders, except that they will be functions of time. Their derivation is straightforward and is omitted here. Also note that modelling of mounts in the form of springs (bar elements) is not a unique approach which can be appropriate. In general, the components of stiffness matrix K can be obtained through a finite element model, or can be obtained analytically, but using other element; e.g., beam elements. Also, the assumption of two-plane symmetry (Figure 4) can be dropped.

Now having collected the terms resulting from equation (16) which depend on generalized co-ordinates, velocities and accelerations on the left side of the equations and having transferred all the other terms to the right side, the equation of motion of the engine–mount system in matrix form can be written as

$$M(t)\ddot{q} + D(t)\dot{q} + Kq = F(t), \tag{23}$$

where matrices M, D and K were given respectively by equations (18), (19) and (22). Note that matrices M and D are periodic functions of time: $M(t) = M(t+T)$, $D(t) = D(t+T)$ and $T = 2\pi/\omega$.

The right side of equation (23) will be

$$F(t) = [F_1(t) \quad F_2(t) \quad \cdots \quad F_6(t)]^{\mathrm{T}},$$

where

$$F_1(t) = \sum_{j=1}^{N_{cyl}} F_{jx}^{gas}(t), \qquad F_2(t) = m_B \rho \omega^2 \cos \omega t + \sum_{j=1}^{N_{cyl}} F_{jy}^{gas}(t),$$

$$F_3(t) = m_B \rho \omega^2 \sin \omega t - (m_A + m_B + M_p) g - \sum_{i=1}^{N_{cyl}} m_{pi} \dot{v}_{Gri} + \sum_{j=1}^{N_{cyl}} F_{jz}^{gas}(t),$$

$$F_4(t) = -m_B \rho s_3 \omega^2 \cos \omega t + m_B \rho s_2 \omega^2 \sin \omega t - \sum_{i=1}^{N_{cyl}} m_{pi} s_2 \dot{v}_{Gri} + \sum_{j=1}^{N_{cyl}} M_{xA}[F_j^{gas}(t)],$$

$$F_5(t) = \tfrac{1}{2} G_{11}^{(5)} \omega^2 - \dot{J}_{12}^B(t)\omega - m_B \rho s_1 \omega^2 \sin \omega t + \sum_{i=1}^{N_{cyl}} m_{pi} a_{1i} \dot{v}_{Gri} + \sum_{j=1}^{N_{cyl}} M_{yA}[F_j^{gas}(t)],$$

$$F_6(t) = \tfrac{1}{2} G_{11}^{(6)} \omega^2 - \dot{J}_{13}^B(t)\omega + m_B \rho s_1 \omega^2 \cos \omega t + \sum_{j=1}^{N_{cyl}} M_{zA}[F_j^{gas}(t)], \quad (24)$$

where v_{Gri} is given by equation (12), and

$$\tfrac{1}{2} G_{11}^{(5)} = J_{12}^{B_0} \sin \gamma + J_{13}^{B_0} \cos \gamma, \qquad \tfrac{1}{2} G_{11}^{(6)} = -J_{12}^{B_0} \cos \gamma + J_{13}^{B_0} \sin \gamma.$$

F_{jx}^{gas}, F_{jy}^{gas} and F_{jz}^{gas} are the X, Y and Z projections of the resulting gas force in the jth cylinder, and $M_{xA}[F_j^{gas}]$, $M_{yA}[F_j^{gas}]$ and $M_{zA}[F_j^{gas}]$ are the resulting moments of the gas forces in the jth cylinder with respect to the axes X, Y and Z passing through the point A. The expressions for $\dot{J}_{12}^B$, $\dot{J}_{13}^B$ are obtained by differentiation of

$$J_{12}^B = J_{12}^{B_0} \cos \gamma - J_{13}^{B_0} \sin \gamma, \qquad J_{13}^B = J_{12}^{B_0} \sin \gamma + J_{13}^{B_0} \cos \gamma.$$

Remark 1. Note that forces and moments related with the gas pressure should be only taken into account when there is a gradient of pressure in the combustion chamber. If the pressure is uniformly distributed inside the chamber at each instant of time, then the resulting force is zero at each instant of time, and all terms with superscript "gas" in the right side of equation (24) vanish.

Remark 2. Note that for some engines, e.g., slow, or medium speed marine engines, there is an intermediate shaft which joins the crankshaft and the propeller. In the dynamic model considered here the interaction between the crankshaft and the intermediate shaft is reduced to a torque moment (M_x) upon the assumption that a flexible coupling will minimize all other forces and moments of interaction. Therefore its contributions to the generalized forces are neglected. The moment M_x does not contribute into the generalized forces (21), because the rotation angle of the crankshaft is prescribed $\gamma = \omega t + \phi_0$; i.e., γ is not considerd as a generalized co-ordinate.

3. INVESTIGATION OF PARAMETRIC RESONANCE

Consider a general equation

$$M(t)\ddot{q} + D(t)\dot{q} + Kq = 0, \quad (25)$$

where matrices M and D are of order n, and T-periodic, and q is $n \times 1$ vector. In general, K can be T-periodic as well.

A general theory of linear differential equations with periodic coefficients [7] allows a treatment of equations like equation (25). To investigate parametric resonance phenomena (dynamic instability) it is sufficient to consider the conditions that give rise to unbounded solutions $q(t)$ of equation (25).

Equation (25) can be represented in equivalent state space form as

$$A(t)\dot{X} + B(t)X = 0,$$

where

$$A = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}, \qquad X = \begin{bmatrix} q \\ \dot{q} \end{bmatrix},$$

or, in abbreviated form,

$$\dot{X} = P(t)X, \tag{26}$$

where $P = -A^{-1}B$, a $2n \times 2n$ matrix, and $P(t+T) = P(t)$. Note also that matrix A is non-singular, upon the presumption that the mass matrix M is non-singular for all $t \subset [0, T]$.

Now consider X as a $2n \times 2n$ matrix as well. Assuming the initial conditions $X(0) = I$, the matrix function $X(t)$ will be called a matrix of fundamental solutions (or matrizant), and the matrix $X(T)$ will be called the monodromy matrix.

According to the Floquet–Lyapunov theorem [7], the matrix of fundamental solutions can be expressed as

$$X(t) = S(t)\,\mathrm{e}^{tR},$$

where $S(0) = I$, $R = (1/T)\ln X(T)$, $S(t+T) = S(t)$. Eigenvalues of R, α_i, are called characteristic exponents, and eigenvalues of $X(T)$, β_i, are called multipliers of the system. Note the relation between them:

$$\beta_i = \mathrm{e}^{\alpha_i T}, \qquad i = 1, 2n.$$

Therefore $X(T)$, the monodromy matrix (namely its eigenvalues), yields all of the information required to analyze the stability of trivial solutions of equation (26), and consequently of equation (25), namely [7]: (1) all solutions bounded on $[0, \infty]$, if multipliers are inside, or on the unit circle (the latest case with simple elementary divisors); (2) asymptotic stability, if multipliers are inside of the unit circle; (3) instability of the solution, if at least one multiplier is either outside the unit circle, or on the unit circle with a multiple elementary divisor.

There are several methods to obtain regions of stability and instability for systems described by differential equations with periodic coefficients. For example, for the Mathieu equation such methods as straightforward expansion (in power series of the parameter), the method of strained parameters, Whittakers's method and the method of multiple scales are described in the literature [11, 7, 13].

In this study it is proposed to conduct the stability analysis by obtaining the eigenvalues of the monodromy matrix $X(T)$. Note that in the case of a general system of coupled differential equations with periodic coefficients determination of the matrix of fundamental solutions $X(t)$ is not a trivial task. Here it is proposed to conduct a numerical integration on the interval $[0, T]$, for equation (26) to obtain $X(T)$, starting from $X(0) = I$. An explicit four-stage Runge–Kutta method will be used as a numerical integrator.

At first, for illustration purposes, a rotor–mount system with a single degree of freedom is considered.

3.1. NUMERICAL RESULTS FOR A SDOF SYSTEM

An example of a mechanical system is shown in Figure 5, where there is a rotor (body B) rotating about the axis *b–b* and the framework (body A) is allowed to rotate about axis

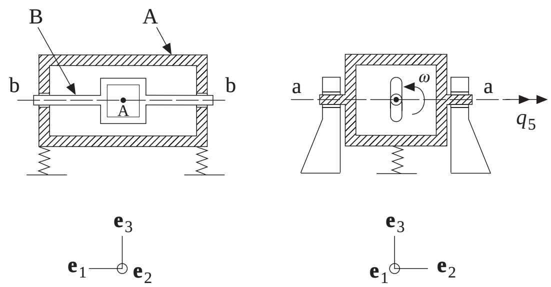

Figure 5. An example of a SDOF system.

a–a which passes through the mass centre of the framework (point A). The angle of rotation q_5 is assumed to be very small. The framework (body A) is attached to the ground by elastic springs which create the rotational stiffness denoted by K_{55}.

The equation of motion for this system is obtained from the general equation (23) by constraining all degrees of freedom except q_5 and assuming that $M_p = 0$ (no pistons). Assume also that the mass center of body A coincides with point D (see Figure 2). Recall that point D is the projection of the mass centre of body B on the axis of the crankshaft (in this example rotor) rotation.

Then the motion of this system (with forcing functions set to zero) can be described by the following equation:

$$(1 - \varepsilon \sin^2 \omega t)\ddot{q}_5 - \tfrac{3}{2}\varepsilon\omega \sin 2\omega t \dot{q}_5 + \lambda^2 q_5 = 0, \tag{27}$$

with parameters

$$\varepsilon = \frac{J_{22}^{B_0} - J_{33}^{B_0}}{J_{22}^{A} + J_{22}^{B_0}}, \qquad \lambda = \sqrt{K_{55}/(J_{22}^{A} + J_{22}^{B_0})}, \tag{28}$$

where $J_{22}^{B_0}$ and $J_{33}^{B_0}$ are components of the tensor of moments of inertia defined with respect to the axes $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ rigidly connected to the body B. From equation (28) it follows that $\varepsilon < 1$, because $J_{22}^{B_0} > 0$, $J_{33}^{B_0} > 0$ and $J_{22}^{A} \geqslant 0$.

The rigidly connected system $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ is chosen in the following way. The axis $\mathbf{g}_1$ is taken parallel to $\mathbf{e}_1$. As far as, $\mathbf{g}_2$, $\mathbf{g}_3$ are concerned, they are taken in such a way that the components $J_{23}^{B_0} = J_{32}^{B_0} = 0$. Without loss of generality, it also assumed that, at the instant $t = 0$, bases $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ coincide.

Note that the unbalance radius ρ that contributes to the external forcing function does not appear in equation (27), because it is not involved in consideration of the homogeneous solutions.

One can rewrite equation (27) using a new variable: $\gamma = \omega t$; thus $t = \gamma/\omega$: i.e., t can be considered as function of γ, $t = t(\gamma)$. Then $q_5(t) = q_5(\gamma/\omega) = \tilde{q}_5(\gamma)$ and

$$\tilde{q}_5'(\gamma) = \dot{q}_5 \frac{1}{\omega}, \qquad \tilde{q}_5''(\gamma) = \ddot{q}_5 \frac{1}{\omega^2}.$$

Thus

$$\dot{q}_5 = \omega \tilde{q}_5'(\gamma), \qquad \ddot{q}_5 = \omega^2 \tilde{q}_5''(\gamma).$$

Substituting this into equation (27), one obtains

$$(1 - \varepsilon \sin^2 \gamma)\omega^2 \tilde{q}_5''(\gamma) - \tfrac{3}{2}\varepsilon\omega^2 \sin 2\gamma \tilde{q}_5'(\gamma) + \lambda^2 \tilde{q}_5(\gamma) = 0.$$

Denoting $\delta = \lambda^2/\omega^2$, one can write

$$(1 - \varepsilon \sin^2 \gamma)\tilde{q}_5''(\gamma) - \tfrac{3}{2}\varepsilon \sin 2\gamma \tilde{q}_5'(\gamma) + \delta \tilde{q}_5(\gamma) = 0. \quad (29)$$

Note that the coefficients of this equation have a period π. This equation was numerically integrated and the monodromy matrix $X(\pi)$ (2×2 in state space form) was obtained. Then its eigenvalues (multipliers) were evaluated. Regions of stability and instability (dark areas) are presented in the plane of the non-dimensional parameters δ and ε in Figure 6. For the unstable (dark) area the absolute value of one of the multipliers was >1.

One can see that the greater the parameter $|\varepsilon|$, which reflects nonsymmetry of the body B, the wider the frequency region is where resonance can occur. Note that the line $\delta = 1$ corresponds to the frequency rotation $\omega = \lambda$, where λ is the natural frequency of the system.

3.2. NUMERICAL RESULTS FOR AN ENGINE–MOUNT SYSTEM

The parametric resonance analysis of a six-degree-of-freedom engine–mount system will be conducted on the basis of equations (23), where the right side is assumed to be zero, because homogeneous solutions are of interest in this section. The equations (23) will be represented in state space form (26). Thus there will be a system of 12 coupled differential equations with periodic coefficients.

The input data for an example of an engine–mount system (a medium-speed diesel type multi-cylinder engine was considered [5]) are presented in Table 1. It is assumed in this section that no damping properties are present in the system which means that the viscous damping matrix $C = 0$. This represents the case when parametric resonance arises more easily. The stiffnesses of the springs (Figure 4) were assumed to be the same: $k_1 = k_2 = k_3 = 0{\cdot}206E + 07$ N/m.

Without loss of generality, at the instant when $t = 0$, it is assumed that the unbalance radius has zero phase, i.e., it is oriented as the axis Y (the vector $\mathbf{e}_2$ in Figure 2). The phase angles of the pistons are determined according to the configuration of the cranks and with respect to the unbalance radius direction. The components of $J_{ij}^{B_0}$ are calculated in the co-ordinate system (fixed with the crankshaft) parallel to the reference system (at $t = 0$)

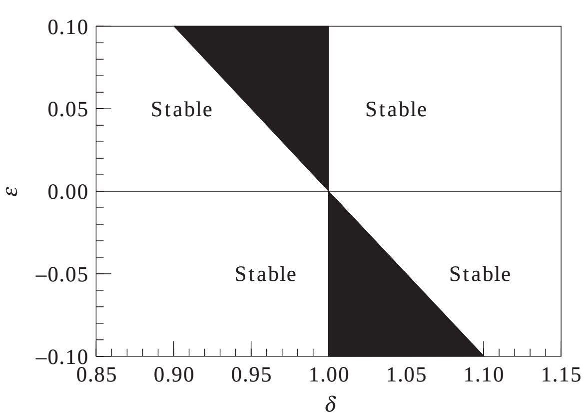

Figure 6. Regions of stability and instability for the SDOF system (Figure 5).

TABLE 1

Input parameters of the engine–mount system. Units: length [m], *mass* [kg], *angle* [degree], J^A_{ij}, $J^{B_0}_{ij}$ [kg m^2]

m_A	m_B	s_1	s_2	s_3	l	l_2	N_{cyl}	m_p	A_1	A_2	H
12 200	4100	0	0	0	0·2286	1·162	6	81	0·72	1·8	1·2
f_1	f_2	f_3	f_4	f_5	f_6	γ_{po1}	γ_{po2}	γ_{po3}	γ_{po4}	γ_{po5}	γ_{po6}
1·6	0·96	0·32	−0·32	−0·96	−1·6	0	−120	−240	−240	−120	0
J^A_{11}	J^A_{12}	J^A_{13}	J^A_{22}	J^A_{23}	J^A_{33}	$J^{B_0}_{11}$	$J^{B_0}_{12}$	$J^{B_0}_{13}$	$J^{B_0}_{22}$	$J^{B_0}_{23}$	$J^{B_0}_{33}$
12 000	0	0	17 100	0	17 100	300	0	0	8500	0	8500

and taken at the crankshaft's centre of mass. The components of J_A are taken at the centre of mass (point A) and in the system parallel to the reference system (they are assumed constant).

A numerical integration on the interval [0, T], for the equation of motion represented in the state space form (26) was produced and $X(T)$ (12×12 monodromy matrix) was obtained. An explicit four-stage Runge–Kutta method was used as a numerical integrator. The eigenvalues β_i, $i = 1$, 12, of the monodromy matrix are presented below for two examples.

In the first example (the crankshaft with $J^{B_0}_{22} = J^{B_0}_{33}$) all eigenvalues (Table 2) are on the unit circle and not repeated, which means a stable case; i.e., the homogeneous solutions of equation (23) are bounded.

For the second example, just one parameter was changed in the input data (Table 1); namely, a crankshaft with $J^{B_0}_{33} = 5667$ kg m^2 (asymmetry ratio $J^{B_0}_{22}/J^{B_0}_{33} = 1{\cdot}5$) was considered which gives a greater contribution of the time-dependent components of matrices M and D (see equations (18) and (19)). For this example one can see that absolute values of some eigenvalues are greater than 1 (outside the unit circle), which means instability, eventually leading to unboundness of the homogeneous solutions. In other words, parametric resonance arises. The numerical integrations were produced with different step sizes to verify the obtained values of the monodromy matrix $X(T)$, and consequently its

TABLE 2

Eigenvalues of monodromy matrix (*multipliers*); *stable and unstable cases*

First example			Second example		
Real part	Imaginary part	Absolute value	Real value	Imaginary part	Absolute value
0·9628309	0·2701047	0·9999999	1·2783248	0·0000000	1·2783248
0·9628309	−0·2701047	0·9999999	−0·6836327	0·7298257	0·9999999
−0·6058155	0·7956047	0·9999999	−0·6836327	−0·7298257	0·9999999
−0·6058155	−0·7956047	0·9999999	−0·9816487	0·1906977	1·0000000
−0·9852860	0·1709135	0·9999999	−0·9816487	−0·1906977	1·0000000
−0·9852860	−0·1709135	0·9999999	0·7822736	0·0000000	0·7822736
0·2286707	0·9735036	0·9999999	0·2286707	0·9735036	0·9999999
0·2286707	−0·9735036	0·9999999	0·2286707	−0·9735036	0·9999999
−0·2279639	0·9736695	1·0000000	−0·2279639	0·9736695	1·0000000
−0·2279639	−0·9736695	1·0000000	−0·2279639	−0·9736695	1·0000000
−0·6012910	0·7990300	0·9999999	−0·6012910	0·7990300	0·9999999
−0·6012910	−0·7990300	0·9999999	−0·6012910	−0·7990300	0·9999999

eigenvalues (multipliers). The results in Table 2 are presented for the engine's rotation frequency, 5·5 Hz ($T = 0{\cdot}18182$ s).

The influence of the asymmetry parameter of the crankshaft,

$$\varepsilon = \frac{J_{22}^{B_0} - J_{33}^{B_0}}{J_{22}^{B_0}},$$

on the dynamic instability of the system was then investigated. Namely, the component $J_{22}^{B_0}$ was fixed equal to 8500 kg m^2 (see Table 1), and component $J_{33}^{B_0}$ was changed. Three values of the parameter ε were chosen, $\varepsilon = 0$; 1/5 and 1/3.

The frequency range [0, 10] Hz was numerically tested in terms of evaluation of the maximum value of the multipliers for a discrete set of frequencies. The frequency increment was equal to 0·025 Hz. For each frequency the maximum absolute value of the multipliers (which will be further referred to as the "instability factor") was computed. We denote this function as $\xi(\omega)$, so according to the mentioned definition the instability factor is

$$\xi(\omega) = \operatorname*{Max}_{i=1,12} \|\beta_i(\omega)\|.$$

With $\varepsilon = 0$ the numerical results were $\xi(\omega) = 1$ for any $\omega \subset [0, 10]$; i.e., the system is stable. The obtained graphs of $\xi(\omega)$ for $\varepsilon = 1/5$ and 1/3 are presented in Figure 7, where portions of the interval [0, 10] not shown have $\xi(\omega) = 1$.

One can see that the greater the parameter ε (i.e., the greater contribution of the time-dependent components in matrices M and D) the greater is the instability factor. The width of instability intervals increases with the increase of ε as well. The homogeneous solutions for these instability intervals will be unbounded.

As an illustration of the character of an unbounded homogeneous solution, the angular displacement q_6 (or ϕ_3) as a function of time is shown in Figure 8. For this example, zero initial conditions, except for $\dot{q}_6(0) = \dot{\phi}_3(0) = 1$, were prescribed. This graph corresponds to the maximum instability factor $\xi = 1{\cdot}2808$ at the frequency $\omega = 5{\cdot}4836$ Hz, and $\varepsilon = 1/3$ (see Figure 7).

Note that the quantity ε (the asymmetry parameter of the crankshaft) can be particularly large in the case of small engines (with one, or two cylinders), because of their arrangement of cranks. Also note that the reciprocating parts (pistons) also yield contributions to time-dependent components of the matrices M and D, hence also can affect the instability factor. The investigation of this effect will be a subject of future analysis.

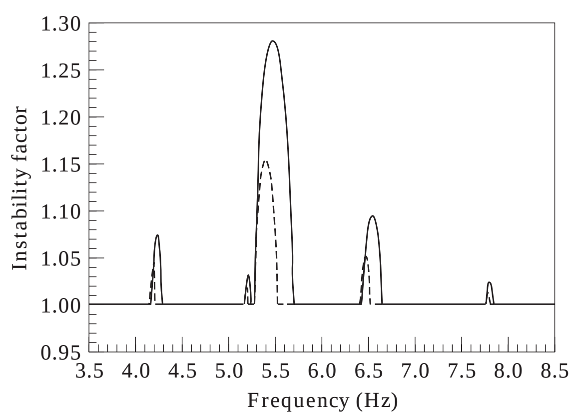

Figure 7. Instability factors: - - - -, $\varepsilon = 1/5$; ——, $\varepsilon = 1/3$.

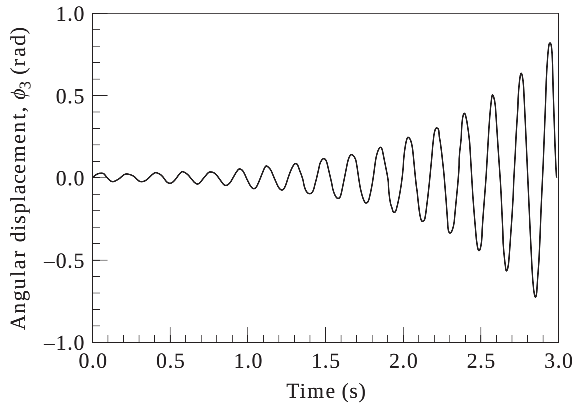

Figure 8. The angular displacement ϕ_3 of the unbounded homogeneous solution; instability factor $\xi = 1{\cdot}2808$.

4. SUMMARY

A formulation of a dynamic model of an engine–mount system has been developed. The formulated equation of motion contains the time-periodic mass and velocity matrices, which may lead to such phenomena as parametric resonance (dynamic instability). The investigation of the parametric resonance conditions on an example of an engine–elastic mount system has shown that in the case of a specific crankshaft (when certain diagonal components of the tensor of moments of inertia are non-equal), instability can occur at certain rotation frequencies.

Application of the developed model to some rotor–mount systems is also possible. The numerical results for a particular case of a rotor–mount system have shown that there is a domain in the space of system's parameters which yields dynamic instability

In general, one can say that when time-dependent components in the mass and velocity matrices become larger, then more possibilities exist for instability to take place. A full parametric analysis of the effect of different input parameters on the system's stability can be viewed as future work.

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APPENDIX A: DERIVATION OF MATRIX *R*

The vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ of the auxiliary rotating basis (Figure 1) are expressed in terms of the ground-basis vectors as

$$\begin{bmatrix}\mathbf{i}_1\\ \mathbf{i}_2\\ \mathbf{i}_3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\gamma & \sin\gamma\\ 0 & -\sin\gamma & \cos\gamma\end{bmatrix}\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{bmatrix}, \qquad (30)$$

where $\gamma=\omega t+\phi_0$ (angle of crankshaft rotation).

Assume that at the moment t (or for the angle γ) the engine framework has the angular displacements ϕ_1, ϕ_2, ϕ_3; then the basis $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ (rigidly fixed with the crankshaft) is actually the basis $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, which is given the angular displacements ϕ_1, ϕ_2, ϕ_3; i.e.,

$$\mathbf{g}_k=\mathbf{i}_k+\sum_{n=1}^{3}\Delta\mathbf{i}_{kn}, \qquad k=1,2,3, \qquad (31)$$

where $\Delta\mathbf{i}_{kn}$ is the change to the $\mathbf{i}_k$ basis vector due to its rotation about the basis vector $\mathbf{e}_n$. These changes are computed as follows (note that angles ϕ_1, ϕ_2, ϕ_3 are assumed to be very small):

$$\Delta\mathbf{i}_{11}=\phi_1\mathbf{e}_1\times\mathbf{i}_1=\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ \phi_1 & 0 & 0\\ 1 & 0 & 0\end{vmatrix}=0,$$

and, analogously,

$$\Delta\mathbf{i}_{12}=\phi_2\mathbf{e}_2\times\mathbf{i}_1=\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ 0 & \phi_2 & 0\\ 1 & 0 & 0\end{vmatrix}=-\phi_2\mathbf{e}_3,$$

$$\Delta\mathbf{i}_{13}=\phi_3\mathbf{e}_3\times\mathbf{i}_1=\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ 0 & 0 & \phi_3\\ 1 & 0 & 0\end{vmatrix}=\phi_3\mathbf{e}_2,$$

$$\Delta\mathbf{i}_{21}=\phi_1\mathbf{e}_1\times\mathbf{i}_2=\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ \phi_1 & 0 & 0\\ 0 & \cos\gamma & \sin\gamma\end{vmatrix}=\phi_1\cos\gamma\,\mathbf{e}_3-\phi_1\sin\gamma\,\mathbf{e}_2,$$

$$\Delta \mathbf{i}_{22} = \phi_2 \mathbf{e}_2 \times \mathbf{i}_2 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & \phi_2 & 0 \\ 0 & \cos\gamma & \sin\gamma \end{vmatrix} = \phi_2 \sin\gamma\, \mathbf{e}_1,$$

$$\Delta \mathbf{i}_{23} = \phi_3 \mathbf{e}_3 \times \mathbf{i}_2 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & 0 & \phi_3 \\ 0 & \cos\gamma & \sin\gamma \end{vmatrix} = -\phi_3 \cos\gamma\, \mathbf{e}_1,$$

$$\Delta \mathbf{i}_{31} = \phi_1 \mathbf{e}_1 \times \mathbf{i}_3 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \phi_1 & 0 & 0 \\ 0 & -\sin\gamma & \cos\gamma \end{vmatrix} = -\phi_1 \sin\gamma\, \mathbf{e}_3 - \phi_1 \cos\gamma\, \mathbf{e}_2,$$

$$\Delta \mathbf{i}_{32} = \phi_2 \mathbf{e}_2 \times \mathbf{i}_3 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & \phi_2 & 0 \\ 0 & -\sin\gamma & \cos\gamma \end{vmatrix} = \phi_2 \cos\gamma\, \mathbf{e}_1,$$

$$\Delta \mathbf{i}_{33} = \phi_3 \mathbf{e}_3 \times \mathbf{i}_3 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & 0 & \phi_3 \\ 0 & -\sin\gamma & \cos\gamma \end{vmatrix} = \phi_3 \sin\gamma\, \mathbf{e}_1.$$

Therefore, substituting the obtained $\Delta \mathbf{i}_{ik}$ in equation (31), one can rewrite it in matrix form, as

$$\begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{bmatrix} + \begin{bmatrix} 0 & \phi_3 & -\phi_2 \\ \phi_2 \sin\gamma - \phi_3 \cos\gamma & -\phi_1 \sin\gamma & \phi_1 \cos\gamma \\ \phi_3 \sin\gamma + \phi_2 \cos\gamma & -\phi_1 \cos\gamma & -\phi_1 \sin\gamma \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix},$$

and substituting equation (30) in the above expression one obtains

$$\begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{bmatrix} = \begin{bmatrix} 1 & \phi_3 & -\phi_2 \\ \phi_2 \sin\gamma - \phi_3 \cos\gamma & \cos\gamma - \phi_1 \sin\gamma & \sin\gamma + \phi_1 \cos\gamma \\ \phi_3 \sin\gamma + \phi_2 \cos\gamma & -\sin\gamma - \phi_1 \cos\gamma & \cos\gamma - \phi_1 \sin\gamma \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

or, in abbreviated form,

$$\mathbf{g}_i = R_{ik} \mathbf{e}_k, \qquad i = 1, 2, 3,$$

where the matrix R will be called the transformation matrix.