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TRAVELLING WAVE PACKETS IN AN INFINITE THIN CYLINDRICAL SHELL UNDER INTERNAL PRESSURE

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The Flügge type basic equations for an infinitely long thin cylindrical shell, including the effect of initial tensions due to non-uniform internal pressure, are employed, and by using the complex WKB method the solution of the basic equations is constructed in the form of superposition of the packets of short bending, longitudinal and torsional waves. The dependence of frequencies, group velocities, amplitudes and other dynamic characteristics upon variable pressure is examined.

1. INTRODUCTION

Thin long circular cylindrical shells are used in many engineering structures, such as trunk pipelines or drill pipes. Vibrations of long or infinitely long cylindrical shells under initial pressure are of great practical interest [1–4]. In particular, the effect of initial uniform circumferential stress on the dynamic response of an infinitely long circular cylindrical shell has been studied in reference [1], where motion has been supposed to be independent of the axial co-ordinate. Vibration analysis of a long rotating cylindrical shell which includes both the effect of uniform external loading and the influence of uniform initial stresses caused by rotation has been presented in references [2, 4]. In these and other papers solutions of basic equations with constant coefficients have been found in the form of harmonic waves.

The general goal of the present paper is to study running short waves in a thin infinitely long cylindrical shell subjected to internal non-uniform pressure. A shell is supposed to have in its surface local perturbations which are treated as the initial conditions. This problem does not admit solutions in the form of harmonic waves. The specific goal defined herein is to state the modified complex WKB method [5] for constructing solutions of the governing equations with variable coefficients in the form of superposition of the localized families (packets) of bending, longitudinal and torsional waves travelling in the axial direction. Earlier, this method has been used to study the running packets of bending waves in a non-circular cylindrical shell with slanting edges [6], and in an infinite shell of revolution [7]. The present investigation examines also the dependence of frequencies, group velocities, amplitudes and other dynamic characteristics of the travelling wave packets upon variable pressure.

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2. THE GOVERNING EQUATIONS

The co-ordinate system is considered as shown in Figure 1. The circular cylindrical shell is assumed to be elastic, isotropic, infinite and sufficiently thin for applicability of the assumptions of classic shell theory. For analysis of wave propagation in the shell the Flügge type basic equations [8], including the effect of the initial stresses, are used:

$$\frac{\partial \tilde{N}_x}{\partial \tilde{x}} + \frac{1}{R}\frac{\partial \tilde{N}_{\varphi x}}{\partial \varphi} + N_\varphi^* \frac{\partial^2 \tilde{u}_1}{\partial \tilde{x}^2} + \frac{1}{R^2} N_\varphi^* \frac{\partial^2 \tilde{u}_1}{\partial \varphi^2} - \rho h \frac{\partial^2 \tilde{u}_1}{\partial \tilde{t}^2} = 0,$$

$$\frac{1}{R}\frac{\partial \tilde{N}_\varphi}{\partial \varphi} - \frac{1}{R^2}\frac{\partial \tilde{M}_\varphi}{\partial \varphi} + \frac{\partial \tilde{N}_{x\varphi}}{\partial \tilde{x}} - \frac{1}{R}\frac{\partial \tilde{M}_{x\varphi}}{\partial \tilde{x}} + N_x^* \frac{\partial^2 \tilde{u}_2}{\partial \tilde{x}^2}$$

$$+ \frac{1}{R} N_\varphi^* \left(\frac{\partial^2 \tilde{u}_2}{\partial \varphi^2} - \tilde{u}_2 + 2\frac{\partial \tilde{u}_3}{\partial \varphi} \right) - \rho h \frac{\partial^2 \tilde{u}_2}{\partial \tilde{t}^2} = 0,$$

$$\frac{\partial^2 \tilde{M}_x}{\partial \tilde{x}^2} + \frac{1}{R^2}\frac{\partial^2 \tilde{M}_\varphi}{\partial \varphi^2} + \frac{1}{R}\tilde{N}_\varphi + \frac{1}{R}\frac{\partial^2 \tilde{M}_{x\varphi}}{\partial \tilde{x}\,\partial \varphi} + \frac{1}{R}\frac{\partial^2 \tilde{M}_{\varphi x}}{\partial \tilde{x}\,\partial \varphi}$$

$$- N_x^* \frac{\partial^2 \tilde{u}_3}{\partial \tilde{x}^2} + \frac{1}{R^2} N_\varphi^* \left(2\frac{\partial \tilde{u}_2}{\partial \varphi} + \tilde{u}_3 - \frac{\partial^2 \tilde{u}_3}{\partial \varphi^2} \right) + \rho h \frac{\partial^2 \tilde{u}_3}{\partial \tilde{t}^2} = 0. \quad (1)$$

Here $\tilde{N}_x, \tilde{N}_\varphi, \tilde{N}_{\varphi x}, \tilde{M}_x, \tilde{M}_\varphi, \tilde{M}_{x\varphi}$ are the stress and moment resultants, $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ are the components of displacement (see Figure 1), R, ρ, h are the radius density and thickness of the shell, respectively, and N_x^*, N_φ^* are the initial tensions due to load. The case when the shell experiences the normal and internal non-uniform pressure

$$\tilde{P}(\tilde{x}) = -[Eh/R(1-v^2)]P(Rx) \leqslant 0 \quad (2)$$

is considered here, where v is Poisson's ratio, E is Young's modulus, $x = \tilde{x}/R$, $P(Rx) = f(x, t)$ is an infinitely differentiable non-negative function.

It should be noted that equations (1) represent the state of a shell perturbed from its membrane state. It is assumed here that $f(x)$ is a slowly varying function so that the axial and hoop stresses due to pressure (2) may be found from the equations of the membrane shell theory [9], as follows:

$$N_x^* = 0, \qquad N_\varphi^* = [Eh/(1-v^2)]\, f(x). \quad (3)$$

By substituting equations (3) and also the relationships [8] between the stress and moment resultants and the displacements into the foregoing equations and rewriting these equations in dimensionless form, the equations for the description of wave propagation are obtained as follows

$$(\mathbf{L} - \partial^2/\partial t^2)\mathbf{U}^{\mathrm{T}} = 0, \quad (4)$$

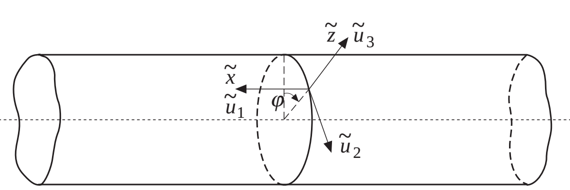

Figure 1. The neutral surface of the infinite thin cylindrical shell and the co-ordinate system.

where $\mathbf{U} = (u_1, u_2, u_3)$ is the 3-vector, the superscript T denotes a transposition, $u_j = \tilde{u}_j / R$ $(j = 1, 2, 3)$, $t = \sqrt{E/[(1 - \nu^2)\rho R^2]}\tilde{t}$ is non-dimensional time, and $\mathbf{L}$ is the 3×3 matrix of which the elements are the differential operators:

$$l_{11} = \frac{\partial^2}{\partial x^2} + [\tfrac{1}{2}(1-\nu)(1+\varepsilon^4) + f(x)]\frac{\partial^2}{\partial \varphi^2}, \qquad l_{12} = \frac{1+\nu}{2}\frac{\partial^2}{\partial x\,\partial\varphi},$$

$$l_{13} = \nu\frac{\partial}{\partial x} - \varepsilon^4\left[\frac{\partial^3}{\partial x^3} - \tfrac{1}{2}(1-\nu)\frac{\partial^3}{\partial x\,\partial\varphi^2}\right], \qquad l_{21} = l_{12},$$

$$l_{22} = [1 + f(x)]\frac{\partial^2}{\partial\varphi^2} + \tfrac{1}{2}(1-\nu)(1+3\varepsilon^4)\frac{\partial^2}{\partial x^2} - f(x),$$

$$l_{23} = [1 + 2f(x)]\frac{\partial}{\partial\varphi} - \tfrac{1}{2}\varepsilon^4(3-\nu)\frac{\partial^3}{\partial x\,\partial\varphi^2},$$

$$l_{31} = -\nu\frac{\partial}{\partial x} + \varepsilon^4\left[\frac{\partial^3}{\partial x^3} - \tfrac{1}{2}(1-\nu)\frac{\partial^3}{\partial x\,\partial\varphi^2}\right], \qquad l_{32} = \tfrac{1}{2}\varepsilon^4(3-\nu)\frac{\partial^3}{\partial x^2\,\partial\varphi} - [1 + 2f(x)]\frac{\partial}{\partial\varphi},$$

$$l_{33} = -\varepsilon^4\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\,\partial\varphi^2} + \frac{\partial^4}{\partial\varphi^4}\right) - [2\varepsilon^4 - f(x)]\frac{\partial^2}{\partial\varphi^2} - [1 + \varepsilon^4 + f(x)]. \tag{5}$$

Here $\varepsilon = \sqrt{h/R}$ is a natural smaller parameter.

3. THE INITIAL CONDITIONS

The wave forms of motion caused by the initial displacements and velocities

$$u_j\,|_{t=0} = \lambda_j^\circ(\zeta, \varepsilon)\exp\{i[m\varphi + \varepsilon^{-1}S^\circ(x)]\}, \qquad j = 1, 2, 3,$$
$$\dot{u}_j\,|_{t=0} = i\varepsilon_j\,\eta_j^\circ(\zeta, \varepsilon)\exp\{i[m\varphi + \varepsilon^{-1}S^\circ(x)]\}, \tag{6}$$

where

$$\lambda_j^\circ = \sum_{k=0}^{\infty}\varepsilon^{k/2}\lambda_{jk}^\circ(\zeta), \qquad \eta_j^\circ = \sum_{k=0}^{\infty}\varepsilon^{k/2}\eta_{jk}^\circ(\zeta), \qquad S^\circ(x) = a^\circ x + \tfrac{1}{2}b^\circ x^2, \qquad \operatorname{Im} b^\circ > 0, \tag{7}$$

$$i = \sqrt{-1}, \qquad \zeta = \varepsilon^{-1/2}x, \qquad \varepsilon_1 = \varepsilon_2 = \varepsilon^{-1}, \qquad \varepsilon_3 = 1,$$
$$a^\circ, m \sim 1, \qquad \lambda_j^\circ(\zeta, \varepsilon), \qquad \eta_j^\circ(\zeta, \varepsilon) \sim 1 \quad \text{for any } \zeta, \tag{8}$$

will be studied below. In equations (7), $a^\circ > 0, b^\circ, m$ are constants, and $\lambda_{jk}^\circ, \eta_{jk}^\circ$ are polnomials of degrees M_{jk} and K_{jk}, respectively, with complex coefficients. The symbol $\sim$ means that two quantities are of the same order at $\varepsilon \to 0$ (see the definition, e.g., in reference [10]).

The real and imaginary parts of the functions (6), with account taken of the last inequality in equations (7), define the two initial packets localized near the line $x = 0$. They approximate perturbations which may be generated in the shell by some transient forces applied along the line $x = 0$. The polynomials $\lambda_{jk}^\circ, \eta_{jk}^\circ$ are introduced in equation (6) to define the possible oscillations in amplitude of wave packets [11]. The wavelengths in these packets are proportional to 1 and ε in the circumferential and axial directions, respectively. Thus, among all possible wave forms of motion, short waves with lengths (along the shell axis) being quantities of the order $\sqrt{h/R}$ will be examined in this paper.

4. CLASSIFICATION OF SOLUTIONS

The presence of a small parameter in the governing equations (4) permits the classification of its solutions to be carried out. A general method to classify the solutions of partial differential equations for thin shells has been developed by A. L. Gol'denveizer [9, 12]. A simple example illustrating this method may be found in reference [13]. The basic idea of this method is as follows. It is necessary to pick out the unknowns which determine the type of solution. But it is desirable that the number of variables selected be as small as possible. In the present problem they are u_1, u_2, u_3. The orders of each of these variables are compared with the main small parameter ε.

$$u_1 \sim \varepsilon^{\alpha_1}, \qquad u_2 \sim \varepsilon^{\alpha_2}, \qquad u_3 \sim \varepsilon^{\alpha_3}, \tag{9}$$

where α_j are indices of the intensity of the functions u_j.

As mentioned above the lengths of waves being analyzed are proportional to 1 and ε in the circumferential and axial directions, respectively. Therefore, in equation (4) it is assumed

$$\partial/\partial x \sim \varepsilon^{-1}, \qquad \partial/\partial\varphi \sim 1. \tag{10}$$

In addition, suppose

$$\partial/\partial t \sim \varepsilon^{\beta}, \qquad f(x) \sim 1, \tag{11}$$

where β is an index of variation of the functions u_j in time. The necessity of inputting this index is explained by the various speeds and frequencies of bending, longitudinal and torsional waves [12, 14].

The problem is to find non-contradictory values for the indices α_j, β. The non-contradiction criterion is the equality of the orders of not less than two main terms in each of the equations of system (4). Depending upon which terms are the main ones, different systems and correspondingly one or another character of the solution will be obtained. Among all possible types of solutions three types will be considered, corresponding to bending, longitudinal and torsional waves.

4.1. BENDING WAVES

In this case the main terms in equation (4) are

(1st equation) $$\partial^2 u_1/\partial x^2, \qquad \nu\, \partial u_3/\partial x,$$

(2nd equation) $$\frac{1+\nu}{2}\frac{\partial^2 u_1}{\partial x\,\partial\varphi}, \qquad \frac{1-\nu}{2}\frac{\partial^2 u_2}{\partial x^2}, \qquad (1+2f)\frac{\partial u_3}{\partial\varphi}$$

(3rd equation) $$-\nu\frac{\partial u_1}{\partial x}, \qquad -\varepsilon^4\frac{\partial^4 u_3}{\partial x^4}, \qquad -u_3, \qquad f\left(\frac{\partial^2 u_3}{\partial\varphi^2} - u_3\right), \qquad -\frac{\partial^2 u_3}{\partial t^2}, \tag{12}$$

and their orders are, respectively, equal to

$$\begin{gathered} \alpha_1 - 2, \qquad \alpha_3 - 1, \\ \alpha_1 - 1, \qquad \alpha_2 - 2, \qquad \alpha_3, \\ \alpha_1 - 1, \qquad 4 + \alpha_3 - 4, \qquad \alpha_3, \qquad \alpha_3, \qquad \alpha_3 - 2\beta. \end{gathered} \tag{13}$$

Because the indices of intensity of the initial displacements are equal to zero (see equations (8)), for bending vibrations it should be assumed that $\alpha_3 = 0$, $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$. Equating the orders in (13) one obtains $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta = 0$. Then, taking into account expressions (9), one has

$$u_1 = \varepsilon u_z, \qquad u_2 = \varepsilon^2 v_z, \qquad u_3 = w_z, \tag{14}$$

where $u_z, v_z, w_z \sim 1$. As a result, system (4) may be replaced by the system

$$(\mathbf{L} - \varepsilon^2 \partial^2/\partial t_1^2)\,(\mathbf{E}_{z\varepsilon}\,\mathbf{U}_z^{\mathrm{T}}) = 0, \tag{15}$$

where,

$$\mathbf{U}_z = (u_z, v_z, w_z), \qquad \mathbf{E}_{z\varepsilon} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $t_1 = \varepsilon t$ is "slow" time which is introduced to emphasize different variability of bending and tangential waves in time. The functions (14) will be the solution of the governing equations if u_z, v_z, w_z satisfy equation (15). The distinctive property of this solution is in the function u_3 being main in the asymptotic sense. The tangential components u_1, u_2 of the point displacement are here "generated" by u_3. Therefore, solution (14) will define mainly bending waves.

4.2. LONGITUDINAL WAVES

To examine the longitudinal waves the main terms in equation (4) are held to be

(1st equation) $\qquad \partial^2 u_1/\partial x^2, \qquad -\partial^2 u_1/\partial t^2,$

(2nd equation) $\qquad \dfrac{1+v}{2}\dfrac{\partial^2 u_1}{\partial x\,\partial\varphi}, \qquad \dfrac{1-v}{2}\dfrac{\partial^2 u_2}{\partial x^2}, \qquad -\dfrac{\partial^2 u_2}{\partial t^2},$

(3rd equation) $\qquad -v\,\partial u_1/\partial x, \qquad -\partial^2 u_3/\partial t^2. \qquad (16)$

Here it is supposed that $\alpha_1 = 0$. Equating the orders of the main terms (16) gives $\alpha_2 = 1$, $\alpha_3 = 1$, $\beta = -1$. Then

$$u_1 = u_x, \qquad u_2 = \varepsilon v_x, \qquad u_3 = \varepsilon w_x, \tag{17}$$

where $u_x, v_x, w_x \sim 1$, will be the solution of system (4) if the functions u_x, v_x, w_x satisfy the system of equations

$$(\mathbf{L} - \partial^2/\partial t^2)\,(\mathbf{E}_{x\varepsilon}\,\mathbf{U}_x^{\mathrm{T}}) = 0, \tag{18}$$

where,

$$\mathbf{U}_x = (u_x, v_x, w_x), \qquad \mathbf{E}_{x\varepsilon} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}.$$

In this case the component u_1 is main, and solution (17) will represent mainly longitudinal waves.

4.3. TORSIONAL WAVES

Here $\alpha_2 = 0$, $\alpha_1 = 1$, $\alpha_3 = 2$, $\beta = -1$. Then the functions

$$u_1 = \varepsilon u_y, \qquad u_2 = v_y, \qquad u_3 = \varepsilon^2 w_y \tag{19}$$

will correspond to torsional waves, where $\mathbf{U}_y = (u_y, v_y, w_y)$ is the solution of the equations

$$(\mathbf{L} - \partial^2/\partial t^2)\,(\mathbf{E}_{y\varepsilon}\,\mathbf{U}_y^{\mathrm{T}}) = 0, \tag{20}$$

where

$$\mathbf{E}_{y\varepsilon} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon^2 \end{pmatrix}.$$

It should be noted that other possible types of solutions are not considered here because they do not satisfy conditions (8), (10).

Taking into account the linearity of the governing equations as well as the classification of its solutions, one seeks the solution of equations (4) in the form of a superposition of longitudinal, torsional and bending waves:

$$\mathbf{U} = \mathbf{E}_{x\varepsilon}\,\mathbf{U}_x^{\mathrm{T}} + \mathbf{E}_{y\varepsilon}\,\mathbf{U}_y^{\mathrm{T}} + \mathbf{E}_{z\varepsilon}\,\mathbf{U}_z^{\mathrm{T}}. \tag{21}$$

Equations (15), (18), (20) contain variable coefficients and do not permit exact analytical solutions to be found. Hence, in this investigation use will be made of the asymptotic method based on expansions in powers of a small parameter ε and occupying a central place in the theory of thin shells.

5. BASIC IDEAS OF THE WKB METHOD

This section is concerned with the history and underlying concepts of the WKB method being used and developed below. As an illustrative example, consider the partial differential equation

$$u''_{tt} = v^2(x)u''_{xx}, \tag{22}$$

describing longitudinal vibrations of a non-homogeneous bar. The substitution of

$$u = y(x)\exp(\mathrm{i}\omega t), \tag{23}$$

where $\omega \to \infty$, leads to an equation with a small parameter $\mu = \omega^{-1}$ in the derivative term:

$$\mu^2 y'' + a(x)y = 0, \qquad a(x) = v^{-2}(x). \tag{24}$$

This is the model example for many more complicated equations and systems. Numerous studies have been devoted to this or similar equations (see, e.g., references [10, 15–17]). In these and other papers, the formal asymptotic solution of equation (24) is assumed to be of the form

$$y \cong \sum_{k=0}^{\infty} \mu^k a_k(x)\exp\{\mathrm{i}\mu^{-1}S(x)\}. \tag{25}$$

The symbol $\cong$ here means that the series is an asymptotic expansion of the function y in the Poincaré sense (see the definitions in reference [10]). Substituting equation (25) into equation (24) and equating the coefficients of μ^k ($k = 0, 1, \ldots$) yields a sequence of equations to define the functions S, a_k. The asymptotic solutions of type (25) are called the WKB approximations, and the method of construction of such solutions is said to be the WKB method. This name comes from the first letters of the authors' names: Wentzel, Kramers and Brillouin, who first applied this method to problems of quantum mechanics. The approximate solution (23), (25) describes high-frequency vibrations, with the wavelengths being proportional to a small parameter $\mu = \omega^{-1}$. Therefore, the solutions of type (23), (25) are often called short-wave asymptotics. Afterwards, this method has been generalized to partial differential equations. In particular, the short wave asymptotic theory of the wave equations was used in reference [18] to study diffraction problems. In the shell theory, the WKB method was applied for an approximate investigation of the stress–strain states of thin shells [9], for studying free vibrations [12] and buckling [19] of thin elastic shells. The distinguishing feature in the majority of investigations mentioned above was that the problem under consideration were either stationary ones or reduced to them.

The significant contribution to the further development of this method for non-stationary problems has been made by V. P. Maslov [11, 20] who has constructed the solutions of the quantum mechanics equations in the form of functions localized near fixed or moving points and lines. The new method has become known as the complex WKB method or the Maslov–WKB method.

Reverting to equations (22), according to the ideas of this approach, one can find the asymptotic solution of equation (22) in the form

$$u \cong u^*(x, t, \mu) \exp\{i\mu^{-1} S(x, t)\}, \tag{26}$$

where $\partial u^*/\partial x, \partial u^*/\partial t \sim 1$ as $\mu \to 0$, and $S(x, t)$ is a complex function satisfying the inequality

$$\operatorname{Im} S(x, t) > 0 \quad \text{for any } t \geqslant 0. \tag{27}$$

Here μ is some small parameter which is introduced artificially to examine short waves with length being proportional to μ. Function (26), with expression (27) in mind, defines, in a bar, short waves localized near some moving point: i.e., a wave packet. Substituting equation (26) into equation (22) yields the Hamilton–Jacobi equations

$$S'_t = \pm v(x) S'_x \tag{28}$$

and the transfer equation with respect to u^* (written out here). One of the problems is to find the complex solution $S(x, t)$ satisfying equation (27). It is necessary to note that applying the complex WKB method to differential equations having a higher order than that of equation (22) generates non-linear Hamilton–Jacobi equations. The approximate method for solving similar equations has been proposed in reference [11]. This approach has been successfully adapted in shell theory to study free vibrations localized near the "weakest" lines and points on the shell surface [21, 22], and also to examine the running packets of short bending waves in thin non-circular medium length cylindrical shells [23]; its application [23] has allowed detection of the possible effects of reflecting some packets from sufficiently curved regions of a shell.

An attempt to modify the complex WKB method has been undertaken in references [5–7]. The basic concepts of this modification lie in introducing the center of the wave packet and a local co-ordinate system connected with this center. This approach permits one to seek the phase function $S(x, t)$ in equation (26) in explicit form and, on the other hand, to avoid integrating very complicated (in the shell theory) equations analogous to equation (28). It is this approach that will be used below to construct the asymptotic solutions of equations (15), (18) and (20).

6. TRAVELLING PACKETS OF BENDING WAVES

As the initial conditions represent wave packets with center $x = 0$, it is natural to seek solutions of system (15) in the form of travelling packets [11]. Let $x = q_z(t_1)$ be the packet center of the bending waves, where $q_z(t_1)$ is a twice differentiable function. In view of the local character of the solutions, it is convenient to introduce a local co-ordinate system connected with the center $q_z(t_1)$:

$$x = q_z(t_1) + \varepsilon^{1/2}\xi_z. \tag{29}$$

Then equations (15) can be rewritten as

$$\left[\mathbf{L}_\xi - \varepsilon^2\left(\frac{\partial^2}{\partial t_1^2} - 2\varepsilon^{-1/2}\dot{q}_z\frac{\partial^2}{\partial \xi_x\,\partial t_1} + \varepsilon^{-1}\dot{q}_z^2\frac{\partial^2}{\partial \xi_z^2} - \varepsilon^{-1/2}\ddot{q}_z\frac{\partial}{\partial \xi_z}\right)\right](\mathbf{E}_{z\varepsilon}\,\mathbf{U}_z^{\mathrm{T}}) = 0. \tag{30}$$

The 3×3 matrix $\mathbf{L}_\xi$ is defined from the matrix $\mathbf{L}$ by replacing $\partial^k/\partial x^k$ in equations (5) by the operators $\varepsilon^{-k/2}\,\partial^k/\partial\xi_z^k$. The dots $(\cdot)$ denote differentiation with respect to t_1.

The function $f(x)$ is expanded into the series

$$f(x) = f[q_z(t_1)] + \varepsilon^{1/2}f'[q_z(t_1)]\xi_z + \tfrac{1}{2}\varepsilon f''[q_z(t_1)]\xi_z^2 + \cdots \tag{31}$$

in a neighbourhood of the center $q_z(t_1)$.

Upon taking into account equations (6)–(8), the solution of system (30) is assumed to be of the form

$$\mathbf{U}_z \cong \sum_{k=1}^{\infty}\varepsilon^{k/2}\mathbf{U}_{z,k}\exp\{\mathrm{i}(m\varphi + \varepsilon^{-1}S_z)\}, \tag{32}$$

$$S_z = \int_0^{t_1}\omega_z(\tau)\,\mathrm{d}\tau + \varepsilon^{1/2}p_z(t_1)\xi_z + \tfrac{1}{2}\varepsilon b_z(t_1)\xi_z^2, \tag{33}$$

$$\mathbf{U}_{z,k} = (u_{z,k}, v_{z,k}, w_{z,k}), \qquad \operatorname{Im} b_z(t_1) > 0 \quad \text{for any } t_1 \geqslant 0,$$

where $u_{z,k}(\xi_z, t_1)$, $v_{z,k}(\xi_z, t_1)$, $w_{z,k}(\xi_z, t_1)$ are polynomials in ξ_z, $\omega_z(t_1)$ is the momentary frequency of vibrations of the shell in a vicinity of the line $x = q_z(t_1)$, the constant m and $p_z(t_1)$ are the wave numbers, and the function $b_z(t_1)$ characterizes the width of the wave packet. All unknown functions in equations (32) and (33) are supposed to be twice differentiable with respect to t_1. The last inequality guarantees attenuation of wave amplitudes within the packet.

One can note that the explicit form (33) of the phase function S, with equation (29) in mind, may be treated as the first three terms in the expansions of S into Taylor series in a neighbourhood of the center $x = q_z(t_1)$. It should also be emphasized that solutions in the forms (32), (33), when $q_z = 0$, and ω_z, p_z, b_z are constants, have been constructed earlier in the problem on the local buckling [13, 19, 21, 24] and vibration [22, 25] of thin medium length cylindrical shells near the "weakest" generator.

The substitution of equations (31–33) into equation (30) produces the sequence of equations

$$\sum_{j=0}^{k}\mathbf{L}_{z,j}\,\mathbf{U}_{z,k-j}^{\mathrm{T}} = 0, \qquad \mathrm{k} = 0, 1, 2, \ldots, \tag{34}$$

for which $\mathbf{U}_{z,k}$, q_z, p_z, ω_z, b_z can be determined. Here $\mathbf{L}_{z,0}$ is the 3×3 matrix with the elements

$$l_{z,11} = -p_z^2, \qquad l_{z,12} = 0, \qquad l_{z,13} = \mathrm{i}\nu p_z, \qquad l_{z,21} = [(1+\nu)/2]mp_z,$$

$$l_{z,22} = [(1-\nu)/2]p_z^2, \qquad l_{z,23} = \mathrm{i}m[1+2f(q_z)], \qquad l_{z,31} = -\mathrm{i}\nu p_z, \qquad l_{z,32} = 0,$$

$$l_{z,33} = -1 - p_z^4 - (1+m^2)f(q_z) + (\omega_z - \dot{q}_z\, p_z)^2, \tag{35}$$

and the matrix operators $\mathbf{L}_{z,j}$ for $j \geqslant 1$ are expresed by the matris $\mathbf{L}_{z,0}$. For example,

$$\mathbf{L}_{z,1} = \left(b_z \frac{\partial \mathbf{L}_{z,0}}{\partial p_z} + \frac{\partial \mathbf{L}_{z,0}}{\partial q_z} + \dot{p}_z \frac{\partial \mathbf{L}_{z,0}}{\partial \omega_z} \right) \xi_z - \mathrm{i}\, \frac{\mathbf{L}_{z,0}}{\partial p_z} \frac{\partial}{\partial \xi_z},$$

$$\mathbf{L}_{z,2} = \tfrac{1}{2} \left(b_z^2 \frac{\partial^2 \mathbf{L}_{z,0}}{\partial p_z^2} + 2 b_z \frac{\partial^2 \mathbf{L}_{z,0}}{\partial p_z\, \partial q_z} + \frac{\partial^2 \mathbf{L}_{z,0}}{\partial q_z^2} + \dot{p}_z^2 \frac{\partial^2 \mathbf{L}_{z,0}}{\partial \omega_z^2} \right.$$

$$\left. + 2 \dot{p}_z\, b_z \frac{\partial^2 \mathbf{L}_{z,0}}{\partial \omega_z\, \partial p_z} + \dot{b}_z \frac{\partial \mathbf{L}_{z,0}}{\partial \omega_z} \right) \xi_z^2 - \tfrac{1}{2} \frac{\partial^2 \mathbf{L}_{z,0}}{\partial p_z^2} \frac{\partial^2}{\partial \xi_z^2}$$

$$- \mathrm{i} \left(b_z \frac{\partial^2 \mathbf{L}_{z,0}}{\partial p_z^2} + \frac{\partial^2 \mathbf{L}_{z,0}}{\partial p_z\, \partial q_z} + \dot{p}_z \frac{\partial^2 \mathbf{L}_{z,0}}{\partial \omega_z\, \partial p_z} \right) \xi_z \frac{\partial}{\partial \xi_z} - \mathrm{i}\, \frac{\partial \mathbf{L}_{z,0}}{\partial \omega_z} \frac{\partial}{\partial t_1}$$

$$- \mathrm{i} \left(\tfrac{1}{2}\, b_z \frac{\partial^2 \mathbf{L}_{z,0}}{\partial p_z^2} + \tfrac{1}{2}\, \dot{\omega}_z \frac{\partial^2 \mathbf{L}_{z,0}}{\partial \omega_z^2} + \dot{p}_z \frac{\partial^2 \mathbf{L}_{z,0}}{\partial \omega_z\, \partial p_z} + \mathbf{G}_z \right), \tag{36}$$

where $\mathbf{G}_z$ is a 3×3 matrix having only one non-zero element $g_{z,33} = -\mathrm{i}p_z\, \ddot{q}_z$. Consider equations (34) for $k = 0, 1, 2, \ldots$.

6.1. ZEROTH AND FIRST ORDER APPROXIMATIONS

In the zeroth order approximation ($k = 0$), one has the homogeneous system of algebraic equations

$$\mathbf{L}_{z,0}\, \mathbf{U}_{z,0}^{\mathrm{T}} = 0. \tag{37}$$

For a non-trivial solution of these equations, the determinant of their coefficients is set equal to zero yielding the relation

$$\omega_z(t_1) = \dot{q}_z(t_1) p_z(t_1) - H_z^{\pm}[p_z(t_1), q_z(t_1)], \tag{38}$$

where

$$H_z^{\pm} = \pm\sqrt{1 - \nu^2 + p_z^4 + (1+m^2) f(q_z)} \tag{39}$$

are Hamiltonian functions. Then the solution of equation (37) may be represented in the form

$$\mathbf{U}_{z,0} = P_{z,0}(\xi_z, t_1)\mathbf{Z}, \tag{40}$$

where $P_{z,0}$ is an unknown polynomial in ξ_z, and $\mathbf{Z} = (z_1, z_2, z_3)$ is any non-zero solution of equations (37). One can put

$$z_3 = 1, \qquad z_1 = -l_{z,13}/l_{z,11}, \qquad z_2 = (l_{z,22}\, l_{z,13} - l_{z,11}\, l_{z,23})(l_{z,11}\, l_{z,22})^{-1}.$$

It is assumed here that $z_j \sim 1$ as $\varepsilon \to 0$.

The signs ($\pm$) in equation (38) indicate the availability of two branches (positive and negative) of the solutions corresponding to the functions $H_z^{\pm}$. These signs are omitted in what follows.

For the first order approximation ($k = 1$), one has the non-homogeneous system of algebraic equations

$$\mathbf{L}_{z,0}\,\mathbf{U}_{z,1}^{\mathrm{T}} = -\mathbf{L}_{z,1}\,\mathbf{U}_{z,0}^{\mathrm{T}}. \tag{41}$$

The condition for solution of equation (41) gives the differential equation

$$\bar{\mathbf{Z}}_c\,\mathbf{L}_{z,1}\,\mathbf{U}_{z,0}^{\mathrm{T}} = 0 \tag{42}$$

with respect to $P_{z,0}$, where $\mathbf{Z}_c$ is any non-trivial solution of the system $\bar{\mathbf{L}}_{z,0}^{\mathrm{T}}\,\mathbf{Z}_c^{\mathrm{T}} = 0$. For equation (42) to have polynomial solutions it is necessary for the functions p_z, q_z to satisfy Hamiltonian system

$$\dot{q}_z = \partial H_z/\partial p_z, \qquad \dot{p}_z = -\partial H_z/\partial q_z. \tag{43}$$

Then the solution of equation (41) has the form

$$\mathbf{U}_{z,1} = P_{z,1}\,\mathbf{Z} + \xi_z\,P_{z,0}\left(b_z\frac{\partial \mathbf{Z}}{\partial p_z} + \frac{\partial \mathbf{Z}}{\partial q_z}\right) - \mathrm{i}\,\frac{\partial P_{z,0}}{\partial \xi_z}\frac{\partial \mathbf{Z}}{\partial p_z}, \tag{44}$$

where $P_{z,1}$ is again an unknown polynomial in ξ_z.

Comparison of equations (6), (7) and (32) gives the initial conditions

$$p_z(0) = a^\circ, \qquad q_z(0) = 0 \tag{45}$$

for the canonical system (43). It is evident that the initial problem (43), (45) has the unique solutions $p_z^+(t_1)$, $q_z^+(t_1)$ and $p_z^-(t_1)$, $q_z^-(t_1)$ corresponding to the Hamiltonians H_z^+ and H_z^-, respectively, if the function $f(x)$ is infinitely differentiable.

6.2. SECOND ORDER AND HIGHER APPROXIMATIONS

In the second order ($k = 2$) approximation, one has the non-homogeneous system

$$\mathbf{L}_{z,0}\,\mathbf{U}_{z,2}^{\mathrm{T}} = -\mathbf{L}_{z,1}\,\mathbf{U}_{z,1}^{\mathrm{T}} - \mathbf{L}_{z,2}\,\mathbf{U}_{z,0}^{\mathrm{T}}. \tag{46}$$

The compatibility condition for this system yields the relation

$$\bar{\mathbf{Z}}_c\,(\mathbf{L}_{z,1}\,\mathbf{U}_{z,1}^{\mathrm{T}} + \mathbf{L}_{z,2}\,\mathbf{U}_{z,0}^{\mathrm{T}}) = 0, \tag{47}$$

which, by means of equations (42) and (44), again consists of differential equations with respect to $P_{z,0}$. This equation has a solution of polynomial form if the function b_z satisfies the Riccati equation

$$\dot{b}_z + \frac{\partial^2 H_z}{\partial p_z^2}\,b_z^2 + 2\,\frac{\partial^2 H_z}{\partial p_z\,\partial q_z}\,b_z + \frac{\partial^2 H_z}{\partial q_z^2} = 0. \tag{48}$$

It is apparent that

$$b_z(0) = b^\circ. \tag{49}$$

One can prove that problem (48), (49) has the unique solutions $b_z^\pm(t_1)$ corresponding to $H_z^\pm$ so that [5] $\operatorname{Im} b^\pm(t_1) > 0$ for any $t_1 \geqslant 0$, if $\operatorname{Im} b^\circ > 0$.

Upon taking into account equation (47), equation (46) is reduced to

$$D_{z,\xi t}\,P_{z,0} = 0, \tag{50}$$

where

$$D_{z,\xi t}=A_{z,2}\frac{\partial^2}{\partial\xi_z^2}+A_{z,1}\,\xi_z\frac{\partial}{\partial\xi_z}+\left(A_{z,0}+\mathrm{i}\frac{\partial}{\partial t_1}\right),$$

$$A_{z,0}=\mathrm{i}\left(\bar{\mathbf{Z}}_c\frac{\partial\mathbf{L}_{z,0}}{\partial\omega_z}\mathbf{Z}^{\mathrm{T}}\right)^{-1}$$

$$\times\,\bar{\mathbf{Z}}_c\left[\left(\tfrac{1}{2}b_z\frac{\partial^2H_z}{\partial p_z^2}\frac{\partial\mathbf{L}_{z,0}}{\partial\omega_z}+\tfrac{1}{2}\dot{\omega}_z\frac{\partial^2\mathbf{L}_{z,0}}{\partial\omega_z^2}+\dot{p}_z\frac{\partial^2\mathbf{L}_{z,0}}{\partial\omega_z\,\partial p_z}+\mathbf{G}_z\right)\mathbf{Z}^{\mathrm{T}}+\frac{\partial\mathbf{L}_{z,0}}{\partial p_z}\frac{\partial\mathbf{Z}^{\mathrm{T}}}{\partial q_z}\right],$$

$$A_{z,1}=\mathrm{i}\left(b_z\frac{\partial^2H_z}{\partial p_z^2}+\frac{\partial^2H_z}{\partial p_z\,\partial q_z}\right),\qquad A_{z,2}=\tfrac{1}{2}\frac{\partial^2H_z}{\partial p_z^2}.\tag{51}$$

The solution of equation (50) is the polynomial

$$P_{z,0}(\xi_z,t_1;c_{00},c_{01},\ldots,c_{0m_0})=\sum_{k=0}^{m_0}C_{z,k}(t_1;c_{00},c_{01},\ldots,c_{0m_0})\xi_z^k\tag{52}$$

of degree m_0 with coefficients

$$C_{z,m_0}(t_1;c_{00})=c_{00}\Xi_0(t_1),\qquad C_{z,m_0-1}(t_1;c_{01})=c_{01}\Xi_1(t_1),$$

$$C_{z,m_0-r}(t_1;c_{0r},c_{0r-2},c_{0r-4},\ldots)=\Xi_r(t_1)\Big[c_{0r}+\mathrm{i}(m_0-r+2)$$

$$\times(m_0-r+1)\int A_{z,2}(t_1)C_{z,m_0-r+2}(t_1;c_{0r-2},c_{0r-4},\ldots)\Xi_r^{-1}(t_1)\,\mathrm{d}t_1\Big],$$

$$\Xi_j(t_1)=\exp\left\{\mathrm{i}\int[m_0-j)A_{z,1}(t_1)+A_{z,0}(t_1)]\,\mathrm{d}t_1\right\},$$

where $r=2,3,\ldots,m_0$, $j=0,1,\ldots,m_0$, and c_{0j} are arbitrary complex numbers. All the coefficients in equation (50) are calculated at $H_z=H_z^\pm$, $p_z=p_z^\pm$, $q_z=q_z^\pm$, $b_z=b_z^\pm$, $\omega_z=\omega_z^\pm$, with the signs $\pm$ being omitted here.

The kth approximations yield the non-homogeneous differential equations

$$D_{z,\xi t}P_{z,k-2}=N_{z,k-2},\qquad k=3,4,\tag{53}$$

with respect to $P_{z,k-2}$, where $N_{z,k-2}$ are some polynomials in ξ_z. For example, $N_{z,1}$ is a polynomial of the (m_0+3)th degree, and then a solution $P_{z,1}=P_{z,1}(\xi_z,t_1;c_{10},\ldots,c_{1m_1})$ is also some polynomial containing arbitrary complex constants $c_{10},\ldots,c_{1m_1}$.

So, system (15) has the two asymptotic solutions

$$\mathbf{U}_z^\pm\cong[P_{z,0}^\pm(\xi_z^\pm,t_1;c_{00}^\pm,c_{01}^\pm,\ldots,c_{0m_0}^\pm)\mathbf{Z}^\pm+O(\varepsilon^{1/2})]$$

$$\times\exp\left\{\mathrm{i}\left[m\varphi+\varepsilon^{-1}\int_0^{t_1}\omega_z^\pm(\tau)\,\mathrm{d}\tau+\varepsilon^{-1/2}p_z^\pm(t_1)\xi_z^\pm+\tfrac{1}{2}b_z^\pm(t_1)(\xi_z^\pm)^2\right]\right\},\tag{54}$$

where $\mathbf{U}_z^{\pm} = (u_z^{\pm}, v_z^{\pm}, w_z^{\pm})$, $\xi_z^{\pm} = \varepsilon^{-1/2}[x - q_z^{\pm}(t_1)]$, $t_1 = \varepsilon t$. The signs $(+)$ and $(-)$ indicate that all functions in equation (54) are calculated at $H_z = H_z^{\pm}$ and $H_z = H_z^{-}$, respectively. The symbol $O(\varepsilon^{1/2})$ represents the quantities, having the order $\varepsilon^{1/2}$, which have not been written out here (compare with equation (32)). To take them into account it is necessary to consider the higher approximations for $k \geqslant 3$.

It should be noted that the error of solution (54) depends on the parameters Im $b_z^{\pm}$ and $p_z^{\pm}$; it grows when Im $b_z^{\pm} \to 0$ and (or) $p_z^{\pm} \to 0$.

7. TANGENTIAL WAVES

Solutions of equations (18) and (20) can also be constructed in the form of wave packets (32) with the centres $x = q_x(t)$ and $x = q_y(t)$, respectively. All the formulas and equations obtained in the previous section are valid for tangential waves. To go over to tangential waves the index z should be changed to x (for longitudinal vibrations) or to y (for torsional vibrations), while the "slow" time t_1 is replaced by t. In these cases the elements of the matrices $\mathbf{L}_{x,0}$ and $\mathbf{L}_{y,0}$ are

$$l_{x,11} = -p_x^2 + (\omega_x - \dot{q}_x p_x)^2, \qquad l_{x,12} = l_{x,13} = 0,$$

$$l_{x,21} = -[(1+\nu)/2]mp_x, \qquad l_{x,22} = -[(1-\nu)/2]p_x^2 + (\omega_x - \dot{q}_x p_x)^2,$$

$$l_{x,23} = 0, \qquad l_{x,31} = -\mathrm{i}\nu p_x, \qquad l_{x,32} = 0, \qquad l_{x,33} = (\omega_x - \dot{q}_x p_x)^2,$$

$$l_{y,11} = -p_y^2 + (\omega_y - \dot{q}_y p_y)^2, \qquad l_{y,12} = -[(1+\nu)/2]mp_y, \qquad l_{y,13} = 0,$$

$$l_{y,21} = l_{y,23} = 0, \qquad l_{y,22} = -[(1-\nu)/2]p_y^2 + (\omega_y - \dot{q}_y p_y)^2,$$

$$l_{y,31} = -\mathrm{i}\nu p_y, \qquad l_{y,32} = -\mathrm{i}m[1 + 2f(q_y)], \qquad l_{y,33} = (\omega_y - \dot{q}_y p_y)^2, \tag{55}$$

The Hamiltonian functions have the forms

$$H_x^{\pm} = \pm p_x, \qquad H_y^{\pm} = \pm\sqrt{(1-\nu)/2}\,p_y, \tag{56}$$

and $\mathbf{G}_x = -\mathrm{i}p_x \ddot{q}_x \mathbf{E}$, $\mathbf{G}_y = -\mathrm{i}p_y \ddot{q}_y \mathbf{E}$, where $\mathbf{E}$ is the identity matrix.

In the case of tangential vibrations, problems (43), (45) and (48), (49), and also equation (50) have solutions in closed form. For longitudinal waves

$$p_x = p_x^{\pm} = a^{\circ}, \qquad q_x = q_x^{\pm} = \pm t, \qquad b_x = b_x^{\pm} = b^{\circ},$$

$$P_{x,0} = P_{x,0}^{\pm}(\xi_x^{\pm}; d_{00}^{\pm}, d_{01}^{\pm}, \ldots, d_{0n_0}^{\pm}) = \sum_{j=0}^{n_0} d_{0j}^{\pm}(\xi_z^{\pm})^j, \tag{57}$$

and for torsional ones

$$p_y = p_y^{\pm} = a^{\circ}, \qquad q_y = q_y^{\pm} = \pm\sqrt{(1-\nu)/2}\,t, \qquad b_y = b_y^{\pm} = b^{\circ},$$

$$P_{y,0} = P_{y,0}^{\pm}(\xi_y^{\pm}; r_{00}^{\pm}, r_{01}^{\pm}, \ldots, r_{0k_0}^{\pm}) = \sum_{j=0}^{k_0} r_{0j}^{\pm}(\xi_x^{\pm})^j, \tag{58}$$

where $\xi_x^{\pm} = \varepsilon^{-1/2}(x \mp t)$, $\xi_y^{\pm} = \varepsilon^{-1/2}(x \mp \sqrt{(1-\nu)/2}\,t)$, and $d_{0j}^{\pm}$, $n_0, r_{0j}^{\pm}, k_0$ are arbitrary constants. Then $\omega_x = 0$, $\omega_y = 0$ and

$$\mathbf{U}_{x,0} = P_{x,0}\mathbf{X}, \qquad \mathbf{U}_{y,0} = P_{y,0}\mathbf{Y}, \tag{59}$$

where

$$\mathbf{X} = (x_1, x_2, x_3), \qquad \mathbf{Y} = (y_1, y_2, y_3),$$

$$x_1 = 1, \qquad x_2 = -l_{x,21}/l_{x,22}, \qquad x_3 = -l_{x,31}/l_{x,33},$$

$$y_2 = 1, \qquad y_1 = -l_{y,12}/l_{y,11}, \qquad y_3 = (l_{y,11}\, l_{y,32} - l_{y,12}\, l_{y,31})/(l_{y,11}\, l_{y,33}).$$

Here $x_j, y_j \sim 1$, as $\varepsilon \to 0$. The components x_j and y_j are calculated in accordance with equations (55) for the positive and negative branches of functions evaluated in equations (57) and (58).

Thus, systems (18) and (20) have the asymptotic solutions

$$\mathbf{U}_x^{\pm} \cong [P_{x,0}^{\pm}(\xi_x^{\pm}; d_{00}^{\pm}, d_{01}^{\pm}, \ldots, d_{0n_0}^{\pm})\mathbf{X}^{\pm} + O(\varepsilon^{1/2})] \times \exp\{\mathrm{i}[m\varphi + \varepsilon^{-1/2}a^{\circ}\xi_x^{\pm} + \tfrac{1}{2}b^{\circ}(\xi_x^{\pm})^2]\} \qquad (60)$$

and

$$\mathbf{U}_y^{\pm} \cong [P_{y,0}^{\pm}(\xi_y^{\pm}; r_{00}^{\pm}, r_{01}^{\pm}, \ldots, r_{0k_0}^{\pm})\mathbf{Y}^{\pm} + O(\varepsilon^{1/2})] \exp\{\mathrm{i}[m\varphi + \varepsilon^{-1/2}a^{\circ}\xi_y^{\pm} + \tfrac{1}{2}b^{\circ}(\xi_y^{\pm})^2]\}, \qquad (61)$$

respectively, where $\mathbf{U}_x^{\pm} = (u_x^{\pm}, v_x^{\pm}, w_x^{\pm})$, $\mathbf{U}_y^{\pm} = (u_y^{\pm}, v_y^{\pm}, w_y^{\pm})$.

8. APPROXIMATE SOLUTION OF THE INITIAL PROBLEM

To satisfy the initial conditions, consider the linear combinations of the positive and negative branches of the solutions constructed above. It is evident that the vector functions $\mathbf{U}_z^{+} + \mathbf{U}_z^{-}$, $\mathbf{U}_x^{+} + \mathbf{U}_x^{-}$ and $\mathbf{U}_y^{+} + \mathbf{U}_y^{-}$ satisfy systems (15), (18) and (20), respectively. Then from equation (21), the vector function

$$\mathbf{U} \cong \mathbf{E}_{x\varepsilon}(\mathbf{U}_x^{+} + \mathbf{U}_x^{-})^{\mathrm{T}} + \mathbf{E}_{y\varepsilon}(\mathbf{U}_y^{+} + \mathbf{U}_y^{-})^{\mathrm{T}} + \mathbf{E}_{z\varepsilon}(\mathbf{U}_z^{+} + \mathbf{U}_z^{-})^{\mathrm{T}} \qquad (62)$$

is the formal asymptotic solution of the governing equations (4). Indeed, the introduction of vector (62) into equations (4), in accordance with the asymptotic constructions carried out above, produces the sequence of identities for $\varepsilon^{k/2}$ $(k = 0, 1, \ldots)$.

Solution (62) contains arbitrary constants $c_{00}^{\pm}, \ldots, c_{0m_0}^{\pm}$, $d_{00}^{\pm}, \ldots, d_{0n_0}^{\pm}$, $r_{00}^{\pm}, \ldots, r_{0k_0}^{\pm}$, m_0, n_0, k_0 which may be determined from the initial conditions. The substitution of equation (62) into equations (6), with regard to the equalities $\xi_x^{\pm}|_{t=0} = \xi_y^{\pm}|_{t=0} = \xi_z^{\pm}|_{t_1=0} = \xi$, yields the equations

$$(P_{x,0}^{+} + P_{x,0}^{-})|_{t=0} = \lambda_{10}^{\circ}(\zeta), \qquad (P_{x,0}^{+} - P_{x,0}^{-})|_{t=0} = -\eta_{10}^{\circ}(\zeta)/a^{\circ},$$

$$(P_{y,0}^{+} + P_{y,0}^{-})|_{t=0} = \lambda_{20}^{\circ}(\zeta), \qquad (P_{y,0}^{+} - P_{y,0}^{-})|_{t=0} = -\sqrt{2/(1-\nu)}\eta_{20}^{\circ}(\zeta)/a^{\circ},$$

$$(P_{z,0}^{+} + P_{z,0}^{-})|_{t_1=0} = \lambda_{30}^{\circ}(\zeta), \qquad (P_{z,0}^{+} - P_{z,0}^{-})|_{t_1=0} = [\mathrm{i}\nu\eta_{10}^{\circ}(\zeta) - a^{\circ}\eta_{30}^{\circ}(\zeta)]/a^{\circ}H_z^{\circ}, \qquad (63)$$

where $H_z^{\circ} = \sqrt{1 - \nu^2 + (a^{\circ})^4 + (1 + m^2)f(0)}$. The equality conditions of the coefficients in equations (63) for the same degree of ζ give $n_0 = \max\{M_{10}, K_{10}\}$, $k_0 = \max\{M_{20}, K_{20}\}$, $m_0 = \max\{M_{30}, K_{30}, K_{10}\}$ and produce a non-homogeneous system of $(n_0 + k_0 + m_0)$ algebraic equations with respect to the constants $c_{0j}^{\pm}, d_{0j}^{\pm}, r_{0j}^{\pm}$.

9. ANALYSIS AND EXAMPLES

To analyze the solution (62) it is convenient to present it component-wise (see $\mathbf{E}_{x\varepsilon}, \mathbf{E}_{y\varepsilon}, \mathbf{E}_{z\varepsilon}$):

$$u_1 \cong (u_x^{+} + u_x^{-}) + \varepsilon(u_y^{+} + u_y^{-}) + \varepsilon(u_z^{+} + u_z^{-}), \qquad (64)$$

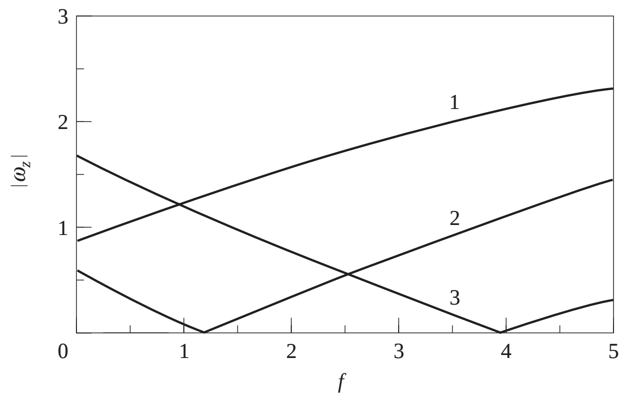

Figure 2. Dimensionless frequencies $|\omega_z|$ of travelling bending vibrations versus parameter $F = (1 + m^2)f$, for $\nu = 0{\cdot}3$ and various a°. 1, $a^\circ = 0{\cdot}5$; 2, $a^\circ = 1{\cdot}2$; 3, $a^\circ = 1{\cdot}5$.

$$u_2 \cong (v_y^+ + v_y^-) + \varepsilon(v_x^+ + v_x^-) + \varepsilon^2(v_z^+ + v_z^-), \tag{65}$$

$$u_3 \cong (w_z^+ + w_z^-) + \varepsilon(w_x^+ + w_x^-) + \varepsilon^2(w_y^+ + w_y^-). \tag{66}$$

Here all the summands are evaluated by equations (54), (60), (61). In equation (64), the terms u_x^+, u_x^- define the positive and negative packets of longitudinal waves with the centres $x = q_x^+(t) = t$ and $x = q_x^-(t) = -t$, respectively. They run in opposite directions with the constant group velocities $V_{gx}^{\pm} = \pm E^{1/2}/[(1 - \nu^2)\rho]^{1/2}$. In equation (65), the summands $v_y^{\pm}$ represent packets of torsional waves with the centres $x = q_y^{\pm}(t) = \pm[(1 - \nu)/2]^{1/2}t$, which travel with the group velocities $V_{gy}^{\pm} = \pm E^{1/2}/[2(1 + \nu)\rho]^{1/2}$. Finally, in equation (66) the terms $w_z^{\pm}$ correspond to packets of bending waves with the centres $x = q_z^{\pm}(\varepsilon t)$, the group velocities being

$$V_{gz}^{\pm} = \pm\sqrt{E/(1 - \nu^2)\rho}\,2[p_z^{\pm}(\varepsilon t)]^3/H_z^\circ. \tag{67}$$

In the packets of tangential waves, the wave numbers $p_x^{\pm} = p_y^{\pm} = a^\circ$ and the parameters $b_x^{\pm} = b_y^{\pm} = b^\circ$, which characterize the packet width, are constants; as regards bending waves, the quantities $p_z^{\pm}(\varepsilon t)$, $b_z^{\pm}(\varepsilon t)$ are functions of "slow" time.

The other terms, being proportional to ε and ε^2, represent the packets "generated" by the main ones which have been enumerated above. Since, the wave amplitudes in the "generated" packets are small, they are not taken into consideration on calculations.

Formulas (55–59) indicate the independence of tangential waves of the pressure $f(x)$ in the zeroth order approximation. Conversely, the behaviour of the bending wave packets is more complicated; it depends on the pressure $f(x)$.

Let f be a constant at first. Then

$$p_z = p_z^{\pm} = a^\circ, \qquad q_z = q_z^{\pm}(\varepsilon t) = \pm(2(a^\circ)^3/H_z^\circ)\varepsilon t,$$
$$b_z = b_z^{\pm}(\varepsilon t) = (H_z^\circ)^3 b^\circ/[(H_z^\circ)^3 + 2(a^\circ)^2 b^\circ[3(H_z^\circ)^2 - 2(a^\circ)^4]\varepsilon t],$$
$$\omega_z = \omega_z^{\pm} = [(a^\circ)^4 - (1 - \nu^2) - (1 + m^2)f]/H_z^\circ. \tag{68}$$

Analysis of the functions $b_z^{\pm}(\varepsilon t)$ shows that the higher the internal pressure is, the slower the travelling packets of bending waves become dissolved. The dependence of the dimensionless frequencies $|\omega_z|$ of the travelling bending vibrations on the parameter $F = (1 + m^2)f$ for various a° is shown in Figure 2. In the numerical computation, Poisson's ratio was taken as $\nu = 0{\cdot}3$, and $a^\circ = 0{\cdot}5$, $1{\cdot}2$, $1{\cdot}5$. It may be seen that, for $F = (a^\circ)^4 - 1 + \nu^2$, this frequency equals zero. When $F > (a^\circ)^4 - 1 + \nu^2$, the frequency increases with initial internal pressure. This effect agrees with results obtained in references [1, 2].

It is of interest to study the behaviour of bending waves in the shell subjected to variable internal pressure. It is assumed $f(x) > 0, f'(x) < 0$ for any $x, f(x) \to f_{\pm\infty}$, as $x \to \pm\infty$, where $0 \leqslant f_{\pm\infty} < (a^\circ)^4/(1 + m^2) + f(0)$ are some constants. The analysis of the Hamiltonian system (43) gives

$$p_z^+ > 0, \quad \dot{p}_z^+ > 0, \quad V_{gz}^+ > 0, \quad \dot{V}_{gz}^+ > 0,$$
$$p_z^- > 0, \quad \dot{p}_z^- < 0, \quad V_{gz}^- < 0, \quad |\dot{V}_{gz}^-| < 0, \tag{69}$$

for any $t_1 > 0$. These inequalities show that the wave parameter p_z^+ and the group velocity V_{gz}^+ in the positive packet, travelling in the direction of pressure diminution, increase; and the parameters p_z^-, V_{gz}^- in the negative packet, moving in the opposite direction, decrease. Moreover, $p_z^\pm \to p_{\pm\infty}$, $V_{gz}^\pm \to V_{\pm\infty}$, as $t_1 \to +\infty$, where

$$p_{\pm\infty} = \{(a^\circ)^4 + (1 + m^2)[f(0) - f_{\pm\infty}]\}^{1/4}, \tag{70}$$

and $V_{\pm\infty}$ are determined from equation (67).

As an example, consider the shell subjected to constant internal and variable external pressures, so that

$$\tilde{P}(Rx) = -[Eh/R(1 - \nu)][f_{in} + f_{ex}(x)] < 0, \tag{71}$$

where $f_{in} > 0, f_{ex} < 0$. Similar combined action may be experienced, e.g., by an underwater trunk pipeline lying at various depths and transporting gas or liquid under high pressure. Numerical computations for $f_{in} = 2, f_{ex}(x) = -(1 + \tanh x), \nu = 0{\cdot}3, h/R = 0{\cdot}004, a^\circ = 1{\cdot}1, b^\circ = i, \lambda_3^\circ = 1, \eta_3^\circ = 0, m = 0$ were performed. Figure 3 shows the manner in which the frequencies $|\omega_z^\pm|$ of travelling positive and negative packets of bending waves vary with the course of time. Here $\omega_z^+ \to 0{\cdot}63$ and $\omega_z^- \to 1{\cdot}4$, as $t_1 \to +\infty$. In Figure 4, the parameter $\operatorname{Im} b_z^\pm$ and the maximum amplitude $w_{\max}^\pm$ of bending waves in positive and negative packets are plotted as functions of t_1. It may be concluded, that the positive packet of bending waves travelling in the direction of pressure diminution becomes dissolved faster than the negative packets. Moreover, one can see the possible effect of slight focusing of the negative packet, which moves in the direction of internal pressure growth.

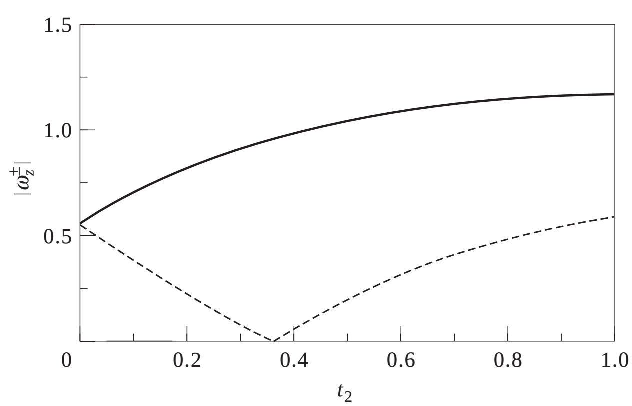

Figure 3. Dimensionless frequencies $|\omega_z^\pm|$ of travelling bending vibrations versus dimensionless time t_1 in the case of the non-uniform internal pressure $f(x) = 1 - \tanh x$, for $\nu = 0{\cdot}3, h/R = 0{\cdot}004, m = 0, a^\circ = 1{\cdot}1$. ——, Negative packet; - - -, positive packet.

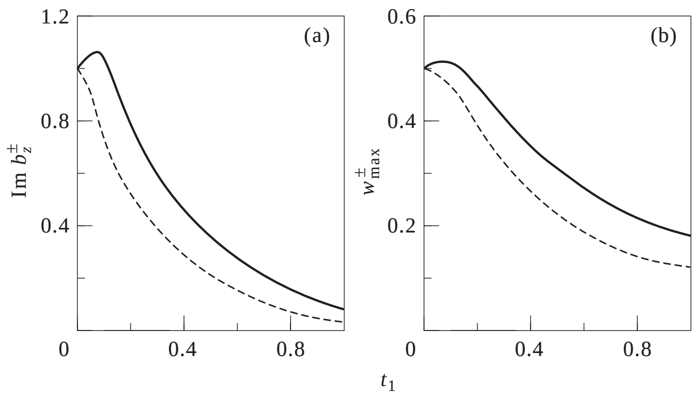

Figure 4. (a) Parameters Im $b_z^{\pm}$ and (b) maximum amplitudes $w_{max}^{\pm}$ of travelling packets of bending waves versus dimensionless time t_1 in the case of the non-uniform pressure $f(x) = 1 - \tanh x$, for $\nu = 0{\cdot}3$, $h/R = 0{\cdot}004$, $m = 0$, $a^{\circ} = 1{\cdot}1$, $b^{\circ} = i$, $\lambda_3^{\circ} = 1$, $\eta_3^{\circ} = 0$. Key as Figure 3.

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