Journal of Sound and Vibration (1998) **209**(4), 561–569

NON-LINEAR NORMAL MODES OF A CONTINUOUS SYSTEM

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(*Received* 14 *March* 1997)

A power series method is presented for the computation of normal modes and frequencies of an elastic beam resting on a non-linear foundation. The equation of motion is first discretized by using the Galerkin procedure. The time-dependent generalized co-ordinates are obtained by transforming the time variable into an oscillating time which transforms the discretized equations into a form solvable by the power series method. Results are obtained for simply supported and clamped beams, and good agreement is shown for the simply supported case with the result given by the invariant manifold approach.

1. INTRODUCTION

The problem of computing the normal modes and frequencies of non-linear continuous systems has received much attention in recent years. In general, three approaches are employed in the solution of such problems. In the first approach, the time dependence is assumed *a priori* to be simple harmonic and the harmonic balance principle is then used to obtain boundary value problems for the normal modes. Benamar *et al*. [1] used this approach to compute the non-linear normal modes of simply supported and clamped beams. Szemplinska-Stupnicka [2] used the generalized Ritz method in conjunction with the harmonic balance principle to determine the mode shapes of some non-linear beam systems and considered the effects of non-linear boundary conditions.

In the second approach, the motion is assumed to be a combination of the linear normal modes, the individual contributions of which are represented by time-dependent generalized co-ordinates. The Galerkin procedure is then employed to obtain a set of non-linear coupled ordinary differential equations for the generalized co-ordinates. The discretized equations are normally solved approximately by a perturbation technique [3, 4] or the center manifold reduction method [5].

In the third approach, no assumption is made regarding the spatial and temporal behavior of the system and the governing partial differential equation is treated directly. Shaw and Pierre [6–8] developed a method based on the invariant manifold theory for constructing non-linear normal modes of conservative and non-conservative systems. The normal mode for a non-linear system was defined as a motion which takes place on a two-dimensional invariant manifold in the phase space. Nayfeh *et al*. [9] determined the non-linear modes of a continuous system with cubic inertia and geometric non-linearities using several methods. The invariant manifold and perturbation methods were applied to the discretized equations and the multiple scales method was employed directly to the governing differential equation and boundary conditions. It was found that all methods yielded equivalent results. This finding, which was supported by a recent study of a

0022–460X/98/040561 + 09 $25.00/0/sv971246

continuous system with quadratic non-linearity [10], can be explained by the fact that all these methods involve asymptotic expansions about the equilibrium position.

In this paper, the non-linear normal modes and frequencies of a continuous system, namely, an elastic beam resting on non-linear foundation, are computed, as in the second approach, by first discretizing the system by using the Galerkin procedure. However, the resulting non-linear differential equations are solved by transforming the time variable into an oscillating time. This transformation permits power series expansion of the generalized co-ordinates about the maximum displacement position, rather than the equilibrium position.

2. FORMULATION

The problem considered is the free vibration of a linear Euler–Bernoulli beam resting on a foundation with cubic non-linearity. The governing partial differential equation is

$$EI\,\partial^4 v/\partial x^4 + \rho A\,\partial^2 v/\partial t^{*2} + \alpha_1 v + \alpha_3 v^3 = 0, \tag{1}$$

subject to the boundary conditions

$$v = \partial^2 v/\partial x^2 = 0 \quad \text{at } x = 0, L \qquad \text{for simply supported beams} \tag{2}$$

and $v = \partial v/\partial x = 0$ at $x = 0, L$ for clamped beams, where $v(x, t^*)$ is the transverse displacement at position x in the domain $(0, L)$ and time t^*, EI is the flexural rigidity, ρ and A are the beam mass density and cross-sectional area respectively, L is the beam length and α_1 and α_3 are the foundation coefficients. Introducing the non-dimensional quantities

$$\zeta = x/L, \qquad t = (t^*/L^2)\sqrt{EI/\rho A}, \qquad w = v/L \tag{3}$$

into equations (1) and (2) results in the non-dimensional form

$$\ddot{w} + w'''' + \alpha w + \beta w^3 = 0, \tag{4}$$

subject to

$$w = w'' = 0 \quad \text{at } \zeta = 0, 1 \qquad \text{for simply supported beams} \tag{5}$$

and $w = w' = 0$ at $\zeta = 0, 1$ for clamped beams, where the overdot and the prime denote partial derivative with respect to t and ζ respectively, $\alpha = \alpha_1 L^4/EI$ and $\beta = \alpha_3 L^6/EI$.

The solution of the governing equation (4) and boundary conditions (5) begins by discretizing the continuous system using the complete set of linear mode shapes as a basis,

$$w(\zeta, t) = \sum_{i=1}^{\infty} q_i(t)\phi_i(\zeta), \tag{6}$$

where $q_i(t)$ are time-dependent generalized co-ordinates and $\phi_i(\zeta) = \psi_i(\zeta)/\psi_i(\bar{\zeta})$ are the mode shapes for the linearized problem, normalized at a freely chosen position $\zeta = \bar{\zeta} = 1/2i$. For simply supported beams

$$\psi_i(\zeta) = \sin\gamma_i\zeta, \qquad \gamma_i = i\pi, \tag{7}$$

and for clamped beams

$$\psi_i(\zeta) = \sin\gamma_i\zeta - \sinh\gamma_i\zeta - \eta_i(\cos\gamma_i\zeta - \cosh\gamma_i\zeta), \tag{8}$$

where $\eta_i = (\sin\gamma_i - \sinh\gamma_i)/(\cos\gamma_i - \cosh\gamma_i)$ and γ_i are the positive roots of $\cos\gamma\cosh\gamma = 1$.

The assumed form of the solution, equation (6), ensures that the boundary conditions, equation (5), are satisfied at any time during the motion. Substituting equation (6) into equation (4) leads to

$$\sum_{i=1}^{\infty} (\ddot{q}_i \phi_i + q_i \phi_i'''' + \alpha q_i \phi_i) + \beta \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} q_k q_l q_m \phi_k \phi_l \phi_m = \varepsilon. \tag{9}$$

The Galerkin procedure determines the generalized co-ordinates by requiring that the error ε resulting from the discretization be orthogonal to every eigenmode ϕ_j for $j = 1, 2, \ldots$. Thus, multiplying equation (9) by ϕ_j, integrating over the beam domain and using the orthogonality of the eigenmodes results in

$$a_j \ddot{q}_j + b_j q_j + \beta \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} c_{jklm} q_k q_l q_m = 0, \qquad j = 1, 2, \ldots, \tag{10}$$

where

$$a_j = \int_0^1 \phi_j^2 \, \mathrm{d}\zeta, \qquad b_j = (\alpha + \gamma_j^4) a_j, \qquad c_{jklm} = \int_0^1 \phi_j \phi_k \phi_l \phi_m \, \mathrm{d}\zeta. \tag{11}$$

Equations (10) represent a set of non-linear coupled ordinary differential equations in terms of the generalized co-ordinates $q_j(t)$ which may be solved by a perturbation method or the invariant manifold approach. Here, a technique based on the power series method is developed. First the time variable t is transformed as

$$\tau = \sin \omega t, \tag{12}$$

which reduces the infinite time domain $0 \leqslant t < \infty$ into a finite time scale $-1 \leqslant \tau \leqslant 1$ within which the new variable oscillates harmonically at a frequency ω to be determined. Introducing equation (12) into equations (10) results in

$$\omega^2 (1 - \tau^2) \bar{q}_j^{\tau\tau} - \omega^2 \tau \bar{q}_j^{\tau} + \frac{b_j}{a_j} \bar{q}_j + \frac{\beta}{a_j} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} c_{jklm} \bar{q}_k \bar{q}_l \bar{q}_m = 0, \qquad j = 1, 2, \ldots, \tag{13}$$

in which $\bar{q}_j(\tau) = q_j(t)$ to indicate the change in functional form of the generalized co-ordinates as a result of the transformation, and the superscript τ denotes differentiation with respect to τ. The transformation of the independent variable from an infinite time t to a finite time τ permits power series expansion of the generalized co-ordinates in terms of τ. According to the theory of ordinary differential equations [11], equations (13) have one ordinary point at $\tau = 0$ and two regular singular points at $\tau = \pm 1$. It is appropriate to write the power series expansion for $\bar{q}_j(\tau)$ about the ordinary point $\tau = 0$ as

$$\bar{q}_j(\tau) = \sum_{n=1}^{\infty} P_{nj} \tau^{2n-2}, \qquad j = 1, 2, \ldots, \tag{14}$$

where the P_{nj} are constants to be determined. Equations (14) show that the generalized co-ordinates can represent periodic motion since they repeat themselves every time $\tau = 0$. Furthermore, only even powers are admitted in the series, so that the periodic motion is captured repeatedly every half-cycle of the oscillating time (positive or negative). This condition places a requirement on the oscillating time frequency to be equal to one-half the vibration frequency:

$$\omega = \Omega/2. \tag{15}$$

The product $\bar{q}_k\,\bar{q}_l\,\bar{q}_m$ in equations (13) can also be written as a single power series given by

$$\bar{q}_k\,\bar{q}_l\,\bar{q}_m = \sum_{n=1}^{\infty} R_{nklm}\,\tau^{2n-2}. \tag{16}$$

Substituting equations (14) and (16) in equations (13) and introducing a shift of indices so that all terms have the same power, one obtains

$$\sum_{n=1}^{\infty}\left[\omega^2(2n)\,(2n-1)P_{n+1,j} - \omega^2(2n-2)^2\,P_{nj} + \frac{b_j}{a_j}P_{nj} + \frac{\beta}{a_j}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} c_{jklm}\,R_{nklm}\right]\tau^{2n-2} = 0, \qquad j = 1, 2, \ldots . \tag{17}$$

If equations (13) are to be satisfied at any time, all the bracketed coefficients in equations (17) must vanish. This condition gives the recurrence relation

$$P_{n+1,j} = \frac{\left[\omega^2(2n-2)^2 - \dfrac{b_j}{a_j}\right]P_{nj} - \dfrac{\beta}{a_j}\displaystyle\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} c_{jklm}\,R_{nklm}}{2n(2n-1)\omega^2}, \qquad n, j = 1, 2, \ldots, \tag{18}$$

between the successive rows in the coefficient matrix P_{nj}.

The initial value problem as formulated by equations (13) requires that the beam initial displacement and velocity be specified. It is convenient to assume that the beam motion starts at $\tau = t = 0$ from the maximum displacement position when the beam velocity vanishes. The non-linear normal mode is computed by assuming the motion to start with a linear mode shape of a given amplitude at $\tau = 0$. The elements of the first row P_{1j}, representing maximum displacement, are therefore assigned zero values except for one element that corresponds to the mode number. The first row in the P_{nj} matrix is a fundamental one on which successive rows depend recursively in accordance with equation (18). Furthermore, the form of expansions (14) ensures that the condition of vanishing initial velocity is satisfied. The present approach differs from the perturbation and invariant manifold methods in that the generalized co-ordinates are expanded about the maximum displacement position at $\tau = 0$ instead of the equilibrium position. The subsequent motion, obtained from equation (18), depends on the oscillating time frequency as a motion parameter. This frequency can be determined by invoking Rayleigh's energy principle which states that, for a conservative system, the maximum potential and kinetic energies are equal. The maximum potential energy is reached at the maximum displacement position, which is assumed to occur at the start of the motion. For the beam considered, this is given by

$$U_{max} = \tfrac{1}{2}\int_0^1 (u''^2 + \alpha u^2 + \tfrac{1}{2}\beta u^4)\,\mathrm{d}\zeta, \tag{19}$$

where the maximum displacement $u(\zeta) = \lambda\phi_i(\zeta)$ is assumed to be a normalized linear mode shape factored by the amplitude of vibration λ. The kinetic energy of the beam is given

$$T = \frac{1}{2}\int_0^1 \dot{w}^2\,\mathrm{d}\zeta = \frac{1}{2}\,\omega^2(1-\tau^2)\int_0^1 (w^\tau)^2\,\mathrm{d}\zeta. \tag{20}$$

The maximum kinetic energy occurs when the equilibrium position is reached, for which $\omega t = \pi/4, 3\pi/4, 5\pi/4$, etc. From equation (12), this position is reached at $\tau = \pm 1/\sqrt{2}$. Consequently, the maximum kinetic energy is computed as

$$T_{max} = \frac{1}{4}\,\omega^2\int_0^1 (w^\tau|_{\tau=1/\sqrt{2}})^2\,\mathrm{d}\zeta. \tag{21}$$

3. RESULTS AND DISCUSSION

The first three non-linear normal modes and frequencies were computed for both simply supported and clamped beams by using the recurrence relation (18) in conjunction with Rayleigh's principle. In each mode, the motion was assumed to start at $\tau = 0$ from the maximum displacement position with the linearized mode shape. As a result, the first row in the coefficient matrix P_{nj}, being the basis on which subsequent motion depends, had zero elements except for one element that corresponded to the mode number and was assigned the value of the amplitude of vibration λ. A frequency search was made for the natural frequency by computing, for each oscillating time frequency ω, the subsequent motion from the recurrence relation (18). The non-linear coefficients R_{nklm} were evaluated based on the linearized mode shape, as represented by the first row, vibrating harmonically with a frequency of $\Omega = 2\omega$. Consequently, all these coefficients were zero except those for which $k = l = m = i$, where $i =$ mode number. This reduced the triple summation in equation (18) to a single term. The resulting motion was used to evaluate the maximum kinetic energy from equation (21). The error function ($U_{max} - T_{max}$), which depends on ω, always crossed the ω-axis at only one value of the oscillating time frequency for which Rayleigh's energy principle was satisfied. The non-linear normal frequency of vibration

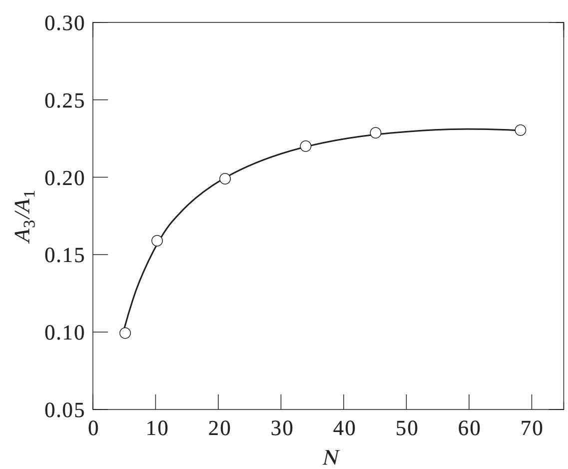

Figure 1. Convergence of the modal ratio (A_3/A_1) for the first non-linear mode of a simply supported beam versus the number of terms N ($\lambda = 0{\cdot}063$).

TABLE 1

Coefficient matrix P_{nj} of the first non-linear mode for the simply supported beam ($\alpha = 0$, $\beta = 10^6$, $\lambda = 0{\cdot}063$)

n	P_{n1}	P_{n2}	P_{n3}
1	0·6300E − 1	0·0000	0·00000
2	−0·1398E00	0·2347E − 8	0·4511E − 1
3	0·9038E − 1	−0·2005E − 8	−0·7288E − 1
4	−0·6048E − 1	0·9589E − 9	0·2488E − 1
5	−0·6517E − 4	−0·9273E − 10	−0·1952E − 2
6	−0·4624E − 4	−0·6363E − 10	−0·1140E − 2
7	−0·3498E − 4	−0·4712E − 10	−0·7662E − 3
8	−0·2765E − 4	−0·3670E − 10	−0·5583E − 3
9	−0·2257E − 4	−0·2963E − 10	−0·4294E − 3
10	−0·1887E − 4	−0·2457E − 10	−0·3433E − 3
11	−0·1608E − 4	−0·2080E − 10	−0·2824E − 3
12	−0·1392E − 4	−0·1791E − 10	−0·2376E − 3

was, from equation (15), twice that value. The associated non-linear mode was then approximated by the resulting maximum displacement on the other side of the equilibrium position. This involved evaluating the generalized co-ordinates, equations (14), at $\tau = 1$ which immediately reflected the degree of participation of various modes in the construction of the non-linear mode. It is worth mentioning that the modal dynamics thus determined satisfies approximately the equation of motion and Rayleigh's energy principle. The system time behavior is generated from the equations governing its motion.

To demonstrate the applicability of the present method to the linear vibration of beams, the formulation was first applied to the linear vibration ($\beta = 0$) of a simply supported beam without foundation ($\alpha = 0$). The first natural frequency was computed to within 0·03 percent of its exact value (π^2). Naturally, no coupling was present between the various modes because of the system linearity. The associated generalized co-ordinate had the coefficients $P_{11} = 1$, $P_{12} = -2$, $P_{13} = P_{14} = \cdots = 0$, which represent the harmonic time variation $1 - 2\sin^2 \omega t = \cos \Omega t$.

The non-linear normal modes were obtained and presented by a set of values ($A_1, A_2, A_3, \ldots$) of the generalized co-ordinates normalized with respect to the value of the corresponding mode. In all the results the values used for α and β were $\alpha = 0$ and $\beta = 10^6$. The first non-linear mode was computed for the simply supported beam for $\lambda = 0{\cdot}063$ by including the first three modes ($j = 1, 2, 3$) in its construction. The modal ratio (A_3/A_1) computed with various numbers of terms N included in series (14) is shown in Figure 1. The modal ratio is seen to converge to the correct value with $N = 45$. This number of terms

TABLE 2

Vibration frequency ratio Ω/Ω_L for the simply supported beam ($\alpha = 0$, $\beta = 10^6$)

	First mode		Second mode		Third mode	
Amplitude λ	Reference [8]	Present	Reference [8]	Present	Reference [8]	Present
0·01666	1·6133	1·7028	1·0501	1·0565	1·0099	1·0042
0·0333	2·7209	2·9775	1·2001	1·2107	1·0395	1·0402
0·05	3·9289	4·3415	1·4511	1·4336	1·0891	1·0983
0·063	4·8907	5·3341	1·7162	1·6371	1·1415	1·1526

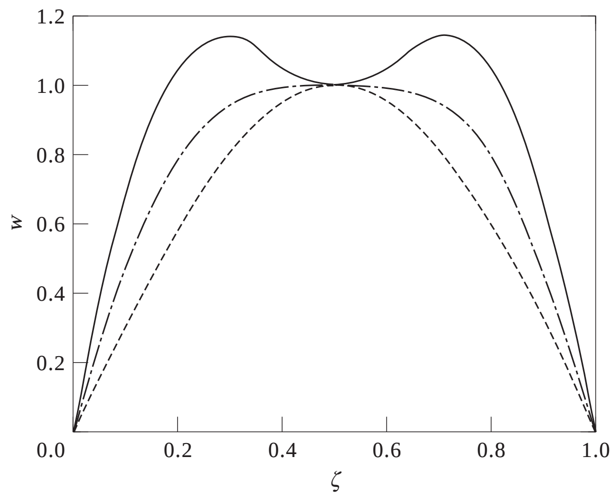

Figure 2. The first non-linear mode for the simply supported beam ($\lambda = 0{\cdot}063$). ——, Present method; —·—·—, invariant manifold; - - - -, linear.

was therefore used in all subsequent computation. With the first three modes included, the first non-linear mode was obtained as $(1, -0{\cdot}7311\mathrm{E}-8, 0{\cdot}2286)$ which is significantly affected by the third modal component, with the second component having negligible contribution because of its asymmetry. When the number of components was increased to five, the first non-linear mode was given by $(1, -1{\cdot}6596\mathrm{E}-8, 0{\cdot}2290, -0{\cdot}3621\mathrm{E}-9, 0{\cdot}5562\mathrm{E}-9)$ which shows a negligible effect from the fifth modal component. Consequently, at amplitude of vibration $\lambda = 0{\cdot}063$, the first non-linear mode is influenced only by the third modal component. The first 12 rows of the coefficient matrix P_{nj} of the first non-linear mode with three modal components are shown in Table 1. The progressive decrease in absolute value of the series coefficients characterizes a convergent solution. In Table 2 the non-linear frequency ratio Ω/Ω_L of the first three modes for the simply supported beam for various amplitudes is compared with those of the invariant manifold approach [8]. Ω_L is the linear frequency of vibration. Good agreement is seen between the two methods, although the present method predicts greater effect of the non-linearity on the first normal frequency. The results show an increase in the first three frequencies, with amplitude of vibration with the second and third frequencies being much less affected.

In Figure 2 the first non-linear mode is compared with that predicted by the invariant manifold approach [8] for $\lambda = 0{\cdot}063$. The present method is seen to attach greater importance to the third modal component. This may be explained by the fact that the invariant manifold approach involves expansion of the generalized co-ordinates up to third

TABLE 3

Vibration frequency ratio Ω/Ω_L for the clamped beam ($\alpha = 0$, $\beta = 10^6$)

Amplitude λ	First mode	Second mode	Third mode
0.01666	1·0638	1·0102	1·0033
0·0333	1·5333	1·0410	1·0116
0·05	2·0930	1·1221	1·0256
0·063	2·5221	1·2709	1·0388

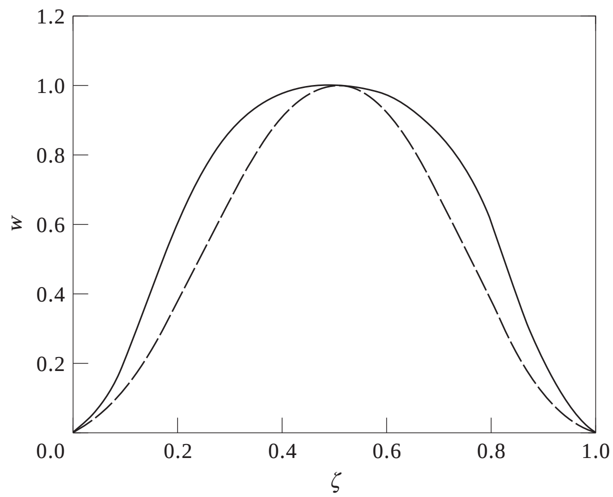

Figure 3. First normal mode of the clamped beam, ——, Non-linear; — —, linear ($\lambda = 0{\cdot}063$).

order only and the normal mode thus generated compares well with that obtained by a fifth order power series expansion ($N = 5$), as can be verified from Figure 1. The second and third non-linear modes were unaffected by the non-linearity, which is in agreement with the invariant manifold findings. Four modal components for the second mode and five components for the third mode were used. The computed second and third modes were $(-0{\cdot}2455\mathrm{E}-7,\ 1,\ 0{\cdot}2731\mathrm{E}-7,\ -0{\cdot}9682\mathrm{E}-9)$ and $(0{\cdot}3527\mathrm{E}-7,\ -0{\cdot}2039\mathrm{E}-7,\ 1,\ -0{\cdot}1016\mathrm{E}-9,\ -0{\cdot}1435\mathrm{E}-12)$, respectively.

The results for the clamped beam show that the vibration frequencies are significantly less affected by the non-linearity than those of the simply supported beam, with the first frequency being most affected, as shown in Table 3. The first non-linear mode for the clamped beam, with $\lambda = 0{\cdot}063$, is compared with the linear mode shape in Figure 3. Again, only the third modal component showed significant influence, although to a smaller degree than in the simply supported beam. The second and third non-linear modes for $\lambda = 0{\cdot}063$ were almost unaffected, as given by $(0{\cdot}3941\mathrm{E}-5, 1, 0{\cdot}1066\mathrm{E}-7, 0{\cdot}1636\mathrm{E}-4)$ and

TABLE 4

Coefficient matrix P_{nj} of the first non-linear mode for the clamped beam ($\alpha = 0$, $\beta = 10^6$, $\lambda = 0{\cdot}063$)

n	P_{n1}	P_{n2}	P_{n3}
1	0·063	0·0	0.0
2	−0·1352E00	−0·1443E − 5	0·4126E − 1
3	0·7755E − 1	0·1537E − 5	−0·9065E − 1
4	−0·5274E − 1	−0·5798E − 6	0·4016E − 1
5	−0·2877E − 3	0·8916E − 7	0·8573E − 3
6	−0·2027E − 3	0·5866E − 7	0·4351E − 3
7	−0·1525E − 3	0·4231E − 7	0·2687E − 3
8	−0·1202E − 3	0·3232E − 7	0·1855E − 3
9	−0·9780E − 4	0·2578E − 7	0·1373E − 3
10	−0·8163E − 4	0·2117E − 7	0·1066E − 3
11	−0·6946E − 4	0·1778E − 7	0·8578E − 4
12	−0·6003E − 4	0·1521E − 7	0·7085E − 4

$(0{\cdot}1322\text{E}-3, 0{\cdot}8123\text{E}-5, 1, -0{\cdot}2903\text{E}-4, -0{\cdot}5172\text{E}-3)$ respectively. The first 12 rows in the coefficient matrix P_{nj} for the first mode shape of the clamped beam $\lambda = 0{\cdot}063$ are given in Table 4.

4. CONCLUSIONS

A method for the computation of normal modes and frequencies of elastic beams resting on a non-linear foundation has been presented. The Galerkin procedure was used to discretize the equation of motion and the discretized equations were solved by the power series method upon transforming the time variable into an harmonically oscillating time. For simply supported and clamped beams, the first non-linear mode was found to be significantly influenced by the third modal component only, while the second and third non-linear modes were practically unchanged.

The vibration frequencies were increased by the non-linearity, although the first frequency was the most affected. Whereas existing perturbation and invariant manifold techniques involve asymptotic expansions about the equilibrium position, the present method expands the solution about the maximum displacement position and facilitates the inclusion of a greater number of terms in the expansion. A significant advantage of the present method is that it determines the degree of participation of various modes in the construction of non-linear mode shapes. The computational labor is also significantly reduced, regardless of the type of boundary conditions.

REFERENCES

1. R. Benamar, M. Bennouna and R. G. White 1991 *Journal of Sound and Vibration* **149**, 179–195. The effects of large vibration amplitudes on the mode shapes and natural frequencies of thin elastic structures, Part I: simply-supported and clamped–clamped beams.
2. W. Szemplinska-Stupnicka 1983 *International Journal of Nonlinear Mechanics* **18**, 149–165. Nonlinear normal modes and the generalized Ritz method in the problems of vibrations of nonlinear elastic continuous systems.
3. A. H. Nayfeh 1973 *Perturbation Methods*. New York: John Wiley.
4. A. H. Nayfeh 1981 *Introduction to Perturbation Techniques*. New York: John Wiley.
5. J. Carr 1981 *Application to Center Manifold Theory*. New York: Springer-Verlag.
6. S. W. Shaw and C. Pierre 1991 *Journal of Sound and Vibration* **150**, 170–173. Nonlinear normal modes and invariant manifolds.
7. S. W. Shaw and C. Pierre 1993 *Journal of Sound and Vibration* **164**, 85–121. Normal modes for nonlinear vibratory systems.
8. S. W. Shaw and C. Pierre 1994 *Journal of Sound and Vibration* **169**, 319–347. Normal modes of vibration for nonlinear continuous systems.
9. A. H. Nayfeh and S. A. Nayfeh 1994 *Transactions of the American Society of Mechanical Engineers*, *Journal of Vibration and Acoustics* **116**, 129–136. On nonlinear modes of continuous systems.
10. A. H. Nayfeh and S. A. Nayfeh 1995 *Transactions of the American Society of Mechanical Engineers*, *Journal of Vibration and Acoustics* **117**, 199–205. Nonlinear normal modes of a continuous system with quadratic nonlinearities.
11. W. Kaplan 1958 *Ordinary Differential Equations*. Reading, Massachusetts: Addison-Wesley.