



# ANALYSIS OF NON-LINEAR PIEZOTHERMOELASTIC LAMINATED BEAMS WITH ELECTRIC AND TEMPERATURE EFFECTS

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Conventional analyses of piezoelectric laminated structures are based on linear theories. Investigations of non-linear characteristics are still relatively scarce. In this paper, static, dynamic, and control effects of a piezothermoelastic laminated beam with an initial non-linear large static deflection (the von Karman type geometric non-linear deformation) and temperature and electric inputs are studied. It is assumed that the piezoelectric layers are uniformly distributed on the top and bottom surfaces of the beam. Beam equations incorporating the non-linear deflections, piezoelectric layers, temperature and electric effects are simplified from the generic piezothermoelastic shell equations. Analytical solutions of non-linear static deflection and eigenvalue problems of the non-linearly deformed beam including temperature and electric effects are derived. Active control effects on non-linear static deflections and natural frequencies imposed by the piezoelectric actuators via high control voltages are investigated. A numerical example is provided and response behavior is investigated.

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## 1. INTRODUCTION

Recent studies and analyses of piezoelectric laminated structures and systems are mostly carried out using small deflection linear theories [1–3]. Investigations of non-linear characteristics are still relatively scarce. This study is concerned with an investigation of static, dynamic, and control characteristics of a non-linear piezoelectric laminated beam subjected to mechanical, temperature, and electric excitations. Pai *et al.* [4] proposed a refined model for non-linear composite plate laminated with piezoelectric layers. Yu [5] reviewed recent studies of linear and non-linear theories of elastic and piezoelectric plates. Sreeram *et al.* [6] investigated a non-linear hysteresis modelling of a piezoelectric actuator. Librescu [7] proposed a refined geometrical non-linear theory of anisotropic laminated shells. Linear thermo–electromechanical behavior of distributed piezoelectric sensors and actuators has also been recently studied [8, 9]. A theory on geometrical non-linearity of piezothermoelastic shell laminates simultaneously exposed to mechanical, electric, and thermal fields has been recently proposed [10]. Tzou and Zhou [11] investigated static and dynamic control of a circular plate with geometrical non-linearity. In this study, static, dynamic and control effects of a piezothermoelastic laminated beam with an initial non-linear large static deflection (the von Karman type geometric non-linear deformation)

and temperature/electric inputs are studied. System equations are defined and simplified to compare them with a classical non-linear beam equation first. Analytical solutions, including all design parameters, of non-linear static deflection and free vibrations of the non-linearly deformed beam are derived. Active control effects on non-linear static deflections and natural frequencies imposed by the piezoelectric actuators via high control voltages are investigated.

## 2. NON-LINEAR PIEZOTHERMOELASTIC LAMINATED BEAM

A piezothermoelastic laminated beam is shown in Figure 1, in which two piezoelectric layers are perfectly bonded on the top and the bottom surfaces of a steel beam. It is assumed that the laminated beam undergoes a von Karman type geometrical non-linearity and temperature and electric inputs. Dimensions of the beam are:  $L$  the beam length,  $b$  the beam width,  $h_e$  the steel layer thickness,  $h_p$  the piezoelectric layer thickness. Thus, the total laminated beam thickness is  $h = h_e + 2h_p$ .

Simplifying the governing equations of the nonlinear piezothermoelastic shell laminate [12], one obtains the non-linear piezothermoelastic beam equations in the longitudinal ( $x$ ) direction and the transverse ( $z$ ) direction, respectively:

$$\frac{\partial N_{xx}}{\partial x} + q_x = \rho h \ddot{u}_x, \quad \frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial N_{xx}}{\partial x} \frac{\partial u_z}{\partial x} + N_{xx} \frac{\partial^2 u_z}{\partial x^2} + q_z = \rho h \ddot{u}_z, \quad (1a, b)$$

where the mass per unit length  $\rho h = \Sigma \int \rho_k dz = 2\rho_p h_p + \rho_e h_e$ ;  $\rho_p$  and  $\rho_e$  are the densities of the piezoelectric layer and the elastic steel layer, respectively.  $N_{xx}$  and  $M_{xx}$  are the membrane force and bending moment per unit width. Since the beam width  $b$  is constant, one can define  $N_x = bN_{xx}$  and  $M_x = bM_{xx}$ . Note that the forces and moments include all elastic, electric, and temperature effects:

$$N_x = \left( Y\tilde{A} + 2Y_p\tilde{A}_p + 2\tilde{A}_p \frac{e_{31}^2}{\epsilon_{33}} \right) s_{xx}^0 + e_{31} b (\phi_{33}^e + \phi_{31}^e) + \left[ \left( \frac{e_{31} p_3}{\epsilon_{33}} - \lambda_p \right) \left( \int_{-(h_e/2 + h_p)}^{-h_e/2} \theta b dz + \int_{h_e/2}^{h_e/2 + h_p} \theta b dz \right) - \lambda \int_{-h_e/2}^{h_e/2} \theta b dz \right], \quad (2a)$$

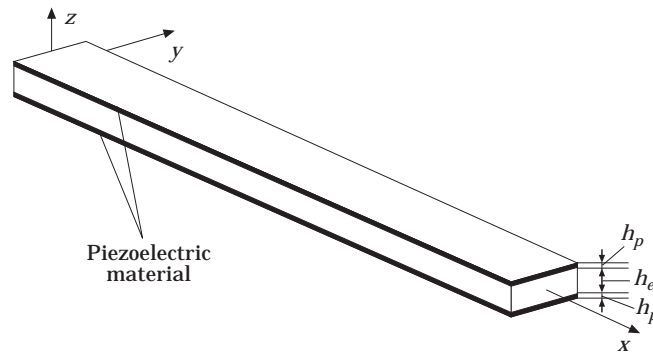


Figure 1. A piezothermoelastic laminated beam.

$$M_x = \left( YI + 2Y_p I_p + 2I_p \frac{e_{31}^2}{\epsilon_{33}} \right) \kappa_{xx} + e_{31} br^a (\phi_{33}^c - \phi_{31}^c) + \left[ \left( \frac{e_{31} P_3}{\epsilon_{33}} - \lambda_p \right) \left( \int_{-(h_e/2 + h_p)}^{-h_e/2} \theta z b \, dz + \int_{h_e/2}^{h_e/2 + h_p} \theta z b \, dz \right) - \lambda \int_{-h_e/2}^{h_e/2} \theta z b \, dz \right], \quad (2b)$$

where  $\tilde{A} = bh_e$ ,  $\tilde{A}_p = bh_p$  are the cross-sectional areas of the elastic steel layer and the piezoelectric layers, respectively;  $Y$  and  $Y_p$  are Young's moduli of the steel and the piezoelectric material;  $I = bh_e^3/12$ ,  $I_p = bh_p^3/12 + bh_p(h_e + h_p)^2/2$  are the area moments of the steel layer and the piezoelectric layer, respectively;  $\phi_{33}^c$  and  $\phi_{31}^c$  are the control voltages applied to the top and the bottom piezoelectric layers;  $\theta$  is the temperature variation;  $e_{31}$ ,  $\epsilon_{33}$ ,  $p_3$  and  $\lambda_p$  are the piezoelectric stress coefficient, the dielectric coefficient, the pyroelectric coefficient, and the stress-temperature coefficient for the piezoelectric material, respectively;  $\lambda$  is the steel stress-temperature coefficient;  $s_{xx}^0$  and  $\kappa_{xx}$  are the membrane strain and bending strain; and  $r^a = (h_e + h_p)/2$  is the actuator moment arm. Also, one can define  $p_x = bq_x$ ,  $p_z = bq_z$  (mechanical excitations per unit length), and  $\bar{m} = \rho bh = 2\rho_p \tilde{A}_p + \rho \tilde{A}$  (mass per unit beam length). Then, equations (1a) and (1b) can be rewritten as

$$\frac{\partial N_x}{\partial x} + p_x = \bar{m} \ddot{u}_x, \quad \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial N_x}{\partial x} \frac{\partial u_z}{\partial x} + N_x \frac{\partial^2 u_z}{\partial x^2} + p_z = \bar{m} \ddot{u}_z. \quad (3a, b)$$

Using equations (2a) and (2b), one can write the axial force and bending moment in a compact form:

$$N_x = \tilde{K} s_{xx}^0 + N_x^c + N_x^t, \quad M_x = \tilde{D} \kappa_{xx} + M_x^c + M_x^t, \quad (4a, b)$$

where  $\tilde{K}$  is the membrane stiffness  $\tilde{K} = (Y\tilde{A} + 2Y_p \tilde{A}_p + 2\tilde{A}_p (e_{31}^2/\epsilon_{33}))$ ;  $\tilde{D}$  is the bending stiffness  $\tilde{D} = (YI + 2Y_p I_p + 2I_p (e_{31}^2/\epsilon_{33}))$ ;  $s_{xx}^0$  is the membrane strain and  $\kappa_{xx}$  is the bending strain with the von Karman type non-linearity:

$$s_{xx}^0 = \partial u_x / \partial x + \frac{1}{2} (\partial u_z / \partial x)^2, \quad \kappa_{xx} = \partial \beta_x / \partial x = -\partial^2 u_z / \partial x^2. \quad (5a, b)$$

$N_x^t$  is the axial force induced by the temperature rise;  $M_x^t$  is the moment due to the temperature rise;  $N_x^c$  is the axial force induced by the control potential; and  $M_x^c$  is the control moment induced by the control potential:

$$N_x^c = e_{31} b (\phi_{33}^c + \phi_{31}^c), \quad M_x^c = e_{31} br^a (\phi_{33}^c - \phi_{31}^c), \quad (6a, b)$$

$$N_x^t = \left[ \left( \frac{e_{31} P_3}{\epsilon_{33}} - \lambda_p \right) \left( \int_{-(h_e/2 + h_p)}^{-h_e/2} \theta b \, dz + \int_{h_e/2}^{h_e/2 + h_p} \theta b \, dz \right) - \lambda \int_{-h_e/2}^{h_e/2} \theta b \, dz \right], \quad (6c)$$

$$M_x^t = \left[ \left( \frac{e_{31} P_3}{\epsilon_{33}} - \lambda_p \right) \left( \int_{-(h_e/2 + h_p)}^{-h_e/2} \theta z b \, dz + \int_{h_e/2}^{h_e/2 + h_p} \theta z b \, dz \right) - \lambda \int_{-h_e/2}^{h_e/2} \theta z b \, dz \right]. \quad (6d)$$

Boundary conditions at the two ends of the laminated beam,  $x = 0$  and  $x = L$ , are

$$N_x = N_x^* \quad \text{or} \quad u_x = u_x^*; \quad M_x = M_x^* \quad \text{or} \quad \beta_x = \beta_x^*; \quad (7a, b)$$

$$\partial M_x / \partial x + N_x \partial u_z / \partial x = Q_z^* \quad \text{or} \quad u_z = u_z^*, \quad (7c)$$

where the quantities with the asterisk \* are the prescribed values on the boundaries. Note that usually either force boundary conditions or displacement boundary conditions are selected for a given physical boundary condition.

### 2.1. SIMPLIFICATION OF THE BEAM GOVERNING EQUATIONS

It is assumed that the mechanical excitations in the longitudinal and transverse directions are zero, i.e.,  $p_z = p_x = 0$ . Substituting the axial force and bending moment into equations (3a) and (3b), one can write

$$\begin{aligned} & \tilde{K} \frac{\partial}{\partial x} \left[ \frac{\partial u_x}{\partial x} + \frac{1}{2} \left( \frac{\partial u_z}{\partial x} \right)^2 \right] + e_{31} b \frac{\partial}{\partial x} (\phi_{33}^c + \phi_{31}^c) + \left[ \left( \frac{e_{31} p_3}{\epsilon_{33}} - \lambda_p \right) \right. \\ & \left. \times \left( \int_{-(h_e/2+h_p)}^{-h_e/2} \frac{\partial \theta}{\partial x} b \, dz + \int_{h_e/2}^{h_e/2+h_p} \frac{\partial \theta}{\partial x} b \, dz \right) - \lambda \int_{-h_e/2}^{h_e/2} \frac{\partial \theta}{\partial x} b \, dz \right] = \bar{m} \ddot{u}_x, \end{aligned} \quad (8a)$$

$$\begin{aligned} & \left\{ -\tilde{D} \frac{\partial^4 u_z}{\partial x^4} + e_{31} b r^a \frac{\partial^2}{\partial x^2} (\phi_{33}^c - \phi_{31}^c) + \left[ \left( \frac{e_{31} p_3}{\epsilon_{33}} - \lambda_p \right) \right. \right. \\ & \left. \left. \times \left( \int_{-(h_e/2+h_p)}^{-h_e/2} \frac{\partial^2 \theta}{\partial x^2} z b \, dz + \int_{h_e/2}^{h_e/2+h_p} \frac{\partial^2 \theta}{\partial x^2} z b \, dz \right) - \lambda \int_{-h_e/2}^{h_e/2} \frac{\partial^2 \theta}{\partial x^2} z b \, dz \right] \right\} \\ & + \left\{ \tilde{K} \frac{\partial}{\partial x} \left[ \frac{\partial u_x}{\partial x} + \frac{1}{2} \left( \frac{\partial u_z}{\partial x} \right)^2 \right] + e_{31} b \frac{\partial}{\partial x} (\phi_{33}^c + \phi_{31}^c) + \left[ \left( \frac{e_{31} p_3}{\epsilon_{33}} - \lambda_p \right) \right. \right. \\ & \left. \left. \times \left( \int_{-(h_e/2+h_p)}^{-h_e/2} \frac{\partial \theta}{\partial x} b \, dz + \int_{h_e/2}^{h_e/2+h_p} \frac{\partial \theta}{\partial x} b \, dz \right) - \lambda \int_{-h_e/2}^{h_e/2} \frac{\partial \theta}{\partial x} b \, dz \right] \right\} \frac{\partial u_z}{\partial x} \\ & + \left\{ \tilde{K} \left[ \frac{\partial u_x}{\partial x} + \frac{1}{2} \left( \frac{\partial u_z}{\partial x} \right)^2 \right] + e_{31} b (\phi_{33}^c + \phi_{31}^c) + \left[ \left( \frac{e_{31} p_3}{\epsilon_{33}} - \lambda_p \right) \right. \right. \\ & \left. \left. \times \left( \int_{-(h_e/2+h_p)}^{-h_e/2} \theta b \, dz + \int_{h_e/2}^{h_e/2+h_p} \theta b \, dz \right) - \lambda \int_{-h_e/2}^{h_e/2} \theta b \, dz \right] \right\} \frac{\partial^2 u_z}{\partial x^2} = \bar{m} \ddot{u}_z. \end{aligned} \quad (8b)$$

Since the longitudinal inertia is negligible, i.e.,  $\bar{m} \ddot{u}_x \simeq 0$ , factoring the partial derivatives and regrouping the force and moments gives

$$\partial N_x / \partial x = 0, \quad \partial^2 M_x / \partial x^2 + N_x \partial^2 u_z / \partial x^2 = \bar{m} \ddot{u}_z. \quad (9a, b)$$

Equation (9a) implies that the axial force  $N_x$  is not a function of  $x$ , i.e.,  $N_x = \text{constant}$ . Considering individual elastic, control, and temperature effects, one can further write equation (9b) as

$$-\tilde{D} \partial^4 u_z / \partial x^4 + \partial^2 M_x^c / \partial x^2 + \partial^2 M_x^t / \partial x^2 + N_x \partial^2 u_z / \partial x^2 = \bar{m} \ddot{u}_z, \quad (10)$$

where  $N_x = \tilde{K} [\partial u_x / \partial x + \frac{1}{2} (\partial u_z / \partial x)^2] + N_x^c + N_x^t$ . The non-linear piezothermoelastic beam equation can be further simplified when boundary conditions are specified. In the following two cases, one is used to compare with the standard equation and the other is for a detailed parametric study.

### 2.2. FREE EXPANSION/CONTRACTION

If the longitudinal motion either at  $x = 0$  or at  $x = L$  is not constrained (free

expansion/contraction), the axial force  $N_x$  vanishes when the boundary conditions are imposed. The differential equation then can be simplified to

$$-\tilde{D} \partial^4 u_z / \partial x^4 + f(x, t) = \bar{m} \ddot{u}_z, \quad (11)$$

where  $f(x, t) = \partial^2 M_x^c / \partial x^2 + \partial^2 M_x^t / \partial x^2$ . This is a standard form of the beam transverse vibration [13]. However, note that the physical meaning is much more complicated than the conventional form, due to the coupling of mechanical, electric, and temperature fields in the non-linear piezothermoelastic laminated beam.

### 2.3. SIMPLY SUPPORTED WITH BOTH ENDS FIXED

Boundary conditions for a simply supported piezothermoelastic laminated beam with both ends fixed are

$$u_z = u_x = 0 \quad \text{and} \quad M_x = 0, \quad (12)$$

at both beam ends:  $x = 0$  and  $x = L$ . Furthermore, it is assumed the voltage  $\phi$  and temperature variation  $\theta$  are uniform in the  $x$  direction. This implies that  $\phi$  and  $\theta$  are not functions of co-ordinate  $x$ . Then, the transverse equation becomes

$$-\tilde{D} \partial^4 u_z / \partial x^4 + N_x \partial^2 u_z / \partial x^2 = \bar{m} \ddot{u}_z, \quad (13)$$

where the axial force  $N_x = \tilde{K}[\partial u_x / \partial x + \frac{1}{2}(\partial u_z / \partial x)^2] + N_x^c + N_x^t$ . Solution procedures for the simply supported non-linear piezothermoelastic beam equation are presented next. Numerical results and control effectiveness are presented in case studies.

## 3. SOLUTION PROCEDURES

In order to investigate the coupling among elastic, electric, temperature and control effects of the piezothermoelastic laminated beam, analytical solutions, including all design and control variables, are derived. The solution procedures are divided into two parts. The first step is to solve for non-linear static solutions and the second step is to solve for dynamic solutions with respect to the non-linear static equilibrium position. To separate the static and dynamic solutions, one can write the displacement and axial force as

$$u_x = u_{x,s} + u_{x,d}, \quad u_z = u_{z,s} + u_{z,d}, \quad (14)$$

$$N_x = N_{x,s} + N_{x,d}, \quad \phi_{3j}^c = \phi_{3j,s}^c + \phi_{3j,d}^c, \quad (15)$$

where the subscript  $_s$  is the static component and  $_d$  is the dynamic component;  $u_{x,s}$ ,  $u_{z,s}$ ,  $u_{x,d}$  and  $u_{z,d}$  are, respectively, the longitudinal/transverse static and corresponding dynamic displacements;  $N_{x,s}$  and  $N_{x,d}$  are the static and dynamic axial forces;  $\phi_{3j,s}^c$  and  $\phi_{3j,d}^c$  ( $j = 1, 3$ ) are the static and dynamic applied voltages, respectively. Then, the equation of motion and the axial force equation can be expressed as

$$-\tilde{D} \left( \frac{d^4 u_{z,s}}{dx^4} + \frac{\partial^4 u_{z,d}}{\partial x^4} \right) + (N_{x,s} + N_{x,d}) \frac{d^2 u_{z,s}}{dx^2} + \frac{\partial^2 u_{z,d}}{\partial x^2} = \bar{m} \ddot{u}_{z,d}, \quad (16)$$

$$(N_{x,s} + N_{x,d}) = \tilde{K} \left[ \left( \frac{du_{x,s}}{dx} + \frac{\partial u_{x,d}}{\partial x} \right) + \frac{1}{2} \left( \frac{du_{z,s}}{dx} + \frac{\partial u_{z,d}}{\partial x} \right)^2 \right] + (N_{x,s}^c + N_{x,d}^c) + N_x^t, \quad (17)$$

where the static and dynamic axial forces induced by the applied control voltages are  $N_{x,s}^c = e_{31} b(\phi_{33,s}^c + \phi_{31,s}^c)$ ,  $N_{x,d}^c = e_{31} b(\phi_{33,d}^c + \phi_{31,d}^c)$ . Usually, the static axial force  $N_{x,s}$  is more inferential than the dynamic axial force  $N_{x,d}$  [14]. Thus, neglecting the dynamic axial force

$N_{x,d}$ , one can define the static displacement and force equilibrium equations of the piezothermoelastic laminated beam subjected to static electric and temperature loads:

$$\tilde{D}(d^4u_{z,s}/dx^4) - N_{x,s}(d^2u_{z,s}/dx^2) = 0, \quad N_{x,s} = \tilde{K}[(du_{x,s}/dx) + \frac{1}{2}(du_{z,s}/dx)^2] + N_{x,s}^c + N_x^t. \quad (18, 19)$$

The partial differential equation for the dynamic motion becomes

$$-\tilde{D}(\partial^4u_{z,d}/\partial x^4) + N_{x,s}\left(\frac{\partial^2u_{z,d}}{\partial x^2}\right) = \bar{m}\ddot{u}_{z,d}. \quad (20)$$

Equations (18–20) and equation (12) constitute the governing equations and boundary conditions for the simply supported non-linear deflection and vibration of the simply supported non-linear piezothermoelastic laminated beam. Since the vibration analysis is based on the statically non-linear deformed position, static solutions of the non-linear piezoelectric laminated beam are presented first. Dynamic analysis of the beam with the initial non-linear deflection is investigated next.

### 3.1. STATIC SOLUTION OF A NON-LINEAR PIEZOTHERMOELASTIC BEAM

For the static problem defined by equations (18) and (19), the static deflection  $u_{z,s}$  and the static axial force  $N_{x,s}$  are evaluated. Recall that the longitudinal displacement boundary conditions are

$$u_{x,s} = 0, \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L. \quad (21)$$

Integrating the slope over the beam length and using the boundary conditions results in

$$\int_0^L \left(\frac{du_{x,s}}{dx}\right) dx = u_{x,s}|_{x=L} - u_{x,s}|_{x=0} = 0. \quad (22)$$

Integrating the static axial force  $N_{x,s}$ , equation (19), over the beam length and applying equation (22), one can derive

$$\int_0^L \left\{ \frac{1}{\tilde{K}} [N_{x,s} - N_{x,s}^c - N_x^t] - \frac{1}{2} \left(\frac{du_{z,s}}{dx}\right)^2 \right\} dx = 0. \quad (23)$$

Since  $N_{x,s} = \text{constant}$ , the axial force equation becomes

$$N_{x,s} = \frac{\tilde{K}}{2L} \int_0^L \left(\frac{du_{z,s}}{dx}\right)^2 dx + \frac{1}{L} \int_0^L (N_{x,s}^c + N_x^t) dx. \quad (24)$$

Defining an eigenvalue  $\lambda = \sqrt{N_{x,s}/\tilde{D}}$  and simplifying the static differential equation (18) of the non-linear piezothermoelastic beam gives

$$d^4u_{z,s}/dx^4 - \lambda^2 d^2u_{z,s}/dx^2 = 0. \quad (25)$$

The general solution for the static differential equation is

$$u_{z,s} = C_1 + C_2 x + C_3 \sinh \lambda x + C_4 \cosh \lambda x, \quad (26)$$

where  $C_i$  ( $i = 1, 2, 3, 4$ ) are the integration constants determined by the boundary conditions. For the simply supported beam defined above, the boundary conditions at

$x = 0$  and  $x = L$  are:  $u_{z,s} = 0$  and  $M_x = 0$  where  $M_x = -\tilde{D} d^2u_{z,s}/dx^2 + (M_{x,s}^c + M_x^t)$ . Then, imposing the boundary conditions to the general solution gives the four equations:

$$C_1 + C_4 = 0, \quad C_1 + C_2 L + C_3 \sinh \lambda L + C_4 \cosh \lambda L = 0, \quad (27a, b)$$

$$-\tilde{D}(C_4 \lambda^2) + (M_{x,s}^c + M_x^t) = 0, \quad (27c)$$

$$-\tilde{D}(C_3 \lambda^2 \sinh \lambda L + C_4 \lambda^2 \cosh \lambda L) + (M_{x,s}^c + M_x^t) = 0. \quad (27d)$$

Solving these four equations, one can determine the constants  $C_i$ . Upon substitutions, the static transverse displacement can be derived as

$$\begin{aligned} u_{z,s} &= [(M_{x,s}^c + M_x^t)/\tilde{D}\lambda^2] [\cosh \lambda x - (\tanh \lambda L/2) \sinh \lambda x - 1] \\ &= [(M_{x,s}^c + M_x^t)/\tilde{D}\lambda^2] (\cosh \lambda(L/2 - x)/(\cosh \lambda L/2) - 1). \end{aligned} \quad (28)$$

Assuming a dimensionless quantity  $v = \lambda(L/2) = (L/2)\sqrt{N_{x,s}/\tilde{D}}$ ,  $\lambda^2 = 4v^2/L^2$ , and  $N_{x,s} = 4v^2\tilde{D}/L^2$ , one can redefine the static transverse displacement equation as

$$u_{z,s} = \frac{(M_{x,s}^c + M_x^t)L^2}{4\tilde{D}v^2} (\cosh v(1 - 2x/L)/\cosh v - 1). \quad (29)$$

Since

$$\begin{aligned} \int_0^L \left( \frac{du_{z,s}}{dx} \right)^2 dx &= \left[ \frac{(M_{x,s}^c + M_x^t)L}{2\tilde{D}v \cosh v} \right]^2 \int_0^L \sinh^2 v(1 - 2x/L) dx \\ &= [(M_{x,s}^c + M_x^t)^2 L^3 / 8\tilde{D}^2 v^2] (\tanh v/v - 1/\cosh^2 v), \end{aligned} \quad (30)$$

one can obtain the static axial force

$$N_{x,s} = \frac{\tilde{K}}{2L} \frac{(M_{x,s}^c + M_x^t)^2 L^3}{8\tilde{D}^2 v^2} \left( \frac{\tanh v}{v} - \frac{1}{\cosh^2 v} \right) + (N_{x,s}^c + N_x^t). \quad (31)$$

Furthermore, since  $N_{x,s} = (4v^2\tilde{D})/L^2$ , the above equation can be simplified to

$$\frac{\tilde{K}(M_{x,s}^c + M_x^t)^2 L^2}{16\tilde{D}^2 v^2} \left( \frac{\tanh v}{v} - \frac{1}{\cosh^2 v} \right) + (N_{x,s}^c + N_x^t) = \frac{4v^2\tilde{D}}{L^2}. \quad (32)$$

Accordingly, with specified electrical control potential  $\phi$  and the temperature rise  $\theta$ , one can calculate the corresponding bending moments  $M_{x,s}^c$ ,  $M_x^t$  and the axial forces  $N_{x,s}^c$ ,  $N_x^t$  and then solve the dimensionless quantity  $v$  in equation (32). Next, one can solve the total static axial force  $N_{x,s}$  by equation (31) and the static transverse displacement  $u_{z,s}$  by equation (28) where  $\lambda^2 = (2v/L)^2$ .

### 3.2. DYNAMIC ANALYSIS OF BEAM WITH AN INITIAL NON-LINEAR DEFLECTION

It is assumed that the vibration analysis is in the vicinity of the non-linear (static) deformed equilibrium position. Frequency control of the piezothermoelastic laminated beam with an initial non-linear deflection is studied in this section. From equation (20), the free dynamic partial differential equation can be written as

$$\tilde{D} \partial^4 u_{z,d} / \partial x^4 - N_{x,s} \partial^2 u_{z,d} / \partial x^2 + \bar{m} \ddot{u}_{z,d} = 0. \quad (33)$$

Examining the equation reveals that this is the case of a transversely vibrating beam with axial force effect. Note that the electric and temperature effects are added to the classical

elastic problem [15]. When the beam oscillates at one of its natural modes, the solution of dynamic differential equation has the form of

$$u_{z,d} = U(x) \sin(\omega t + \varphi), \quad (34)$$

where  $U(x)$  is the modal shape function,  $\omega$  is the frequency and  $\varphi$  is the phase. Substitution and simplification gives an ordinary differential equation:

$$\tilde{D} d^4 U/dx^4 - N_{x,s} d^2 U/dx^2 - \tilde{m}\omega^2 U = 0. \quad (35)$$

The solution of this equation, satisfying the prescribed boundary conditions, furnishes the modal shape functions. For the simply supported beam, the dynamic boundary conditions are zero displacements and zero moments at  $x = 0$  and  $x = L$ :

$$U(x) = 0, \quad d^2 U(x)/dx^2 = 0. \quad (36)$$

These conditions are satisfied by taking the modal shape function as

$$U_i(x) = C_i \sin(i\pi x/L), \quad (37)$$

where  $i$  is the mode number and  $i = 1, 2, 3, \dots, \infty$ . Substituting this expression into equation (35) gives the natural frequency corresponding to the mode shape  $U_i(x)$ :

$$\omega_i = (i^2\pi^2/L^2)\sqrt{\tilde{D}/\tilde{m}}\sqrt{1 + N_{x,s}L^2/i^2\tilde{D}\pi^2}. \quad (38)$$

It is observed that the natural frequency is clearly larger than that obtained when the axial force  $N_{x,s}$  is absent. A case study in which non-linear deflections, temperature and control effects of the non-linear piezothermoelastic laminated beam are studied is presented next.

#### 4. CASE STUDIES

It is assumed that a simply supported three layer PZT/steel/PZT beam with dimensions: width  $b = 0.0508$  m, length  $L = 1$  m, steel thickness  $h_e = 0.00635$  m, and lead zirconate titanate (PZT) thickness  $h_p = 254 \times 10^{-6}$  m is used in the case study, Figure 2. Detailed material properties are summarized in Table 1 and Table 2.

Next, the bending stiffness  $\tilde{D}$  and the membrane stiffness  $\tilde{K}$  can be respectively calculated as:  $\tilde{D} = YI + 2Y_p I_p + 2I_p e_{31}^2/\epsilon_{33} = 93.765$  (Nm<sup>2</sup>);  $\tilde{K} = Y\tilde{A} + 2Y_p \tilde{A}_p + 2\tilde{A}_p e_{31}^2/\epsilon_{33} = 23.986 \times 10^6$  (N). Note that the values of  $YI$ ,  $2Y_p I_p$ , and  $2I_p e_{31}^2/\epsilon_{33}$  in the PZT/steel/PZT beam are 80, 18 and 2% of the total bending stiffness  $\tilde{D}$ , respectively, and values of  $Y\tilde{A}$ ,  $2Y_p \tilde{A}_p$ , and  $2\tilde{A}_p e_{31}^2/\epsilon_{33}$  are 92.8, 6.5 and 0.7% of the total membrane stiffness  $\tilde{K}$ . It is assumed that applied control voltages  $\phi_{33}^c$  and  $\phi_{31}^c$  are uniformly distributed and  $\phi_{31}^c = -\phi_{33}^c = \phi$ , and the temperature rise  $\theta$  is also uniform along the  $x$ -axis and is of linear variation through the thickness:  $\theta(z) = \bar{a}z + \bar{c}$ , where  $\bar{a} = (\theta_t - \theta_b)/(h_e + 2h_p)$ ,  $\bar{c} = (\theta_t + \theta_b)/2$ ;  $\theta_t$  is the top surface temperature and  $\theta_b$  is the bottom surface temperature

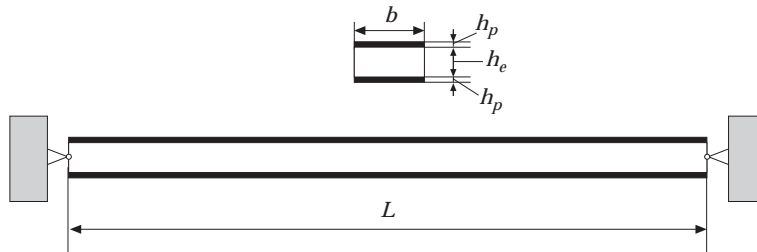


Figure 2. A PZT/steel/PZT laminated beam.



TABLE 1  
*PZT material properties*

Young's modulus (GPa)	$Y_x = Y_y = Y_z = 61$
Shear modulus (GPa)	$G_{xy} = G_{xz} = G_{yz} = 23.64$
Poisson's ratio	$\mu = 0.29$
Density (kg/m <sup>3</sup> )	$\rho = 7.7 \times 10^3$
Thermal expansion coefficient (m/m/°C)	$\alpha = 1.2 \times 10^{-6}$
Thermal stress coefficient (N/m <sup>2</sup> /°C)	$\lambda_p = 1.03 \times 10^5$
Electric permittivity (F/m)	$\epsilon_{33} = 1.65 \times 10^{-8}$
Piezoelectric constant (C/N (m/V))	$d_{31} = 171 \times 10^{-12}$
Piezoelectric stress (C/m <sup>2</sup> )	$e_{31} = 10.43$
Pyroelectric constant (C/m <sup>2</sup> /°C)	$p_3 = 0.25 \times 10^{-4}$

TABLE 2  
*Steel material properties*

Young's modulus (GPa)	$Y_x = Y_y = Y_z = 68.95$
Shear modulus (GPa)	$G_{xy} = G_{xz} = G_{yz} = 26.52$
Poisson's ratio	$\mu = 0.30$
Density (kg/m <sup>3</sup> )	$\rho = 7.75 \times 10^3$
Thermal expansion coefficient (m/m/°C)	$\alpha = 1.1 \times 10^{-5}$
Thermal stress coefficient (N/m <sup>2</sup> /°C)	$\lambda = 1.08 \times 10^6$

of the beam. Note that  $\theta_b = -\theta_t = \theta$  which implies that the total temperature difference between the top and bottom surfaces is  $2\theta$ . Then, the electric control bending moment  $M_x^c$  and the temperature induced moment  $M_x^t$  are

$$M_x^c = 0.003499\phi \text{ (Nm)}, \quad M_x^t = 0.34856\theta \text{ (Nm)}, \quad (39, 40)$$

in which 98% is due to the steel and only 2% is due to the PZT in the temperature induced bending moment. (Recall that the ceramics are less sensitive to temperatures when compared with steels.) Equation (32) then can be simplified to

$$(\tanh v/v - 1/\cosh^2 v) = 64v^4 \tilde{D}^3 / \tilde{K} (M_x^c + M_x^t)^2 L^4. \quad (41)$$

Denoting  $y_1 = (\tanh v/v - 1/\cosh^2 v)$  and  $y_2 = 64v^4 \tilde{D}^3 / [\tilde{K} (M_x^c + M_x^t)^2 L^4]$ , one can plot  $y_1(v)$  and  $y_2(v)$ . Intersections of  $y_1(v)$  and  $y_2(v)$  gives solutions  $v$  of equation (41), such as shown

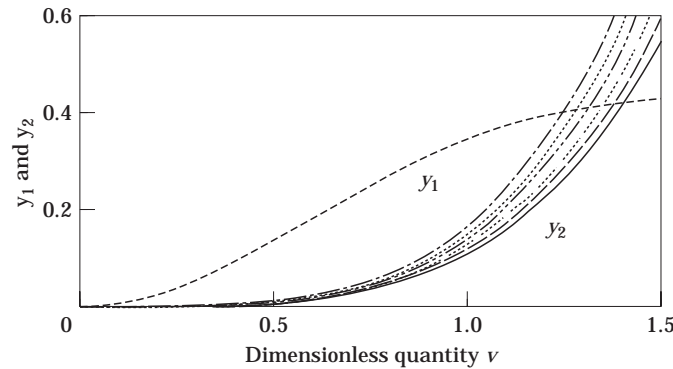


Figure 3. Solution  $v$  for various control voltages:  $\cdots$ , 50;  $\cdots\cdots$ , 100;  $\cdots\cdots\cdots$ , 150;  $\cdots\cdots\cdots\cdots$ , 200;  $\cdots\cdots\cdots\cdots\cdots$ , 250;  $\cdots\cdots\cdots\cdots\cdots\cdots$ , 300 V; temperature  $\theta = 10^\circ\text{C}$ .

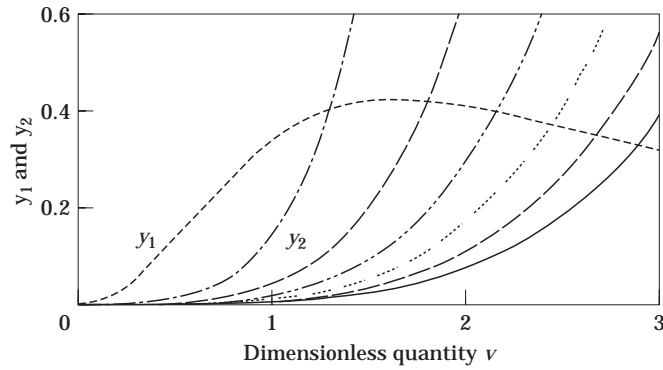


Figure 4. Solution  $v$  for various temperatures (voltage  $\varphi = 100$  V): ---, 10; ----, 20; - · - ·, 30; ·····, 40; ———, 50; - · - ·, 60°C.

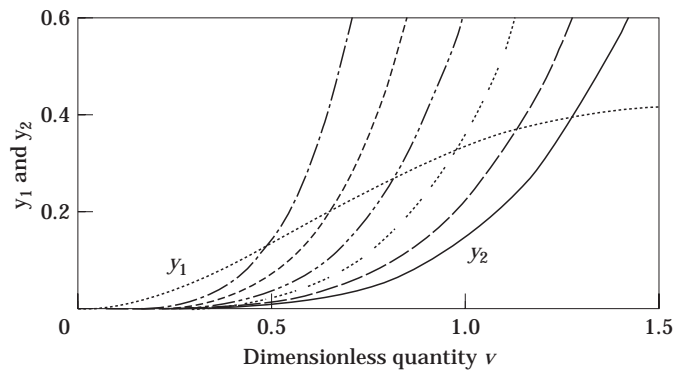


Figure 5. Solution  $v$  for various beam lengths ( $\theta = 10^\circ\text{C}$  and  $\varphi = 100$  V): ---, 0.5; ----, 0.6; - · - ·, 0.7; ·····, 0.8; ———, 0.9; - · - ·, 1.0 m.

in Figures 3–5. Then, the axial force  $N_{x,s}$  and the beam center deflection (at  $x = L/2$ ) can be calculated and its temperature/control effects studied:

$$N_{x,s} = 4v^2\tilde{D}/L^2, \quad u_{z,s}|_{x=L/2} = (M_x^c + M_x^t)L^2/4\tilde{D}v^2(1/\cosh v - 1). \quad (42, 43)$$

Detailed numerical results of the PZT/steel/PZT laminated beam subjected to various temperatures and control voltages are listed in Tables A1–A3 in Appendix A. Static deflection of the beam center ( $x = L/2$ ) with respect to the applied control voltage (at

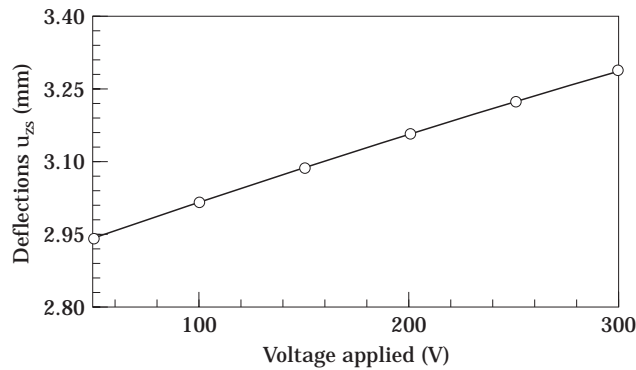


Figure 6. Static deflections versus voltage changes (temperature  $\theta = 10^\circ\text{C}$ ).

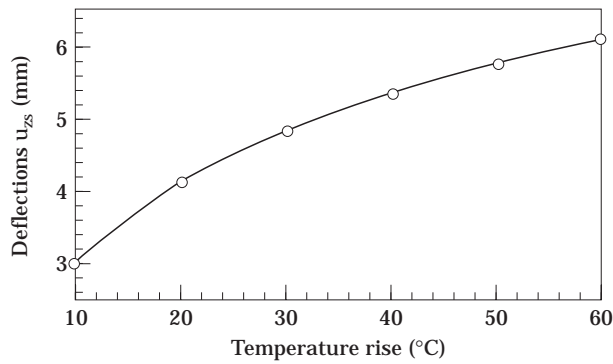


Figure 7. Static deflections versus temperature changes (voltage  $\phi = 100$  V).

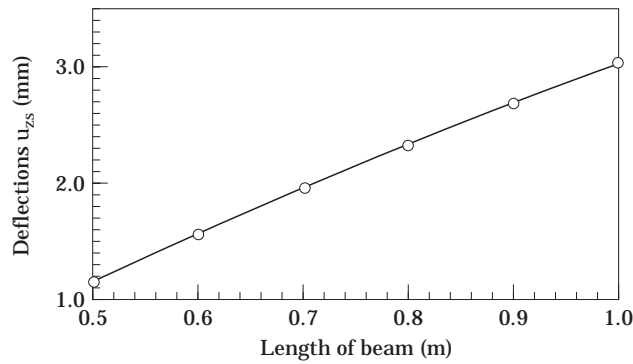


Figure 8. Static deflections for various beam lengths ( $\theta = 10^\circ\text{C}$  and  $\phi = 100$  V).

$\theta = 10^\circ\text{C}$ ), temperature rise (at  $\phi = 100$  V), and beam length (with  $\phi = 100$  V,  $\theta = 10^\circ\text{C}$ ) are plotted in Figures 6–8. Note that the  $10^\circ\text{C}$  temperature represents a total of  $20^\circ\text{C}$  difference between the top and bottom surfaces. The deflection and voltage relation, Figure 6, gives a general guideline that the control voltage induced displacement can be used to compensate the temperature induced deflection or the non-linear deflection. Equivalent axial force with respect to the beam center deflection is presented next.

Figure 9 shows the axial force versus the static deflections of the beam center, which reveals that the induced axial control force stiffens the beam and consequently the natural frequencies of the beam increase. (Note that this force can also be viewed as

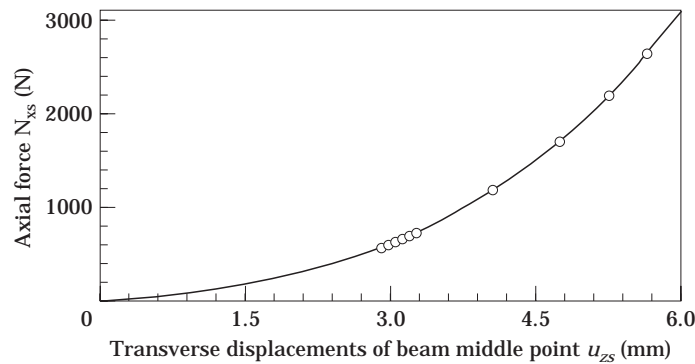


Figure 9. Axial forces versus beam deflections.

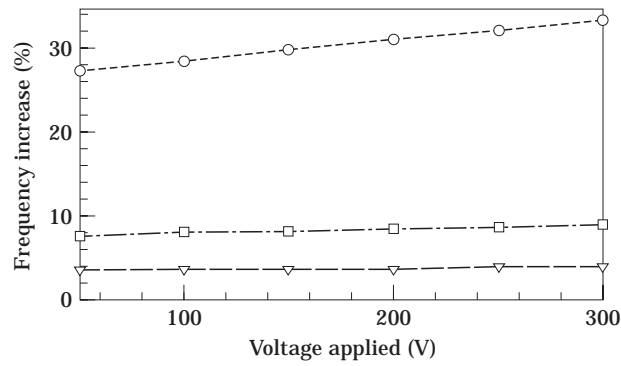


Figure 10. Frequency variations versus control voltages ( $\phi = 10^\circ\text{C}$ ):  $\circ$ ,  $f_1$ ;  $\square$ ,  $f_2$ ;  $\nabla$ ,  $f_3$ .

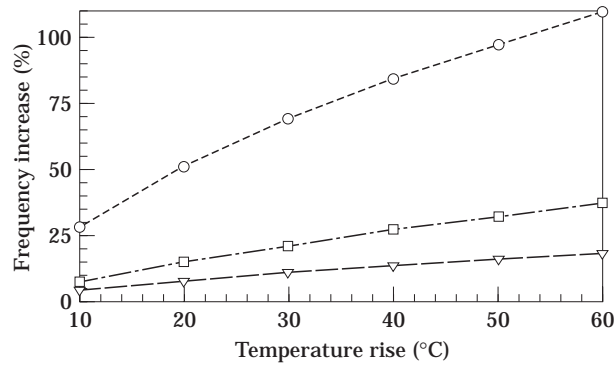


Figure 11. Frequency variations versus temperatures ( $\phi = 100\text{ V}$ ):  $\circ$ ,  $f_1$ ;  $\square$ ,  $f_2$ ;  $\nabla$ ,  $f_3$ .

an axial control force.) The frequency increase can be expressed by the quantity  $\{[1 + (N_{x,s} L^2 / i^2 \tilde{D} \pi^2)]^{1/2} - 1\} \times 100$  percent, where  $i$  is the mode number, and the results are shown in Figures 10–12. The percentage of variation for the first mode is higher than those of the higher modes. The numerical results suggest that both static deflection and dynamic behaviors of the simply supported non-linear PZT/steel/PZT laminated beam are influenced by the temperature and they also can be controlled by the control voltages applied to the piezoelectric actuators.

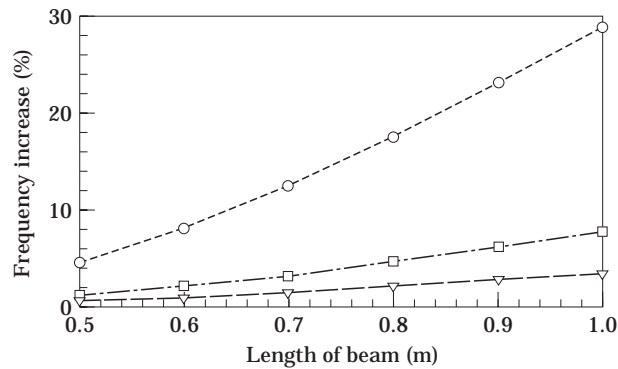


Figure 12. Frequency variations versus lengths ( $\phi = 10^\circ\text{C}$ ,  $\phi = 100\text{ V}$ ):  $\circ$ ,  $f_1$ ;  $\square$ ,  $f_2$ ;  $\nabla$ ,  $f_3$ .

## 5. SUMMARY AND CONCLUSIONS

The conventional analysis of piezoelectric laminated structures and systems often utilizes small deflection linear theories. Response behavior and control characteristics of non-linear piezoelectric systems are relatively unknown. In this study, non-linear static deflections, dynamic characteristic, temperature effects, and control characteristics of a simply supported piezothermoelastic laminated beam with an initial non-linear large static deflection were investigated.

A detailed mathematical model of the piezoelectric laminated beam was defined and analytical solutions including temperature, control voltage, induced axial force and bending moment effects were derived. The reduced non-linear beam equation was identical to the classical non-linear beam equation, if the temperature and electric terms were removed. Non-linear static deflections with the influence of temperature and control voltage were studied. Small amplitude oscillations with respect to the non-linearly deformed static equilibrium position were investigated. It was observed that the total bending stiffness of the PZT/steel/PZT laminated beam is 80% due to the steel and 20% due to the PZT (elasticity: 18% and piezoelectricity: 2%); the total membrane stiffness is 92.8% due to the steel and 7.2% due to PZT (elasticity: 6.5% and piezoelectricity: 0.7%) in the laminated beam. The stiffness (contributed) by piezoelectricity is relatively insignificant. Simulation results also suggested that the voltage induced control displacement/force can be used to compensate the non-linear static deflection, temperature effects, and natural frequencies of the piezoelectric laminated beam.

## ACKNOWLEDGMENTS

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## APPENDIX A: DETAILED NUMERICAL RESULTS

TABLE A1

*Voltage effect of the laminated beam ( $\theta = 10^\circ\text{C}$ ,  $L = 1\text{ m}$ )*

$\phi$ (V)	roots $v$	axial force $N_{x,s}$ (N)	mid. point disp. $u_{z,s}$ (mm)	freq. increase (%)		
				$i = 1$	$i = 2$	$i = 3$
50	1.248	584.2	2.942	27.72	7.60	3.45
100	1.281	615.5	3.018	29.04	7.99	3.63
150	1.314	647.6	3.089	30.37	8.39	3.81
200	1.345	678.5	3.159	31.65	8.78	3.99
250	1.375	709.1	3.226	32.90	9.16	4.17
300	1.405	740.4	3.290	34.17	9.54	4.35

TABLE A2

*Temperature effect of the laminated beam ( $\phi = 100\text{ V}$ ,  $L = 1\text{ m}$ )*

$\theta$ ( $^\circ\text{C}$ )	roots $v$	axial force $N_{x,s}$ (N)	mid. point disp. $u_{z,s}$ (mm)	freq. increase (%)		
				$i = 1$	$i = 2$	$i = 3$
10	1.281	615.5	3.018	29.04	7.99	3.63
20	1.791	1203.1	4.110	51.66	15.11	6.98
30	2.147	1728.9	4.810	69.36	21.12	9.89
40	2.429	2212.9	5.329	84.15	26.40	12.50
50	2.667	2667.8	5.743	97.05	31.17	14.90
60	2.877	3104.4	6.081	108.68	35.60	17.16

TABLE A3

*Effect of beam length ( $\theta = 10^\circ\text{C}$ ,  $\phi = 100\text{ V}$ )*

$L$ (m)	roots $v$	axial force $N_{x,s}$ (N)	mid. point disp. $u_{z,s}$ (mm)	freq. increase (%)		
				$i = 1$	$i = 2$	$i = 3$
0.5	0.483	350.0	1.165	4.62	1.17	0.52
0.6	0.650	440.2	1.564	8.22	2.12	0.95
0.7	0.817	510.9	1.957	12.72	3.33	1.49
0.8	0.979	561.7	2.332	17.83	4.74	2.14
0.9	1.134	595.4	2.685	23.34	6.32	2.85
1.0	1.281	615.5	3.018	29.04	7.99	3.63