



THE ANGULAR RESPONSE OF A PARAMETRIC ARRAY: ANALYTICAL SOLUTION

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A theoretical treatment is formulated for the angular dependence of an absorption-limited parametric array. This allows analytical results for the off-axis far field amplitude and phase responses to be readily obtained. Experimental results show good agreement with the theory.

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1. INTRODUCTION

Berkhay [1] investigated the absorption-limited parametric array using Westervelt's source density method, to describe the self-modulation effects of pulsed sound signal. He developed an expression for the on-axis, far field, time dependent transient pressure produced by the parametric array. He pointed out that a primary carrier, with amplitude modulated by an envelope $f(t)$, produces a secondary signal in which the pattern is described with $\partial^2(f^2)/\partial t^2$. This phenomenon is called self-demodulation.

Since Berkhay's theoretical explanation of self-demodulation, a number of experiments have been carried out. Moffett and co-workers [2] experimentally verified the existence of transient signals generated during the propagation of large-amplitude pulses. Their measurements of the transient signal were performed along the axis of propagation of a 10 MHz pulse in carbon tetrachloride (CCl_4). They also experimentally investigated the angular dependence of the transient parametric array generated by 20 MHz pulses in water [3].

A theoretical investigation of the angular dependence of the transient parametric array by using a frequency domain method, temporal Fourier decomposition, was reported by Moffett and Mellen [4]. Later, Rolfe [5] concluded that, for a spherical primary sound field in which the directivity follows a Gaussian Law, the secondary sound field is represented by the convolution of an input source function $\partial^2(f^2)/\partial t^2$, given by Berkhay, with the impulse response of the parametric array. Pace and Ceen [6] introduced a spatial impulse response function of the parametric array which contains all geometry information. They also showed that the secondary pressure waveform is a convolution of the input source function $\partial^2(f^2)/\partial t^2$ with the spatial response function of the parametric array. In particular, they investigated the transient parametric array where the primary field was discontinuously terminated. Stepanishen and Koenigs [7] employed a time-dependent Green function approach to present a formula for the far field secondary pressure as a time convolution of a source function $\partial^2(f^2)/\partial t^2$ with a spatial and time-dependent impulse response. The impulse response is shown to be a convolution of an aperture-dependent impulse response of the planar projector with the impulse response

of a shaded line array. More generally, Cervenka and Alais [8], using Fourier formalism, presented a theoretical evaluation that gives the analytical expression of the secondary far field generated by self-demodulation of a primary beam in which the space-time spectrum is narrow. This theoretical result is a temporal convolution of the response of a source function $\partial^2(f^2)/\partial t^2$ with another function that depends on the transducer shape and the structure of the interaction zone.

More recently, Averkiou *et al.* [10] investigated the self-demodulation of pulsed sound beams in a thermoviscous fluid by theory and experiment. The theory was based on the Khokhlov–Zabolotskya–Kuznetov (KZK) non-linear parabolic wave equation. Their attention was devoted to the case in which the absorption is sufficiently strong that the non-linear interaction is relatively weak and restricted to the near field of sound beam. Frøysa [11] again used the KZK non-linear parabolic wave equation and analyzed the weakly non-linear propagation of a pulsed sound beam generated by a real sound source in a homogeneous fluid. Both analytical and numerical solutions are presented and the validity of Berkta's model, self-demodulation of pulsed signal, was discussed. Other investigators, Lee and Hamilton [12], using a time-domain algorithm that solves the KZK non-linear parabolic wave equation, calculated waveforms through the shock region and out to the far field locations where the non-linearly generated low frequency components dominate.

The above investigations, both theoretically and experimentally, tended to be focussed on the angular amplitude responses and waveform shapes of the transient parametric endfire array, rather than on its angular phase responses.

It should be noted that Trivett and Rogers [9] considered the arrival time of the secondary signal. However, in their paper, the effects of linear attenuation of primary signals have been neglected. Also their experimental set-up did not favour measuring arrival time of the secondary signal, because they recorded only off-axis secondary waveforms by simply aligning the hydrophone in an arc. So they were unable to observe the difference of the arrival times at different off-axis angles. Both Stepanishen *et al.* [7], and Frøysa [11], using numerical solutions, showed the arrival of a secondary signal which exhibits a time history. However, their main objectives were to confirm that Westervelt's finding [13]: that is, as the off-axis angle increases, the secondary waveforms approach the first derivative of the envelope function squared $\partial(f^2)/\partial t$. Therefore, no analytical and quantitative results were given as far as the angular phase response was concerned.

It is interesting to note that Garrett *et al.* [14] discussed the phase characteristics of parametric arrays. However, their research tended to be focussed on the near field of a parametric array, rather than on the far field.

Parametric transduction is a non-linear process. The secondary signal is generated by scattering of the primary field. From the non-linear acoustics point of view, the phase of an off-axis secondary will not be identical to that of an on-axis secondary, though both are created by the primary waves radiated on the main axis of the projector.

The analysis of on- and off-axis characteristics of a parametric array is important. Not only will this enable one to evaluate the possible impact of this phase shift on the Phase Shift Keying (PSK) modulation, but also will give a detailed physical picture of the forming of the secondary radiation due to scattering of sound by sound.

In this paper, Westervelt's source density approach is used to develop a theoretical formula for the angular dependence of the absorption-limited parametric array. Analytical results for the off-axis amplitude and phase response are presented. Experimental results, which show reasonable agreement with the theory, are presented. A primary frequency of 6.2 MHz was used to generate secondary frequency

parametric sources at 270 kHz. The impact of phase shift of the secondary signal is also addressed.

2. OFF-AXIS FAR FIELD SIMPLIFIED CASE

A projector in a fluid with a planar radiating surface, as shown in Figure 1, is considered. The transmitted pulsed-carrier acoustic waves are collimated, plane and travelling along the x -axis direction. The primary pressure at a point x at time t can be represented in the form

$$p_i(x, t) = P_0 \exp(-\alpha_0 x) f(t - x/c_0) \cos(\omega_0 t - k_0 x). \tag{1}$$

Here, $f(t)$ is the envelope function whose highest Fourier component has a very much lower frequency than the carrier so that α_0 can be used as the absorption coefficient for this band-limited signal.

Non-linear interaction in a primary acoustic field results in a secondary radiation. The source density function is

$$q(x, t) = (1/\rho_0^2 c_0^4) (1 + B/2A) \partial(p_i)^2/\partial t, \tag{2}$$

where

$A = \rho_0 c_0^2$ and $B = \rho_0 (\partial^2 p/\partial \rho^2)_{\rho=\rho_0}$. From equation (1), one obtains $p_i^2(x, t)$:

$$p_i^2(x, t) = (\frac{1}{2})P_0^2 \exp(-2\alpha_0 x) f^2(t - x/c_0) [1 + \cos(2\omega_0 t - 2k_0 x)]. \tag{3}$$

In equation (3), two components are obtained, one of which is a compression term without carrier, which gives rise to the difference frequency pressure. The other component is a band-limited signal at twice the carrier frequency. The scattered wave due to this component can be neglected, since it will be attenuated very quickly. Substitution of $p_i^2(x, t)$ given by equation (3) into the expression of the source density function q , equation (2), gives

$$q(x, t) = \frac{P_0^2 \beta}{2\rho_0^2 c_0^4} \exp(-2\alpha_0 x) \frac{\partial}{\partial t} \left[f^2 \left(t - \frac{x}{c_0} \right) \right]. \tag{4}$$

The difference frequency pressure function, calculated from the general solution of Westervelt's inhomogenous wave equation [15], may be expressed as

$$p_d(\vec{R}, t) = -\frac{\rho_0}{4\pi} \frac{\partial}{\partial t} \iiint \frac{q(t - \tilde{r}/c_0) \exp(-\alpha_d \tilde{r})}{\tilde{r}} dV. \tag{5}$$

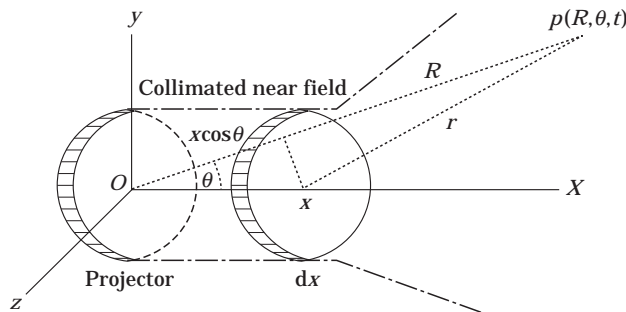


Figure 1. Collimated and plane waves geometry.

Equation (5) differs slightly from Westervelt's equation. Here the absorption of the difference wave is not ignored and α_d is the absorption coefficient at the difference frequency. Consider the far field geometry of the collimated plane wave model illustrated in Figure 1, the difference wave pressure at an observation point in the x - o - y plane can be simplified as

$$p_d(R, \theta, t) = -\frac{S\rho_0}{4\pi} \frac{\partial}{\partial t} \int_0^\infty \frac{\exp(-\alpha_d r)}{r} q\left(x, t - \frac{r}{c_0}\right) dx, \quad (6)$$

where S is the cross-sectional area of the collimated beam and r is the range from the scattering point to the observation point. Substitution of $q(x, t)$ from equation (4) into equation (6) gives

$$p_d(R, \theta, t) = -\frac{p_0^2 S \beta}{8\pi \rho_0 c_0^4} \frac{\partial^2}{\partial t^2} \left[\int_0^\infty \frac{\exp(-2\alpha_0 x - \alpha_d r)}{r} f^2\left(t - \frac{x}{c_0} - \frac{r}{c_0}\right) dx \right]. \quad (7)$$

For the far field assumption, $r \approx R - x \cos \theta$ for the phase and absorption term and $r \approx R$ for the range term in equation (7), which then becomes

$$p_d(R, \theta, t) = -\frac{p_0^2 S \beta \exp(-\alpha_d R)}{8\pi \rho_0 c_0^4 R} \frac{\partial^2}{\partial t^2} \int_0^\infty \exp(-2\alpha_0 + \alpha_d \cos \theta)x \\ \times f^2\left(t - \frac{R}{c_0} - \frac{1 - \cos \theta}{c_0} x\right) dx. \quad (8)$$

Let $u = [(1 - \cos \theta)/c_0]x$, so that $dx = [c_0/(1 - \cos \theta)] du$. If $-2\alpha_0 + \alpha_d \cos \theta \approx -2\alpha_0$, equation (8) can be simplified as

$$p_d(R, \theta, t) = C \frac{\partial^2}{\partial t^2} \int_0^\infty \frac{1}{D} \exp(-\frac{1}{D}u) f^2\left(t - \frac{R}{c_0} - u\right) du, \quad (9)$$

where

$$C = -p_0^2 S \beta \exp(-\alpha_d R) / 16\pi \rho_0 \alpha_0 c_0^4 R, \quad D = \sin^2(\theta/2) / \alpha_0 c_0 \quad (10, 11)$$

If one introduces an impulse response function $h(t, \theta)$ of the parametric array,

$$h(t, \theta) = \begin{cases} (1/D) \exp([1/D]t), & t \geq 0 \\ 0, & t < 0 \end{cases}, \quad (12)$$

then equation (9) can be expressed as

$$p_d(R, \theta, t) = Ch(t, \theta) * \frac{\partial^2}{\partial t^2} [f^2(t - R/c_0)]; \quad (13)$$

that is, the secondary sound pressure can be expressed by a temporal convolution of the parametric source function $\partial^2(f^2)/\partial t^2$ with the impulse response function $h(t, \theta)$ of the parametric array that depends only on the off-axis angle θ , the absorption coefficient α_0 and the sound speed c_0 .

Note that equation (13) may be used to evaluate the absorption-limited on-axis and off-axis far field characteristics of the secondary field. Consider the on-axis far field pressure, in equation (9), when on-axis $\theta \rightarrow 0$ and $D \rightarrow 0$, so obviously

$$(1/D) \exp(-(1/D)u)_{D \rightarrow 0} = \delta(u), \quad (14)$$

where $\delta(u)$ is the Dirac delta function. For on-axis projection, equation (9) can then be expressed as

$$p_d(R, 0, t) = C \frac{\partial^2}{\partial t^2} \int_0^\infty \delta(u) f^2 \left(t - \frac{R}{c_0} - u \right) du = C \frac{\partial^2}{\partial t^2} f^2 \left(t - \frac{R}{c_0} \right). \quad (15)$$

This equation agrees with Berkta's on-axis time dependent result [1], as corrected by Moffett and co-workers [2].

3. SPECIAL CASE AND ANALYTICAL RESULTS

Consider the two-frequency case where the primary wave consists of two separate frequency components with the same amplitude, namely,

$$\cos(\omega_0 + \Omega/2)t + \cos(\omega_0 - \Omega/2)t = 2 \cos(\Omega/2)t \cos \omega_0 t. \quad (16)$$

The envelope function $f(t)$ can then be expressed as

$$f(t) = \cos(\Omega t/2). \quad (17)$$

Let $\tau = t - R/c_0$. Substitution of equation (17) into equation (9) gives

$$p_d(R, \theta, t) = \frac{C}{2D} \frac{\partial^2}{\partial t^2} \int_0^\infty \exp\left(-\frac{u}{D}\right) \cos \Omega(\tau - u) du \quad (18)$$

Since $\exp(ix) = \cos x + i \sin x$, equation (18) can be written in the form

$$p_d(R, \theta, t) = \frac{C}{2D} \frac{\partial^2}{\partial t^2} \operatorname{Re} \left\{ \int_0^\infty \exp\left[-\frac{u}{D} + i\Omega(\tau - u)\right] du \right\}. \quad (19)$$

Here Re denotes the real part. After integration and simplification, equation (19) can be expressed as

$$p_d(R, \theta, t) = A(\theta) \cos[\Omega\tau - \psi(\theta)], \quad (20)$$

where

$$A(\theta) = -(C/2) (1 + D^2\Omega^2)^{-1/2} \Omega^2 = -(C/2) \Omega^2 \{1 + [(\sin^2(\theta/2)/\alpha_0 c_0)\Omega]^2\}^{-1/2}, \quad (21)$$

$$\psi(\theta) = \tan^{-1}(D\Omega) = \tan^{-1}[\sin^2(\theta/2)\Omega/\alpha_0 c_0]. \quad (22)$$

Note that the amplitude $A(\theta)$ in equation (21) agrees with Westervelt's result [15]. The difference between Westervelt's result and that presented here is that equation (20) includes phase information of the difference frequency wave.

Analytical results are presented in Figure 2 for the normalized far field pressure over a range of off-axis angles. All amplitudes are normalized to the on-axis value. A primary frequency of 6.2 MHz was assumed, since this was the centre frequency of the projector used for the experimental studies. A difference frequency of 270 kHz was chosen since it represented a particularly sensitive operating frequency for the hydrophone used in the

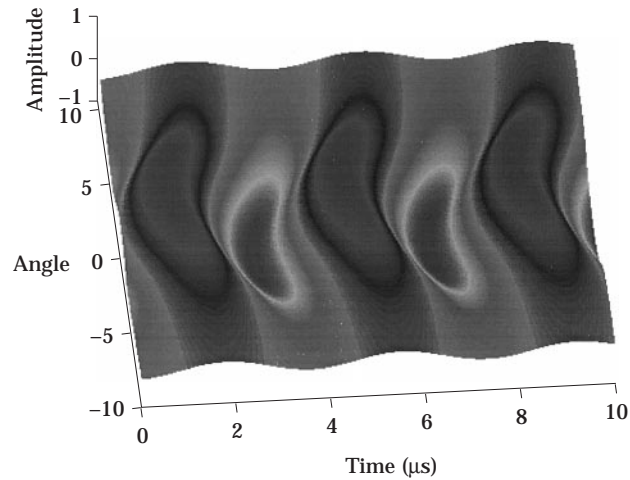


Figure 2. Parametric off-axis analytical result, $f_c = 6.2$ MHz, $f_d = 270$ kHz.

experiments. It can be seen from Figure 2 that the arrival time of the difference wave, off-axis, is delayed compared with that on-axis. As the off-axis angle θ increases, the delay rapidly increases and finally converges to one quarter of the period of the difference wave, which represents a 90° phase shift in equation (22).

4. EXPERIMENT

Experiments were performed in a water tank, with dimension $3\text{ m} \times 5\text{ m} \times 8\text{ m}$, at The University of Birmingham. The arrangement of the experiment is shown in Figure 3.

The projector, a baffled circular ceramic piston of 10 mm active diameter and 6.2 MHz resonance frequency, was mounted in a holder that permitted it to be adjusted in the

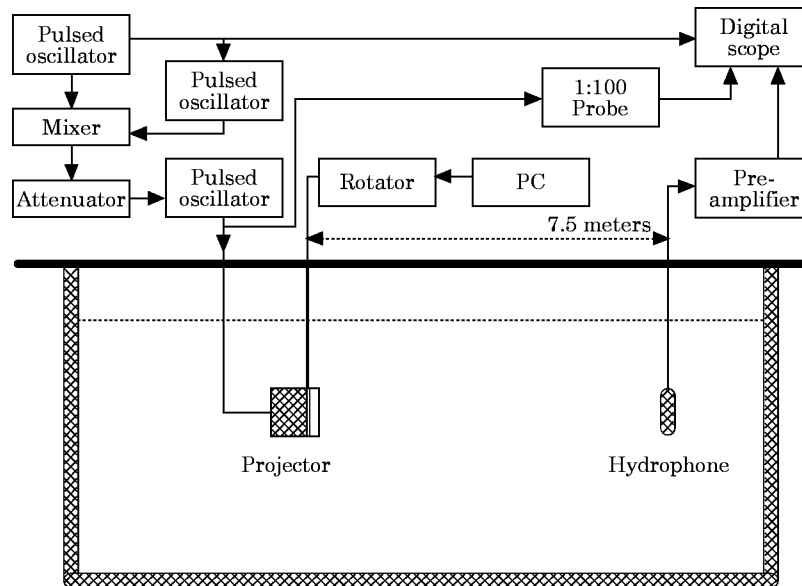


Figure 3. Block diagram of experiment set-up.

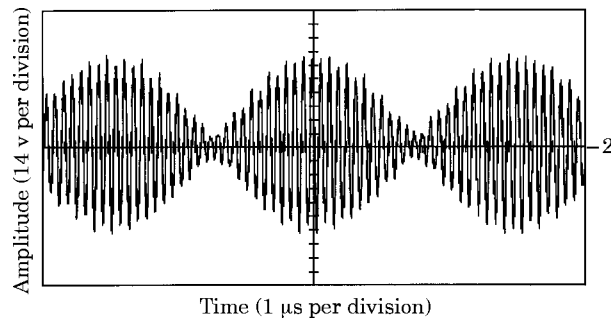


Figure 4. Transmitted modulated primary signal, $f_c = 6.2$ MHz, $f_d = 270$ kHz.

vertical plane. The holder was then mounted on a rotator that could be turned in the horizontal plane by a stepper motor which was controlled by a microprocessor. The oscillator (HP 8116A) provided a pulsed signal at the higher primary frequency and also triggered a second oscillator (HP 8116A) to generate another pulsed signal at the lower primary frequency whose phase is locked on to the higher primary frequency. These two pulsed carrier signals were added together in the mixer to produce the required pulsed modulated signal as defined by equation (16). The modulated signal was amplified (ENI RF amplifier 240L) and finally applied to the projector. The voltage waveform applied to the projector is shown in Figure 4.

The difference signals were received by a B&K hydrophone (B&K 8103) at a distance of 7.5 m, passed through a pre-amplifier (Brookdeal precision AC amplifier 9452) with a 1 MHz active low pass filter, and recorded by a digital oscilloscope (LeCroy 9310L). The received difference signal is shown in Figure 5. In order to increase SNR and thus improve the accuracy of the measurements, a method of averaging 100 received waveforms was employed in the digital oscilloscope. It was found that the average method is very effective in increasing SNR for parametric transduction.

5. RESULTS, DISCUSSIONS AND LIMITATION OF THE THEORY

Figure 6 shows the results for the three-dimensional parametric off-axis amplitude and phase response with a centre frequency of 6.2 MHz and difference frequency of 270 kHz at a distance of 3 m. For better comparison with the theory, Figures 7 and 8 separately show amplitude and phase responses for a difference frequency of 270 kHz at a distance of 7.5 m. It can be seen that the measured amplitude and phase responses are in good agreement with those predicted by the theory.

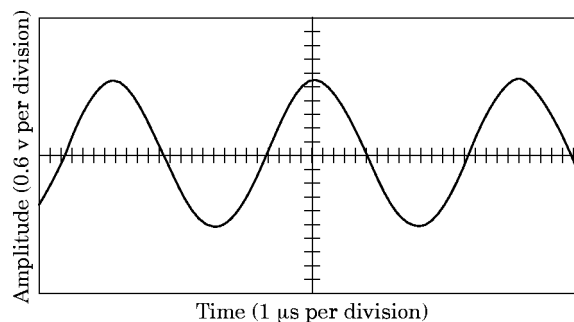


Figure 5. Received secondary signal, $f_d = 270$ kHz.

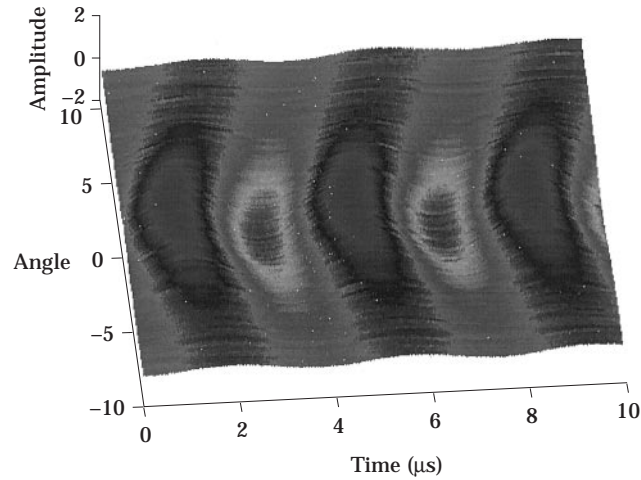


Figure 6. Parametric off-axis experimental result, $f_c = 6.2$ MHz, $f_d = 270$ kHz.

The off-axis phase delay by comparison with the on-axis phase is a unique property of parametric transduction. This can be explained as follows. Parametric transduction is a secondary effect: the secondary sound wave results from the scattering of the primary sound beam. Hence, the distance between scattering source and the off-axis point is larger than that between the scattering source and the on-axis point, this can be clearly seen in Figure 9. Because of this, the scattering off-axis secondary signal will be delayed.

Consider next the influence of the off-axis phase shift on PSK modulation. If the transmitter and receiver employ the same carrier and timing references, even in a “fixed-fixed” link, the steering of the transmit projector would make the maximum received signal phase shift by 90° by comparison with on-axis one. In such a case, Quadrature Phase Shift Keying (QPSK) would not work correctly. However, if the receiver

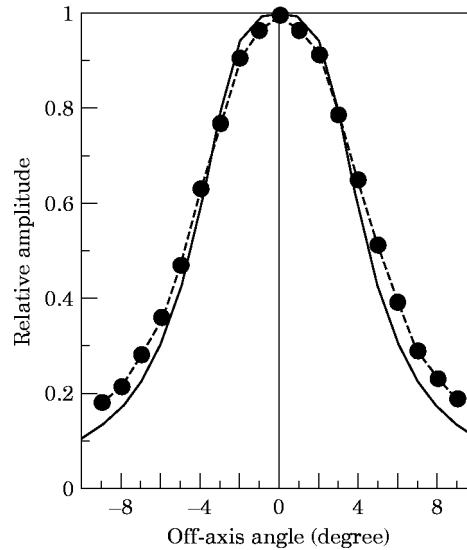


Figure 7. Parametric angular amplitude response; —, analytic results; ----, experimental results.

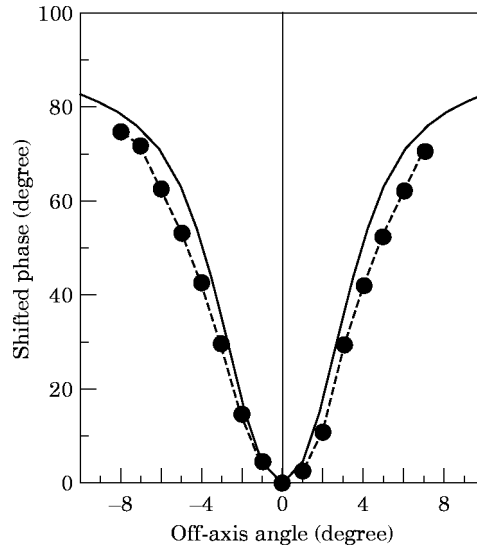


Figure 8. Parametric angular phase response; —, analytic results; ---, experimental results.

system tracks the transmitted signal, this means that the carrier and timing references are recovered by processing the received waveform. In such a case, the phase shift due to rotating the projector would be offset by tracking. Because tracking is generally required for PSK communications, it can be concluded that an off-axis phase shift, a term which is not a function of time, would have negligible effect.

Finally, the validity of applying the present theory can be examined. From the derivation of equation (9), it can be seen that two conditions are assumed to obtain the present theory: (1) the primary waves are collimated and planar, or the virtual array distance R_v is shorter than the Rayleigh distance R_r : that is, the dominant interaction or secondary source scattering takes place in the near field; accordingly, this parametric array should be absorption limited; (2) the observation point should be far away from the transmitting transducer; in such a case, the assumption $r = R - x \cos \theta$ can be satisfied.

In the experimental set-up, the primary frequency was $f_0 = 6.2$ MHz, the radius of the piston $a = 5$ mm, and the observation range $r = 7.5$ m. According to Clay's formula [16], $\alpha_0 = 1.1 \text{ Np m}^{-1}$, then $R_v = 0.46$ m, and also $R_r = 0.34$ m. From the data presented here, it can be seen clearly that condition (2) is sufficiently satisfied, since $r \gg R_r$, however, condition (1) is not, since $R_v \approx R_r$, rather than $R_v < R_r$.

From the fact that the predicted results are generally in agreement with those of the experiment, it can be concluded that condition (1) is not all that essential to the validity of the present theory. The mechanisms behind this could be explained as follows: The key point of condition (1) is that the primary waves are collimated and planar. Within the

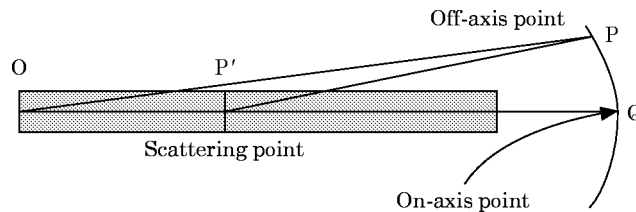


Figure 9. Diagram showing off-axis parametric transduction delay, $OP = OQ$, $PP' > QP'$.

Rayleigh distance, the primary waves can be thought of as collimated and planar. However, in the transient region between Fresnel and Fraunhofer, there may exist a region in which the primary waves can still be thought of as collimated and planar. Therefore, the dominant non-linear interaction occurs in a collimated and planar region, even though $R_v \approx R_r$ or R_r is slightly shorter than R_v .

Because the virtual array distance R_v will be decreased more quickly than Rayleigh distance R_r if the primary frequency is increased, it can be expected that increasing of the primary frequency of the experiment will lead to a better agreement between theoretical and experimental results.

In general, the present theory can be applied to the far field of an absorption-limited parametric array. If the parametric array is not absorption-limited or the observation point is well in the near field, then a simple analytical result may not be obtainable. In such a case, more general solution, numerical solution, should be applied. This will be presented in a further publication.

6. CONCLUSIONS

By using Westervelt's source density approach, an analytical formula has been developed for the far field angular dependence of an absorption-limited parametric array. The formula provides a simple means for evaluating off-axis performances of parametric array. The angular amplitude and phase responses of the far field difference frequency signal, which are in reasonable agreement with experimental results, have been presented for a primary frequency 6.2 MHz and difference frequency of 270 kHz. Some special cases have been discussed. In particular, the present formula is shown to be in agreement with Berktaý's on-axis far field result for a transient parametric array. The angular amplitude response agrees with Westervelt's results. Furthermore, the influence of off-axis phase shift on PSK communications was shown to be negligible if the receiver system obtains its carrier and timing references from the incoming signal.

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