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LETTERS TO THE EDITOR

AN ANALYTICAL MODEL FOR THE FORCED RESPONSE OF CYCLIC STRUCTURES

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1. Introduction

Dynamic models of complicated structures are typically of the finite element type, although considerable work has been done using receptance theory with partial differential equations (see, for example, Ewins [1] or Nicholson [2]). Component mode synthesis has also been associated with large structural models, but is usually of the discrete form (see Hurty [3]). Receptance theory requires the formulation of component receptances, which are generally formed from infinite series of component eigenfunctions. This method may leave the analyst with concerns about the proper form of the cross-receptances or about convergence of the solution. Some problems, such as cyclic structures, are amenable to exact solutions without receptance determinations. It is this class of problems that is of concern here.

In the classical analysis for the forced response of continuous systems (cf. Meirovitch [4], chapter 7), such as beams and plates, the orthogonality of the eigenfunctions is used to uncouple the equation of motion into independent second order ordinary differential equations for the modal responses. As the orthogonality property can be proven for self-adjoint systems, the self-adjointedness of the mass and stiffness operators of the differential equation of motion for the structure is sufficient evidence that an undamped system will uncouple.

The classical theory, as delineated by Meirovitch [4], must be adapted for use with assembled structures. Such structures do not have orthogonal modes in the same sense as an individual component structure. In fact, an assembly of n similar components will have n modes corresponding to each individual mode of the stand-alone component. If the components are identical, the eigenfunctions of each component (every mode of the assembly has an eigenfunction, possibly null, for each component) corresponding to a single mode may be similar or identical. Thus, these eigenfunctions are not orthogonal to each other. Furthermore, the other n-1 modes of the assembly corresponding to the given mode of a stand-alone component also have similar or identical eigenfunctions among all components and modes. Therefore, if one were to (as in the classical method) expand the response of each component into an infinite series of eigenfunctions for that component, substitute that expansion into the differential equation of motion for that component, multiply the resulting equation by the eigenfunction of an arbitrary mode, and integrate over the domain of the component, as many as n terms of the expansion could be non-zero even though the component equation of motion is self-adjoint in the usual sense. There would be n such seemingly insoluble coupled equations for the modal responses.

In this letter, the above dilemma is resolved by assembly of the component equations in a natural way that takes account of the proper contribution of any forces applied to any component. The resulting assembled equations can be treated in the usual manner as uncoupled modal equations, providing a new form of self-adjointedness is satisfied by the

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equations of motion of the assembly. The method is illustrated using an assembly of Euler–Bernoulli beams.

2. ORTHOGONALITY OF MODES IN ASSEMBLED CYCLIC STRUCTURES

The individual modes of a cyclic structure must possess some type of orthogonality in order for the modal equations to uncouple. For each mode of vibration, there is an eigenfunction for each component. The individual eigenfunctions for each component of a given mode will be quite similar, depending upon how well tuned the cyclic structure is (how close the components resemble each other). However, different modes in a set of n modes corresponding to a single mode of a stand-alone component differ in the magnitude and phase of the component eigenfunctions (in the assembly eigensolution, the magnitudes of the eigenfunctions of the components in a given mode cannot be scaled independently, as they are dictated by the eigensolution). It is this difference in the magnitudes and phases of the component eigenfunctions that provides the orthogonality of the modes.

The idea behind the orthogonality of cyclic structures is simple. Assuming the eigensolution is available, the expansion theorem is used to write the response of each component as an infinite series of the eigenfunctions of that component. Adapting the notation of Meirovitch [4], chapter 5, the response of component i, w_i, would be expanded to

$$w_i\,(x, y, t) = \sum_{r=1}^{\infty} \eta_r\,(t) W_{ir}\,(x, y), \tag{1}$$

where the subscript i refers to the component, the subscript r refers to the mode, η_r are the time dependent modal co-ordinates, and W_{ir} are the individual eigenfunctions of each component. These n expansions are substituted into their corresponding component equations of motion. If one again uses Meirovitch's notation for the mass and stiffness operators of a component, one can write these equations as

$$L_i\,[w_i] + M_i \frac{\partial^2 w_i}{\partial t^2} = L_i\left[\sum_{r=1}^{\infty} \eta_r\,(t) W_{ir}\right] + M_i \frac{\partial^2}{\partial t^2}\left(\sum_{r=1}^{\infty} \eta_r\,(t) W_{ir}\right) = f_i, \tag{2}$$

where L_i are the assumed linear stiffness operators for the component equations of motion, M_i are the mass operators, and f_i is the force applied to component i. Next, these n equations are multiplied by the eigenfunction for their respective components of an arbitrary mode. The resulting products are then integrated over the domain of the components. If one also assumes harmonic time dependence at frequency ω, as usual in linear structural systems, these equations may be written as

$$\int_D L_i\left[\sum_{r=1}^{\infty} \eta_r\,(t) W_{ir}\right] W_{is}\,\mathrm{d}D - \omega^2 \int_D M_i \left(\sum_{r=1}^{\infty} \eta_r\,(t) W_{ir}\right) W_{is}\,\mathrm{d}D = \int_D f_i\, W_{is}\,\mathrm{d}D = \bar{f}_{is}. \tag{3}$$

Finally, the n resulting equations are summed. If the modes are othogonal in this assembled sense, only one term in the series for each response (the s term) will survive the summation, and any applied forces will be accounted for through the integration and summation as time dependent modal forces.

For the modes to be orthogonal, the assembly equations must possess an assembly self-adjointedness. As one might now guess, self-adjointedness of the assembly is determined according to the summation of the usual procedure. Thus, for any two sets of comparison functions u_i and v_i, the assembly is said to be self-adjoint if

$$\sum_{i=1}^{n} \int_D L_i\,[u_i]v_i\,\mathrm{d}D = \sum_{i=1}^{n} \int_D L_i\,[v_i]u_i\,\mathrm{d}D \tag{4}$$

and

$$\sum_{i=1}^{n} \int_D M_i\,[u_i]v_i\,\mathrm{d}D = \sum_{i=1}^{n} \int_D M_i\,[v_i]u_i\,\mathrm{d}D \tag{5}$$

The usual proof of orthogonality follows from the assumption of self-adjointedness and non-coincident eigenvalues (see Meirovitch [4, section 5-5] for the proof in the case of a single component). For perfectly tuned cyclic structures, coincident eigenvalues are common, but orthogonal modes for such eigenvalues are generally obtainable through the use of engineering intuition. This claim as well as all of the above developments and more will be made clear through a simple example.

3. EXAMPLE OF A FOUR-BLADED DISK

Here the example of four identical, uniform Euler–Bernoulli beams connected by linear and torsional springs to ground and to each other is used to illustrate the method of obtaining the modal equations of motion for the forced response of a cyclic structure. This model, although highly idealized, is intended to represent a bladed disk. The number of blades was chosen as the smallest model for which all relevant characteristics would be manifest. The blades are mounted symmetrically about a circle and lie in a plane. Vibration is restricted to the same plane. Figure 1 illustrates the model features for an arbitrary number of blades. At the "hub" end, each beam rotates against a grounded torsional spring k_{rg} and translates against a grounded linear spring k_{lg}. A set of coupling springs connects the beams to each other. The beams are free at the other end. The displacement of beam i is designated $y_i\,(x, t)$, and the independent variable x is not distinguished among the beams. As Euler–Bernoulli beam theory assumes small, linear deformations, the rotation of the torsional springs is assumed equal to the slope of the beam displacement at the "hub", $\partial y_i\,/\partial x(0, t)$. Additionally, the angle the linear coupling springs k_{lc} make with the beams is assumed to be a right angle, as proper accounting of the force in the spring would only involve multiplication by a common angle sine, which could be incorporated

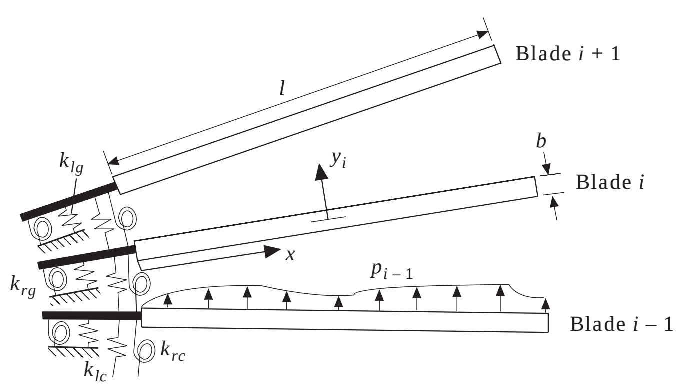

Figure 1. Idealized bladed-disk assembly for an arbitrary number of blades.

into the spring constant k_{lc}. If the beam flexural rigidities are designated $(EI)_i$ (leaving the possibility that each beam may have different properties, although in this example they will be identical), the masses per unit length are designated m_i, and the applied forces are designated p_i, application of Hamilton's principle for uniform beams yields the equations of motion

$$(EI)_i\,\partial^4 y_i/\partial x^4 + m_i \frac{\partial^2 y_i}{\partial t^2} = p_i(x, t) \tag{6}$$

and the boundary conditions

$$(EI)_i \frac{\partial^2 y_i}{\partial x^2}(0, t) - k_{rg}\frac{\partial y_i}{\partial x}(0, t) + k_{rc}\left[\frac{\partial y_{i+1}}{\partial x}(0, t) + \frac{\partial y_{i-1}}{\partial x}(0, t) - 2\frac{\partial y_i}{\partial x}(0, t)\right] = 0, \tag{7}$$

$$(EI)_i \frac{\partial^3 y_i}{\partial x^3}(0, t) + k_{lg}\,y_i(0, t) + k_{lc}\,[2y_i(0, t) - y_{i+1}(0, t) - y_{i-1}(0, t)] = 0, \tag{8}$$

$$\partial^2 y_i/\partial x^2(l, t) = \partial^3 y_i/\partial x^3(l, t) = 0. \tag{9}$$

It can be easily verified that these equations are self-adjoint according to (4) and (5). Note that this holds true even when the blades are "mistuned", such that any or all of the beams may have different values of E, b, h, or l.

The eigenanalysis is performed in the usual way by assuming harmonic motion and separation of variables:

$$y_i(x, t) = Y_i(x)g(t) = (A_i \cos\beta_i x + B_i \sin\beta_i x + C_i \cosh\beta_i x + D_i \sinh\beta_i x)\,e^{j\omega t}, \tag{10}$$

where $\beta_i^4 = m_i\omega^2/(EI)_i$. As there are four boundary conditions for each beam, the eigenvalue problem consists of a 16×16 block-circulant matrix (some properties of square circulant matrices are given by Pierre and Murthy [5]) whose determinant will vanish when ω coincides with a natural frequency. Once a natural frequency, say ω_k, is found, the eigenfunctions Y_{ik} may be determined by assigning a value to one of the coefficients $A_i - D_i$, eliminating the corresponding row from the boundary condition matrix, shifting the corresponding column to the other side of the equation, and inverting the remaining 15×15 matrix. Using rectangular steel beams ($E = 220{\cdot}5$ GPa, density 7500 kg/m^3) of dimension $0{\cdot}01 \times 0{\cdot}02 \times 0{\cdot}20$ m, with the smallest dimension corresponding to the thickness b, the first four modes were determined and are plotted in Figure 2. As modes 2 and 3 have coincident eigenvalues, the third shape was determined by a simple permutation of the second. The spring values used were $k_{rg} = 120$ kNm/rad, $k_{lg} = 120$ MN/m, $k_{rc} = 4$ kNm/rad, and $k_{lc} = 4$ MN/m, and the corresponding natural frequencies were $\omega_1 = 1334{\cdot}0125$ r/s, $\omega_2 = \omega_3 = 1336{\cdot}5454$ r/s, and $\omega_4 = 1338{\cdot}7917$ r/s. In the next four modes, each beam has the approximate shape of the second mode of a single cantilever beam, but the individual beams are phased differently again for different modes in that set, and so on for all higher modes.

To determine the forced response, one must first assign a load. For simplicity, a harmonic point load on the tip of blade 1 was chosen. This load can be represented, using Dirac's delta function, as $f_1 = F\delta(x - l)\sin\omega t$ and $f_2 = f_3 = f_4 = 0$.

To obtain the modal equations for the forced response, one first employs the steps given by equations (1–3) to each beam. The summations in (3) are not eliminated in this step,

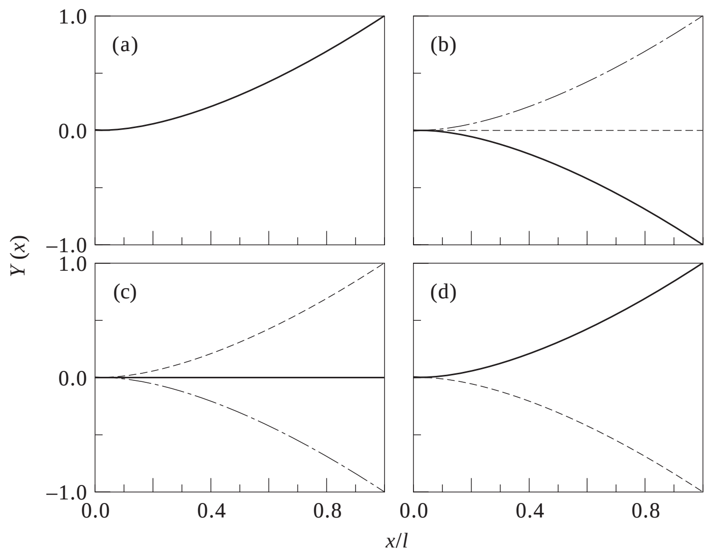

Figure 2. First four modes of a four-bladed disk with $E = 220{\cdot}5$ GPa, $m = 1{\cdot}5$ kg/m, $l = 0{\cdot}2$ m, $b = 0{\cdot}01$ m and $h = 0{\cdot}02$ m. (a) Mode 1: —, $Y_{11} = Y_{21} = Y_{31} = Y_{41}$; (b) Mode 2: —, Y_{12}; ----, $Y_{22} = Y_{42}$; —-—, Y_{32}; (c) Mode 3: —, $Y_{13} = Y_{33}$; ----, Y_{23}; —-— Y_{43}; (d) Mode 4: —, $Y_{14} = Y_{34}$; ----, $Y_{24} = Y_{44}$.

but they are eliminated when these equations are summed together. For example, the first three modal equations are given by

$$\sum_{i=1}^{4}\left((EI)_i\,\beta_{i1}^4\int_0^l Y_{i1}^2\,\mathrm{d}x\right)\eta_1(t) + \sum_{i=1}^{4}\left(m_i\int_0^l Y_{i1}^2\,\mathrm{d}x\right)\ddot{\eta}_1(t) = FY_{11}(l)\sin\omega t, \tag{11}$$

$$\sum_{i=1}^{4}\left((EI)_i\,\beta_{i2}^4\int_0^l Y_{i2}^2\,\mathrm{d}x\right)\eta_2(t) + \sum_{i=1}^{4}\left(m_i\int_0^l Y_{i2}^2\,\mathrm{d}x\right)\ddot{\eta}_2(t) = FY_{12}(l)\sin\omega t, \tag{12}$$

$$\sum_{i=1}^{4}\left((EI)_i\,\beta_{i3}^4\int_0^l Y_{i3}^2\,\mathrm{d}x\right)\eta_3(t) + \sum_{i=1}^{4}\left(m_i\int_0^l Y_{i3}^2\,\mathrm{d}x\right)\ddot{\eta}_3(t) = FY_{13}(l)\sin\omega t = 0, \tag{13}$$

where $\beta_{is}^4 = m_i\,\omega_s^2/(EI)_i$, and the summation terms can be recognized as modal stiffnesses and masses.

The modal responses η_s are easily determined from expressions like equation (11) by assuming a sine function for the particular part. Initial conditions can also be incorporated by transforming physical initial conditions into modal space, if a transient solution is desired. Of course, exact physical responses can only be obtained by consideration of the infinity of modal responses and back-substitution into the expansion theorem (1). The usual convergence rules should be applied in truncating the series of modal responses for engineering approximations.

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