*Journal of Sound and Vibration* (1998) **209**(4), 593–607

# APPLICATION OF MATCHED ASYMPTOTIC EXPANSIONS TO THE FREE VIBRATION OF A HERMETIC SHELL

S. K. Wong

*Department of Physics and Materials Science, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong*

AND

K. Y. Sze

*Department of Mechanical Engineering, University of Hong Kong, Pokfulam Road, Hong Kong*

(*Received* 29 *July* 1996, *and in final form* 3 *July* 1997)

The low frequency branch of the torsion free axisymmetric modes of thin cylindrical shell with hemispherical caps vibrating *in vacuo* is investigated using membrane approximation. The joint conditions needed at the seams are derived by taking the effect of bending into account in narrow bending layers, using the method of matched asymptotic expansions. A closed form equation is obtained for the normal frequencies. The mode structure exhibits finite jumps in the normal displacement across the seams, which are smoothed over in the bending layers. The normal frequencies are obtained numerically for general shapes and approximately for elongated shapes. Comparisons with the finite element method shows good agreement.



## 1. INTRODUCTION

Analytic solutions of vibration problems are useful because the results are immediate, suitable for parametric studies, and can serve as benchmarks for numerical work. They also augment the latter by providing results in extreme parameter regimes which are computationally demanding. Unfortunately, they are available only for simple shapes such as spheres or cylinders. They are almost non-existent for hermetic shells. The present work is concerned with a simple kind of hermetic shell consisting of a thin cylindrical shell with hemispherical endcaps, which can model aircraft, rockets, liquid and gas containers, steam boilers, etc. While numerical solutions for vibrations of this kind of shell exist [1], analytical work does not seem to exist in the literature. In this paper, the low frequency, non-torsional, axisymmetric modes for such shells vibrating *in vacuo* are considered in the membrane approximation.

The membrane approximation for curved shells dates back to the work of Love over a hundred years ago [2]. With the neglect of bending, the resulting equations of motion in terms of the displacement components form a set of second order differential equations which, for cylinders and spheres, can be solved analytically with the help of trigonometric or Legendre functions. Such solutions can be found in textbooks [3]. It is expected to be a good approximation whenever the wavelength of the vibration is much larger than the shell thickness. However, near boundaries, or locations of discontinuous physical or geometrical properties, such as the seams joining the hemispherical caps to the cylindrical

0022–460X/98/040593 + 15 $25.00/0/sv971200 

shell in the case being considered, rapid variations of the displacement will occur, causing the membrane approximation to break down. In a standard reference on shell theory [4], this has been noted for the problem of static loading of boilers of the shape being discussed. A remedy is proposed in the same reference which takes bending into account approximately.

For the vibration problem, the inadequacy of the membrane theory appears as an inability to satisfy the physically reasonable boundary conditions and continuity relations. The failure to satisfy all boundary conditions or continuity requirements simultaneously in the zeroth order approximation is a characteristic feature of singular perturbation theory [5, 6]. One systematic approach to obtaining higher order approximations is the method of matched asymptotic expansion. This method is used in [5] to solve the problem of a hemispherical dome supported under its own weight. Starting from the equations of three-dimensional elasticity, the thickness-over-radius is used as a small parameter for asymptotic expansions, leading in succession to the shell equations, the membrane approximation, and the bending correction at the base. For the vibrations of pre-stressed, clamped cylindrical shells, a similar method known as the method of composite expansions was applied in [7]. Matched asymptotic expansions were employed in [8] for the axisymmetric vibrations of the clamped cylindrical shells, and in [9] for more general vibrations. The references [7–9] all emphasize the importance of deriving appropriate boundary conditions for the membrane equations.

The present work uses the method of matched asymptotic expansions to study the axisymmetric vibrations of a thin cylindrical shell with hemispherical endcaps. By pursuing the simple membrane equations for the cylindrical shell and the spherical shell, it is found that the continuity of axial stress resultants conflicts with the continuity of the normal component of displacement at the seam. The essential result of this work is to point out the correct joint conditions to be applied at the seam, and to provide a derivation using matched asymptotic expansions on the shell equations. The condition is applicable to vibrational modes with frequencies below the ring frequency of the cylinder. Closed form equations are obtained for the natural frequencies. These equations are solved by a simple root-finding method, yielding results which compare favorably with numerical results using the finite element method. The mode shapes also show agreement.

The organization of this paper is as follows. In section 2, the problem of small vibrations is formulated using the membrane approximation, with the joint conditions made plausible but not derived. Closed form equations are obtained for the low frequency, non-torsional axisymmetric modes. In section 3, these equations are solved analytically in the limits of very long and very short cylindrical sections, and numerically for general aspect ratios. The results are compared with the finite element method. The method of matched asymptotic expansions is applied in section 4 to the bending theory of shells, supplying a derivation of the joint conditions. Concluding remarks follow in section 5.

## 2. MEMBRANE APPROXIMATION: FORMULATION

Consider a thin circular cylindrical shell of length $l$ and radius $a$, with hemispherical endcaps of the same thickness $h$. Let $\rho$ be the density, $E$ be the Young's modulus, and $\nu$ be the Poisson's ratio of the material. It is desired to find the frequencies and mode structures of the axisymmetric normal modes of the object *in vacuo*. Only non-torsional modes, which have no azimuthal displacements, are considered. A co-ordinate system as illustrated in Figure 1 is chosen, in which points on the cylindrical shell are described by the co-ordinates $(\varphi, z)$, with $0 \leqslant \varphi \leqslant 2\pi$ and $-l/2 \leqslant z \leqslant l/2$, while points on the hemispherical caps are described by $(\theta, \varphi)$, where $0 \leqslant \theta \leqslant \pi/2$ for the upper cap and

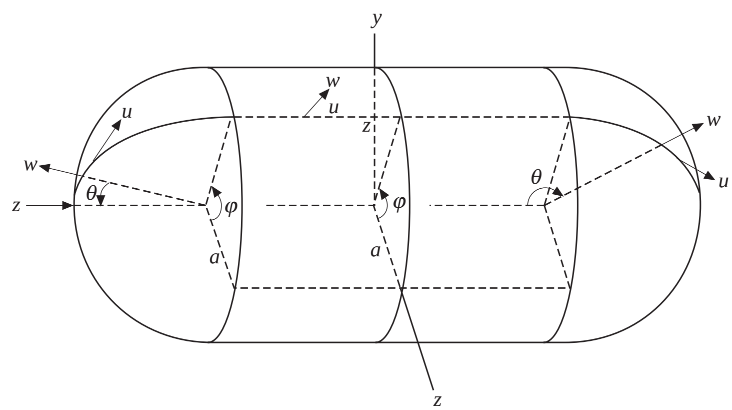


Figure 1. Co-ordinate systems and displacement components.

$\pi/2 \leqslant \theta \leqslant \pi$ for the lower cap. The variable $\eta = \cos\theta$ is also used interchangeably with $\theta$ as a co-ordinate on the hemispheres. The normal component of the displacements, denoted by $w$, is chosen to be directed outward, and the meridional component $u$ is chosen to be directed downward as shown.

It is expected that when the thickness is small, the shell behaves as a membrane in which only tangential stresses need be considered. The precise conditions under which this approximation is justified are given in section 4. For axisymmetric normal modes with the harmonic time dependence $e^{-i\omega t}$ where $\omega$ is the frequency, the displacement components $u$ and $w$ are functions of either $z$ or $\eta$, according to whether points on the cylindrical shell or the hemispherical caps are considered. In the membrane approximation, they satisfy the equation of motion

$$-\omega^2\rho h u = \frac{Eh}{1-\nu^2}\left(\frac{d^2u}{dz^2} - \frac{\nu}{a}\frac{dw}{dz}\right), \qquad -\omega^2\rho h w = \frac{Eh}{1-\nu^2}\left(\frac{\nu}{a}\frac{du}{dz} - \frac{w}{a^2}\right), \tag{1, 2}$$

on the cylindrical shell and

$$-\omega^2\rho h u = \frac{Eh}{1-\nu^2}\frac{1}{a^2}\left[\frac{d}{d\eta}(1-\eta^2)\frac{du}{d\eta} - \frac{u}{1-\eta^2} + (1-\nu)u - (1+\nu)\sqrt{1-\eta^2}\,\frac{dw}{d\eta}\right], \tag{3}$$

$$-\omega^2\rho h w = \frac{Eh}{1-\nu^2}\frac{1}{a^2}\left[(1+\nu)\frac{d}{d\eta}\sqrt{1-\eta^2}u - 2(1+\nu)w\right], \tag{4}$$

on the hemispherical caps. These equations, or their equivalents with different choice of sign conventions and co-ordinates, can be found in many standard textbooks of shell theory. For instance, they can be found from [4] by neglecting the bending terms.

It follows from equations (2) and (4) that the normal displacement can be expressed in terms of the meridional displacement through

$$w = \{\nu a/[1-(1-\nu^2)\Omega^2]\}\, du/dz, \tag{5}$$

for the cylindrical shell and

$$w = [(1+\nu)/(2(1+\nu)-(1-\nu^2)\Omega^2)]\,(d/d\eta)\sqrt{1-\eta^2}u, \tag{6}$$

for the caps, where

$$\Omega = \omega a \sqrt{\rho/E}, \tag{7}$$

is a dimensionless frequency. Substituting equations (5) and (6) into equations (1) and (3) respectively, produces the equation

$$\mathrm{d}^2 u/\mathrm{d}z^2 + (\pi\mu/l)^2 u = 0, \tag{8}$$

for the cylindrical shell and

$$(\mathrm{d}/\mathrm{d}\eta)\,(1-\eta^2)\,(\mathrm{d}u/\mathrm{d}\eta) - [u/(1-\eta^2)] + \lambda(\lambda+1)u = 0, \tag{9}$$

for the hemispherical caps, where

$$\mu = \Omega/\pi\delta[(1-(1-v^2)\Omega^2)/(1-\Omega^2)]^{1/2}, \tag{10}$$

with $\delta = a/l$ denoting the inverse-aspect-ratio, and

$$\lambda = -\tfrac{1}{2} + \{\tfrac{1}{4} + [1+(1+v)\Omega^2]\,[2-(1-v)\Omega^2]/(1-\Omega^2)\}^{1/2}. \tag{11}$$

The reduction of the equations of motion to the second order equations (8) and (9) for the meridional displacement alone represents a great simplification of the membrane approximation.

To complete the description, it is necessary to supply boundary conditions and joint conditions at the seam. Regularity of the solution requires the boundary conditions

$$u = 0 \qquad \text{at} \qquad \eta = \pm 1. \tag{12}$$

However, the conditions to be applied at the seams are not at all obvious. In fact, these conditions cannot be obtained without considering the effect of bending. Postponing the justification to section 4, the following heuristic approach is taken here. It is first assumed that

$$u|_{z=\pm l/2} = u|_{\eta=\pm 0}, \tag{13}$$

i.e., the meridional displacement $u$ is continuous across the seam. It is further assumed that the normal displacement $w$ can suffer at most a finite jump discontinuity. It then follows, on integrating equations (1) and (3) over a narrow gap across the seam and letting the gap width approach zero, that

$$\left(\frac{\mathrm{d}u}{\mathrm{d}z} - \frac{v}{a}\,w\right)\bigg|_{z=\pm l/2} = \left(\frac{1}{a}\frac{\mathrm{d}u}{\mathrm{d}\eta} - \frac{1+v}{a}\,w\right)\bigg|_{\eta=\pm 0}. \tag{14}$$

As shown in section 4, this equation, which follows from consideration of bending, states that the axial stress resultant is continuous.

Eliminating $\mathrm{d}u/\mathrm{d}z$ and $\mathrm{d}u/\mathrm{d}\eta$ from equation (14) by using equations (5) and (6), it follows that

$$w/v|_{z=\pm l/2} = w/(1+v)|_{\eta=\pm 0}. \tag{15}$$

Thus, the normal displacement is discontinuous at the seam. This non-physical result is resolved in section 4 where it is shown that a bending layer exists at the seam, in which this discontinuity is smoothed over. On the other hand, eliminating $w$ from equation (14) by using equations (5) and (6), and employing the assumed continuity of $u$, the following joint condition in terms of logarithmic derivatives is obtained:

$$\frac{1}{1-(1-\nu^2)\Omega^2}\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}z}\bigg|_{z=\pm l/2} = \frac{1}{2(1+\nu)-(1-\nu^2)\Omega^2}\frac{1}{au}\frac{\mathrm{d}u}{\mathrm{d}\eta}\bigg|_{\eta=\pm 0}. \tag{16}$$

It is straightforward to solve equations (8) and (9) under the conditions (12), (13), and (16). First, notice that the system of equations are unchanged under the mirror reflection $z \to -z$, $\eta \to -\eta$, $u \to -u$, $w \to w$. As a result, modes can be classified into those having even or odd parities. Even parity modes have their normal displacement unchanged and meridional displacement reversed by reflection and can be referred to as symmetric modes. The opposite is true for odd parity modes, which can be considered antisymmetric. It is only necessary to consider one half of the object, say the upper half. The solution for the other half can be obtained by appropriate reflection. The only regular solution, aside from a multiplicative constant, of equation (9) in the region $0 \leqslant \eta \leqslant 1$ is the Legendre function $P_\lambda^1(\eta)$. Solutions of equation (8) having even or odd parity are $\sin(\pi\mu z/l)$ and $\cos(\pi\mu z/l)$ respectively. Making use of the relation

$$\frac{[P_\lambda^1(0)]'}{P_\lambda^1(0)} = -\frac{\lambda(\lambda+1)}{2}\left[\frac{\Gamma([\lambda+1]/2)}{\Gamma(\lambda/2+1)}\right]^2 \cot\left(\frac{\pi\lambda}{2}\right),$$

which can be obtained from [10], it is possible to transform equation (16) for the case $z = l/2$ and $\eta = +0$ into the relation

$$C^{(\pm)}(\Omega) = H(\Omega), \tag{17}$$

where

$$C^{(\pm)}(\Omega) = \Omega\left[\frac{1-\Omega^2}{1-(1-\nu^2)\Omega^2}\right]^{1/2}\begin{Bmatrix}\cot(\pi\mu/2)\\ -\tan(\pi\mu/2)\end{Bmatrix}, \tag{18}$$

with the upper or lower factors chosen for $C^{(+)}$ and $C^{(-)}$ respectively, and

$$H(\Omega) = -\frac{1+(1+\nu)\Omega^2}{2(1+\nu)}\left[\frac{\Gamma([\lambda+1]/2)}{\Gamma(\lambda/2+1)}\right]^2 \cot\left(\frac{\pi\lambda}{2}\right). \tag{19}$$

The plus or minus sign corresponds to the symmetric or antisymmetric modes. The normal displacement of these modes are evaluated from equations (5) and (6). The resulting mode structures, normalized to unit meridional displacement at the seam, are given by

$$u = \begin{cases} P_\lambda^1(\eta)/P_\lambda^1(0), \\ \sin(\pi/\mu z/l)/\sin(\pi\mu/2), \end{cases}$$

$$w = \begin{cases} [(1+\nu)/(1-\Omega^2)]H(\Omega)P_\lambda(\eta)/P_\lambda(0), & \text{upper cap}, \\ (\nu/(1-\Omega^2)C^{(+)}\cos(\pi\mu z/l)/\cos(\pi\mu/2), & \text{cylindrical shell}, \end{cases} \tag{20a}$$

for the symmetric modes and

$$u = \begin{cases} P_\lambda^1(\eta)/P_\lambda^1(0), \\ \cos(\pi/\mu z/l)/\cos(\pi\mu/2), \end{cases}$$

$$w = \begin{cases} [(1+\nu)/(1-\Omega^2)]H(\Omega)P_\lambda(\eta)/P_\lambda(0), & \text{upper cap,} \\ (\nu/(1-\Omega^2)C^{(-)}\sin(\pi\mu z/l)/\sin(\pi\mu/2), & \text{cylindrical shell,} \end{cases} \qquad (20b)$$

for the antisymmetric modes. For the lower cap, the replacements $\eta \to -\eta$, $u \to -u$, $w \to w$ should be made for the symmetric modes, and the replacements $\eta \to -\eta$, $u \to u$, $w \to -w$ should be made for the antisymmetric modes.

## 3. MEMBRANE APPROXIMATION: RESULTS

The dimensionless normal frequencies as given by the roots of equation (17) depend only on the inverse aspect ratio $\delta$ and the Poisson's ratio $\nu$. It would appear from the derivation that there is no *a priori* restriction on the range of the roots. The quantities $\lambda$ or $\mu$ could conceivably be complex. However, as discussed in the next section, justification for the membrane approximation can be provided only when $\Omega < 1$. Hence, results are given only for this low frequency case in this section.

Consider first the limit of small inverse aspect ratio $\delta$. In this limit there exist low frequency modes such that $\Omega \sim \delta$. Assuming this ordering, it follows from equations (10) and (11) that $\mu \cong \Omega/\pi\delta$ and $\lambda \cong 1$. Equation (17) becomes

$$\Omega \cot \Omega/2\delta = 0 \qquad \text{or} \qquad \Omega \tan \Omega/2\delta = 0, \qquad (21)$$

for the symmetric or antisymmetric modes respectively. The solutions for both equations can be represented as $\Omega = n\pi\delta$ where $n = 1, 3, 5, \ldots$ give the symmetric modes and $n = 2, 4, 6, \ldots$ give the antisymmetric ones. Since $\mu \cong n$ for these modes, the meridional displacement $u$ is proportional either to $\sin(\pi nz/l)$ for odd $n$ or to $\cos(\pi nz/l)$ for even $n$, and the corresponding normal displacement $w$ is proportional to $\nu\delta\cos(\pi nz/l)$ or $\nu\delta\sin(\pi nz/l)$. Therefore, the derivative $\mathrm{d}u/\mathrm{d}z$ and the displacement component $w$ both vanish at the ends of the cylindrical shell, showing that these are the low frequency modes of a long and thin cylindrical shell with freely supported ends.

In order to obtain a correction to the frequency of these modes due to the hemispherical caps, it is necessary to keep the first non-zero term of the Taylor series of the function $H(\Omega)$ in $\Omega$ in equation (17). Because of the low frequency ordering $\Omega \sim \delta$, the Taylor series is equivalent to an expansion in $\delta$. This is accomplished by substituting the expansion

$$\lambda \cong 1 + (1+\nu)\Omega^2, \qquad (22)$$

of equation (11) into the argument of the cotangent in $H(\Omega)$, and gives rise to the approximation $H \cong \Omega^2$. The improved approximation to equation (17) is now

$$\cot \Omega/2\delta = \Omega \qquad \text{or} \qquad -\tan \Omega/2\delta = \Omega, \qquad (23)$$

for the symmetric and antisymmetric modes respectively. These equations can be solved perturbatively to give the following formula for $\Omega$ valid to first order in $\delta$:

$$\Omega = n\pi\delta(1 - 2\delta) \qquad (24)$$

where, as before, $n$ is odd for the symmetric modes and even for the antisymmetric modes. Because $\lambda \cong 1$, the meridional displacement $u$ on the upper cap can be approximated by $\sin\theta$ according to equation (20a), and transformed to the variable $\theta$ from $\eta$. It is inconvenient to attempt to find the normal displacement $w$ from equation (20a), as it entails

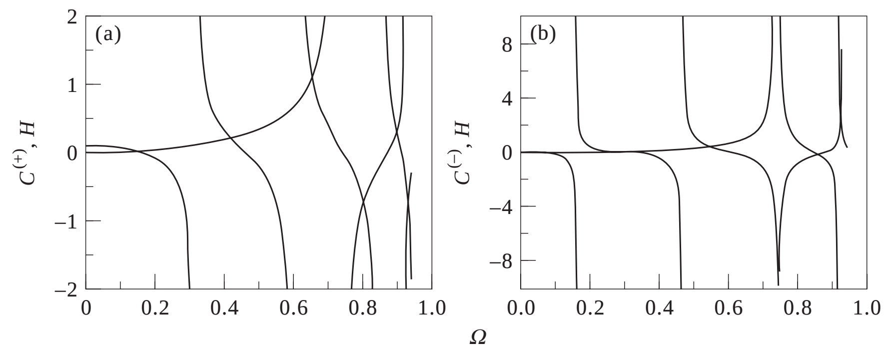


Figure 2. Branch structures of the functions (2a) $C^{(+)}$, $H$ and (2b) $C^{(-)}$, $H$ versus $\Omega$. $C^{\pm)}$ are given by the monotonically decreasing branches while $H$ is given by the monotonically increasing branches. The curves are drawn for $\delta = 0{\cdot}05$ and $v = 0{\cdot}3$.

finding the value of $P_\lambda\,(0)$ when $\lambda$ deviates slightly from unity. Instead, equation (6) can be used to show that $w \cong -\cos\theta$. These forms of the meridional and normal displacements imply that the hemispherical caps vibrate as rigid shells along the axial direction. The vibrations of the two caps are 180° out of phase for the symmetric modes and in phase for the antisymmetric modes.

In the limit of large inverse aspect ratio $\delta$, the effect of the cylindrical shell should become negligible. This follows from taking the limit $\delta \to \infty$ in equation (17) while keeping $\Omega$ fixed. In this limit, $\mu \to 0$, and the left side of equation (17) is either zero or infinity according to whether the mode is antisymmetric or symmetric, so that the equation can be satisfied only if $\lambda$ is an integer. The integer is even for symmetric modes and odd for antisymmetric modes. Writing $n + 1$ for the integer for a reason which becomes apparent later, the equation $\lambda = n + 1$ can be solved for $\Omega$, giving

$$\Omega = [1 + 3v + (n + 1)\,(n + 2)]/2(1 - v^2) \tag{25}$$

$$\{[1 + 3v + (n + 1)\,(n + 2)]^2 - 4(1 - v^2)\,[(n + 1)\,(n + 2) - 2]\}^{1/2}/2(1 - v^2), \tag{26}$$

for the branch which satisfies the condition $\Omega < 1$. These are the frequencies of the low frequency branch of the non-torsional axisymmetric modes of a spherical membrane. The mode structures are given by $u \propto P^1_{n+1}\,(\cos\theta)$ and $w \propto P_{n+1}\,(\cos\theta)$. Numerical calculations to follow show that as $\delta$ increases from zero to infinity, the modes described by equation (24) go over to the modes described by equation (25) with the same integer $n$.

Numerical solution of the equations (17) is facilitated by recognizing the branch structure of both sides of the equation. As $\Omega$ increases from zero to unity, $\lambda$ increases from one to infinity and $\mu$ increases from zero to infinity. The branch of the function $H(\Omega)$ are separated by the values $\Omega^{(H)}_i$ which satisfy $\lambda(\Omega^{(H)}_i) = 2i$ for $i = 1, 2, \ldots$ As illustrated by the solid curves in Figure (2a) and (2b), $H(\Omega)$ increases in each branch from negative to positive infinity, except for the first branch, which starts at the value zero. Figure (2a) shows that the branches of $C^{(+)}(\Omega)$, which are separated by $\Omega^{(C+)}_i$ satisfying $\mu(\Omega^{(C+)}_i) = 2i$ with $i = 1, 2, \ldots$, all decrease from positive to negative infinity except for the first branch which starts at the value $\pi\delta$. Thus, if the sequences $\Omega^{(H)}_i$ and $\Omega^{(C+)}_i$ are merged and rearranged as an increasing sequence, one and only one root of equation (17) will occur between any two successive members of the new sequence. The root can be found by the method of bisection to the desired accuracy. Similarly, as shown in Figure (2b), the function $C^{(-)}\,(\Omega)$ for the antisymmetric modes have branches separated by $\Omega^{(C-)}_i$ satisfying $\mu(\Omega^{(C-)}_i) = 2i - 1$ with $i = 1, 2, \ldots$, decreasing from positive to negative infinity, with the first branch starting at zero. Again the same method can be applied to find the root of the corresponding equation (17).

For the purpose of enumeration, a mode number $n$ is introduced, which denotes the number of zeros of the meridional displacement $u$ other than the two poles. It turns out that $n$ is equal to 1, 3, 5, . . . for the symmetric modes and 2, 4, 6, . . . for the antisymmetric modes, and has the same significance in equations (24) and (25), for a general value of $\delta$.

To obtain the mode structures given by equation (20), the Legendre functions are numerically computed using the integral representation [10]

$$P_\lambda(\cos\theta) = 2\pi \int_0^\theta \cos(\lambda + \tfrac{1}{2})\varphi \, \mathrm{d}\varphi / \sqrt{2(\cos\varphi - \cos\theta)},$$

and the relation

$$P_\lambda^1(\eta) = -\sqrt{1-\eta^2}\, \mathrm{d}P_\lambda / \mathrm{d}\eta$$

As an illustration, the frequencies for the first four modes for the choice $\nu = 0{\cdot}3$ are plotted in Figure 3 against the quantity $\delta' = 2\delta/(1+\delta)$, which is the ratio of the cap portion of the length to the total length of the object. The computed frequencies smoothly connect the results in the small and large aspect ratio limits as given by equations (24) and (25).

The predictions of equation (17) have been compared with the results from the finite element code ABAQUS [11]. The case chosen for the study consist of a cylindrical shell of length 1 m joined to spherical caps of radii 0·2 m, 1 m, and 5 m respectively. The density is 7800 kg/m$^3$, Young's modulus is 207 GPa, and Poisson's ratio is 0·3. For the finite element analysis, the thickness of the shell is chosen to be 0·01 m. The three-node axisymmetric shell element SAX2 available in ABAQUS is used. For the three cases, the shell is meshed into 46, 42, and 68 elements respectively. Convergence has been secured as none of the finite element predictions changes more than 0·032% when the numbers of elements are halved. The four modes with the lowest frequencies are obtained in each

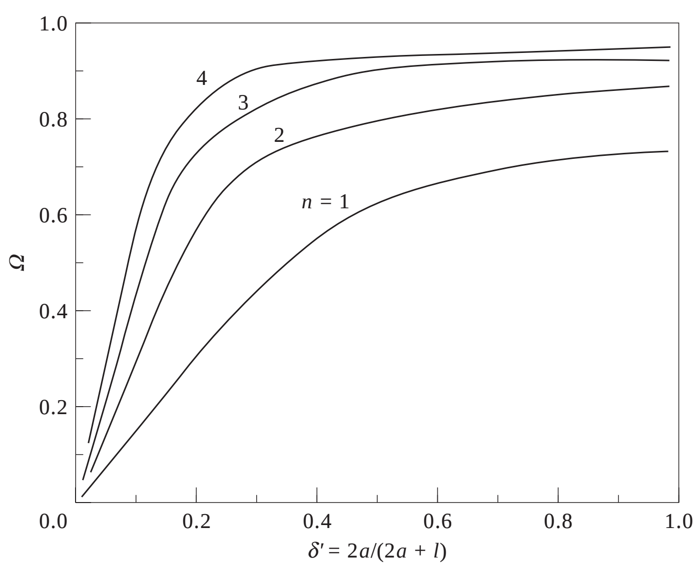


Figure 3. The normalized frequency of the four lowest frequency modes.

TABLE 1

*Comparison between membrane theory and FEM. In all cases the length of the cylinder is 1 m, thickness is 0·01 m, density is 7800 kg/m$^3$, Young's modulus is 207 GPa, Poisson's ratio is 0·3*

| Cap radius (m) | Mode number *n* | Membrane frequency (Hz) | FEM frequency (Hz) | Fractional difference (%) |
|---|---|---|---|---|
| 0·2 | 1 | 1759·0 | 1764·7 | 0·32 |
| 0·2 | 2 | 2880·7 | 2915·5 | 1·19 |
| 0·2 | 3 | 3336·6 | 3368·1 | 0·94 |
| 0·2 | 4 | 3687·7 | 3757·1 | 1·82 |
| 1·0 | 1 | 563·99 | 567·26 | 0·58 |
| 1·0 | 2 | 679·81 | 680·91 | 0·16 |
| 1·0 | 3 | 751·00 | 753·76 | 0·37 |
| 1·0 | 4 | 766·51 | 769·34 | 0·37 |
| 5·0 | 1 | 119·11 | 119·40 | 0·24 |
| 5·0 | 2 | 141·23 | 141·24 | 0·01 |
| 5·0 | 3 | 151·16 | 151·31 | 0·10 |
| 5·0 | 4 | 155·06 | 155·09 | 0·02 |

case. They correspond to the mode numbers ($n$) 1–4 according to the adopted convention. The results of the comparison as shown in Table 1 demonstrate good agreements.

The mode shapes from the present theory and the finite element method are shown in Figure 4 for the four modes corresponding to the case with radius 1 m. The sharp changes at the seam from the present theory arise from the discontinuity in the normal displacement. They are seen to be smoothed out in the finite element approach. Nevertheless, the overall agreement is good.

## 4. BENDING LAYER

The membrane theory does not provide a satisfactory derivation of the joint conditions at the seam between the cylindrical shell and the hemispherical caps. The predicted discontinuity of the normal displacement is obviously non-physical. It is well-known, at least for static loading, that the effect of bending is important at the seam, where stresses

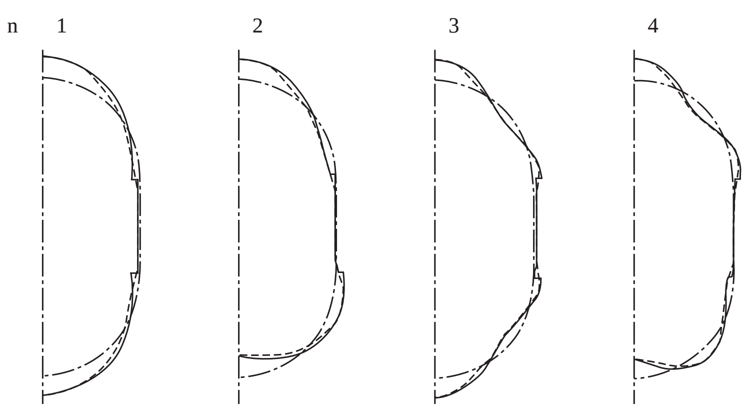


Figure 4. Four modes of vibration shown for a half shell: –·–, undistorted shell; ——, membrane theory; – – – –, FEM.

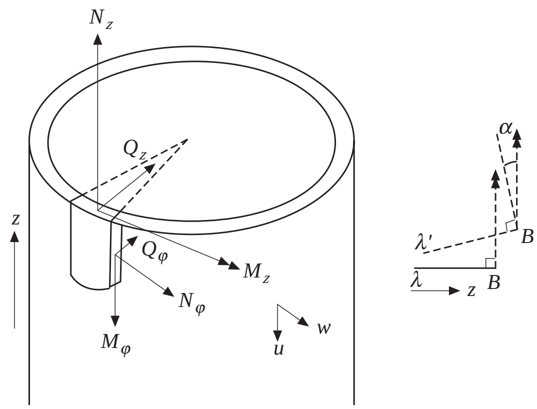


Figure 5. Stress resultants and displacements of a cylindrical shell. The inset illustrates the deflection of the normal.

and displacements can vary rapidly. In this section, the bending theory of shells is used to resolve both difficulties. In the process, the range of validity of the membrane approximation is also obtained. The method of matched asymptotic expansions is used in the analysis, which has previously been applied to the problem of static loading of thin spherical shells [5].

The appropriate equations of motion for non-torsional axisymmetric motion of cylindrical and spherical shells taking bending into account can be obtained from standard textbooks on shell theory such as reference [4]. The sign convention and notations used in this reference are adhered to as much as possible. For a cylindrical shell described in a cylindrical co-ordinate system $(a, \varphi, z)$, the displacement components $u$, $w$ and the non-zero stress resultants $N_z$, $N_\varphi$, $M_z$ and $Q_z$ are indicated in Figure 5. For harmonic oscillations, axial and normal force balance applied to a shell element lead to

$$\rho h \omega^2 u = \mathrm{d}N_z/\mathrm{d}z, \qquad \rho h \omega^2 w = N_\varphi/a + \mathrm{d}Q_z/\mathrm{d}z, \tag{26, 27}$$

while moment equilibrium yields

$$Q_z = \mathrm{d}M_z/\mathrm{d}z. \tag{28}$$

The membrane stress resultants are given in terms of the displacements by

$$N_z = D(-\mathrm{d}u/\mathrm{d}z + v[w/a]), \qquad N_\varphi = D(-v\,\mathrm{d}u/\mathrm{d}z + w/a) \tag{29, 30}$$

where $D = Eh/(1 - v^2)$. The corresponding relation for the bending moment is

$$M_z = \beta^2 D a^2\, \mathrm{d}^2 w/\mathrm{d}z^2 \tag{31}$$

where $\beta^2 = h^2/(12a^2)$. In addition, it is necessary to introduce the angle of deflection of the normal illustrated in the inset of Figure 5, which is given by

$$\alpha = \mathrm{d}w/\mathrm{d}z. \tag{32}$$

For a spherical shell in a spherical co-ordinate system $(a, \theta, \varphi)$, the displacements $u$, $w$ and the non-zero stress resultants $N_\theta$, $N_\varphi$, $M_\theta$, $M_\varphi$ and $Q_\theta$ are depicted in Figure 6. In terms of the variable $\eta = \cos\theta$ instead of $\theta$, the equation corresponding to equations (26)–(32) can be obtained from [4] by performing the appropriate transformation of variables. The results are

$$\rho h \omega^2 u = (1/a)\,[\mathrm{d}/\mathrm{d}\eta(\sqrt{1-\eta^2}N_\theta) + (\eta/\sqrt{1-\eta^2})N_\varphi + Q_\theta]; \tag{33}$$

$$\rho h \omega^2 w = (1/a)\,[N_\theta + N_\varphi - (\mathrm{d}/\mathrm{d}\eta)\,(\sqrt{1-\eta^2}Q_\theta)]; \tag{34}$$

$$Q_\theta = -(1/a)\,[(\mathrm{d}/\mathrm{d}\eta)\,(\sqrt{1-\eta^2}M_\theta) + (\eta/\sqrt{1-\eta^2})M_\varphi]; \tag{35}$$

$$N_\theta = (D/a)\,[-\sqrt{1-\eta^2}\,\mathrm{d}u/\mathrm{d}\eta + (\nu\eta/\sqrt{1-\eta^2})u + (1+\nu)w]; \tag{36}$$

$$N_\varphi = (D/a)\,[-\nu\sqrt{1-\eta^2}\,\mathrm{d}u/\mathrm{d}\eta + (\eta/\sqrt{1-\eta^2})u + (1+\nu)w]; \tag{37}$$

$$M_\theta = \beta^2 D(\sqrt{1-\eta^2}\,\mathrm{d}/\mathrm{d}\eta - (\nu\eta/\sqrt{1-\eta^2}))\,(u + \sqrt{1-\eta^2}\,\mathrm{d}w/\mathrm{d}\eta); \tag{38}$$

$$M_\varphi = \beta^2 D(\nu\sqrt{1-\eta^2}\,\mathrm{d}/\mathrm{d}\eta - (\eta/\sqrt{1-\eta^2}))\,(u + \sqrt{1-\eta^2}\,\mathrm{d}w/\mathrm{d}\eta); \tag{39}$$

$$\alpha = (1/a)\,(u + \sqrt{1-\eta^2}\,\mathrm{d}w/\mathrm{d}\eta). \tag{40}$$

The joint conditions at the seams can be formulated from geometrical considerations and Newton's third law. These conditions for the upper seams, where $z = l/2$ and $\eta = 0+$, are

$$-u^{(+)} = u^{(-)}, \quad w^{(+)} = w^{(-)}, \quad \alpha^{(+)} = \alpha^{(-)}, \quad N_\theta = N_z, \quad Q_\theta = -Q_z, \quad M_\theta = M_z, \tag{41}$$

where the plus sign refers to the hemisphere side and the minus sign refers to the cylinder side. Similar conditions hold at the lower seam. Together with the regularity of the solution at the poles, equations (26)–(41) complete the mathematical description of the capped cylindrical shell.

By eliminating the stress resultants, equations of motion in terms of the displacement components alone are obtained. The membrane equations (1)–(4) of section 2 are recovered if only the membrane stresses are kept. The additional terms arising from bending all contain the factor $\beta^2$, which is small for a thin shell. Comparison with the membrane terms would indicate that these terms are negligible provided the spatial scale of variation of the vibration is of the order of $a$ or larger. However, among these terms are those of the highest order, which contain the fourth derivatives of the normal displacement. Their neglect can cause failure to satisfy the boundary conditions, or joint conditions, as happens in the present case. Thus, a thin layer is expected to exist at the seam, where the bending terms are important. In the terminology of the method of matched asymptotic expansions, this layer is referred to as the inner region, while the outside is referred to as the outer region. Separate approximate solutions are sought for the two regions, which are then matched in an intermediate region.

Following [5], an inner variable $z_*$ is introduced for the upper bending layer through the definition

$$z_* = \begin{cases} (z - l/2)/\sqrt{\beta}, & z < l/2, \\ \alpha\eta/\sqrt{\beta}, & \eta > 0. \end{cases} \tag{42}$$

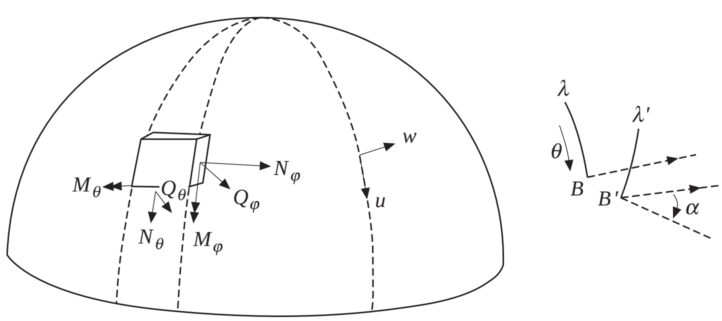


Figure 6. Stress resultants and displacements of a spherical shell. The inset illustrates the deflection of the normal.

Thus, $z_* > 0$ corresponds to the cap and $z_* < 0$ corresponds to the cylinder. The equations of motion are first written in terms of this variable, and the solution is then sought as series of the form

$$u = f_0^{(+)}(z_*) + \sqrt{\beta} f_1^{(\pm)}(z_*) + \cdots,$$
$$w = g_0^{(+)}(z_*) + \sqrt{\beta} g_1^{(\pm)}(z_*) + \cdots, \tag{43}$$

where the signs refer to positive and negative values of $z_*$ respectively. Through two orders in the expansion parameter $\sqrt{\beta}$, equation (26) for the cylinder gives

$$-\mathrm{d}^2 f_0^{(-)}/\mathrm{d}z_*^2 = 0, \tag{44}$$

$$\frac{\mathrm{d}}{\mathrm{d}z_*}(-\mathrm{d}f_1^{(-)}/\mathrm{d}z_* + (\nu/a) g_0^{(-)}) = 0, \tag{45}$$

while equation (27) gives

$$\mathrm{d}f_0^{(-)}/\mathrm{d}z_* = 0, \tag{46}$$

$$a^4\, \mathrm{d}^4 g_0^{(-)}/\mathrm{d}z_*^4 + [1-(1-\nu^2)\Omega^2] g_0^{(-)} = \nu a \frac{\mathrm{d}f_1^{(-)}}{\mathrm{d}z_*}, \tag{47}$$

with the fourth order term due to bending. Similar expansions of equation (33) and (34) for the hemisphere give

$$\mathrm{d}^2 f_0^{(+)}/\mathrm{d}z_*^2 = 0, \tag{48}$$

$$\mathrm{d}/\mathrm{d}z_*[-\mathrm{d}f_1^{(+)}/\mathrm{d}z_* + (1+\nu) g_0^{(+)}/a] = 0, \tag{49}$$

$$\mathrm{d}f_0^{(+)}/\mathrm{d}z_* = 0, \tag{50}$$

$$a^4\, \mathrm{d}^4 g_0^{(+)}/\mathrm{d}z_*^4 + [2(1+\nu) - (1-\nu^2)\Omega^2] g_0^{(+)} = (1+\nu) a\, \mathrm{d}f_1^{(+)}/\mathrm{d}z_*. \tag{51}$$

The joint conditions (41), which are applied at $z_* = 0$, are expanded likewise in $\sqrt{\beta}$. The first of the set of equations (41) implies the continuity of $f_0$. It then follows from equation (46) and (50) that

$$f_0^{(+)} = f_0^{(-)} = A_0, \tag{52}$$

where $A_0$ is a constant. The equations (45) and (49) can be integrated to give

$$-\mathrm{d}f_1^{(+)}/\mathrm{d}z_* + (1+\nu) g_0^{(+)}/a = -\mathrm{d}f_1^{(-)}/\mathrm{d}z_* + \nu g_0^{(-)}/a = B/a, \tag{53}$$

where the fact that $B$ is the same constant for both sides of the layer follows from the equality of $N_\theta$ and $N_z$ at the seam as stated in equation (41). Elimination of $f_1^{(+)}$ and $f_1^{(-)}$ from equations (47) and (51) respectively using equation (53) lead to the pair

$$a^4\, \mathrm{d}^4 g_0^{(+)}/\mathrm{d}z_*^4 + 4m^4 g_0^{(+)} = -(1+\nu)B \qquad z_* > 0, \tag{54a}$$

$$a^4\, \mathrm{d}^4 g_0^{(-)}/\mathrm{d}z_*^4 + 4m^4 g_0^{(-)} = -\nu B \qquad z_* < 0, \tag{54b}$$

where

$$4m^4 = (1-\nu^2)(1-\Omega^2). \tag{55}$$

In anticipation of matching to the solutions in the outer region, only those solutions of equation (54) which remain bounded while moving away from the seam should be kept for $\Omega < 1$, which is the case under consideration. These solutions are

$$g_0^{(+)} = -([1+\nu]/4m^4)B + \mathrm{e}^{-mz_*/a}[C_+ \cos(mz_*/a) + D_+ \sin(mz_*/a)], \tag{56a}$$

$$g_0^{(-)} = -(\nu/4m^4)B + e^{+mz_*/a}[C_- \cos(mz_*/a) + D_- \sin(mz_*/a)], \tag{56b}$$

where $m$ is the positive real root of equation (55) and $C_\pm$, $D_\pm$ are arbitrary constants. These constants are determined using the joint conditions (41). Keeping leading order terms when expanding the joint conditions in $\sqrt{\beta}$, it follows that $g_0$ itself and the first, second, and third derivatives (in $z_*$) of $g_0$ are continuous. The equations (56a) and (56b) simplify into

$$g_0^{(+)} = -(B/4m^4)\,[1 + \nu - \tfrac{1}{2}\,e^{-mz_*/a} \cos(mz_*/a), \tag{57a}$$

$$g_0^{(-)} = -(B/4m^4)\,[\nu + \tfrac{1}{2}\,e^{mz_*/a} \cos(mz_*/a), \tag{57b}$$

Returning to equation (53), integration and the use of continuity at the seam gives

$$f_1^{(+)} = -(B/a)\,[1 + (1+\nu)^2/4m^4]z_* + (aB(1+\nu)/16m^5)\,[1 - e^{mz_*/a}\cos(mz_*/a) + e^{-mz_*/a}\sin(mz_*/a)] + A_1 \tag{58a}$$

$$f_1^{(-)} = -(B/a)\left[1 + \frac{\nu^2}{4m^4}\right]z_* + (aB\nu/16m^5)\,[1 - e^{mz_*/a}\cos(mz_*/a) - e^{-mz_*/a}\sin(mz_*/a)] + A_1 \tag{58b}$$

where $A_1$ is a constant of integration whose value will not be needed.

The constants $A_0$, $B$ which are thus far undetermined can be obtained by matching with the outer solution. In the outer region, the original variable $z$ and $\eta$ are used, and the membrane equations of section 2 apply in the leading order of $\sqrt{\beta}$. The limits of the solution of these equations as the seam is approached from either side should be matched, order-by-order in $\sqrt{\beta}$, with the asymptotic limits as $z_* \to \pm\infty$ of the inner solutions tranformed back to the outer variables. Since the asymptotic limits for $f_0$ and $f_1$ can be written

$$f_0^{(+)} + \sqrt{\beta} f_1^{(+)} + \cdots \cong A_0 - B[1 + (1+\nu)^2/4m^4]\eta + \cdots, \tag{59a}$$

$$f_0^{(-)} + \sqrt{\beta} f_1^{(-)} + \cdots \cong A_0 - (B/a)\,[1 + (1+\nu)^2/4m^4]\,(z - l/2) + \cdots, \tag{59b}$$

while the limits of the corresponding outer solution are given by

$$u \cong u|_{\eta=0+} + \mathrm{d}u/\mathrm{d}\eta|_{\eta=0+}\,\eta + \cdots, \tag{60a}$$

$$u \cong u|_{z=l/2} + \mathrm{d}u/\mathrm{d}\eta|_{z=l/2}\,(z - l/2) + \cdots. \tag{60b}$$

Matching requires

$$u|_{\eta=0+} = u|_{z=l/2} = A_0 \tag{61}$$

and

$$\frac{1}{\nu^2 + 4m^4}\frac{\mathrm{d}u}{\mathrm{d}\eta}\bigg|_{z=l/2} = \frac{1}{(1+\nu)^2 + 4m^4}\frac{1}{a}\frac{\mathrm{d}u}{\mathrm{d}\eta}\bigg|_{\eta=0+} = -\frac{B}{4m^4 a}. \tag{62}$$

These are precisely the joint conditions assumed in the membrane theory in section 2. Thus, the bending theory provides a justification for them. In particular, equation (62) is traceable to equation (53), which, from equations (29) and (36), amounts to the statement of the continuity of axial stress resultant. In view of equation (62), the matching of $g_0^{(\pm)}$ to $w$ simply reproduces the limits of equations (5) and (6), and gives no new information. The inner solution (57) provides the smooth connection between the discontinuous values of the normal displacement in the membrane approximation. An estimate of the thickness of the bending layer is $\sqrt{\beta}a/m$. Setting the latter quantity to $a$ gives an estimate of how close $\Omega$ can approach unity before the membrane approximation loses its validity. This estimate is $1 - \Omega \sim /\sqrt{\beta}$.

Finally, for $\Omega > 1$, it is no longer possible to find two independent solutions of equation (54) which decay to zero moving away from the seam. The matching procedure cannot be applied. For this reason, the present approach is restricted to low frequencies where $\Omega < 1$. The physical reason for this restriction is unclear. Calculations using the finite-element method do not suffer from any difficulty in spanning the full range of frequencies. There is no qualitative change in the mode shape as the critical frequency is traversed. Further investigation would be required to find out if the restriction to low frequencies can be removed.

## 5. CONCLUSIONS

The low frequency, axisymmetric, non-torsional modes of a thin cylindrical shell with hemispherical endcaps vibrating *in vacuo* have been investigated with the help of membrane approximation. The modes are classified into symmetric and antisymmetric modes depending on their properties under mirror reflection about the mid-plane, and are numbered according to the number of nodes in the meridional displacement. The frequency, which increases with the mode number, satisfies a closed form relation, which has been solved numerically as well as analytically for very long and very short cylindrical sections. For very long cylindrical sections, the frequencies for the simple modes approach those for a cylindrical shell with simply-supported edges. The two hemispherical caps oscillate as rigid wholes, being in phase for the antisymmetric modes and out-of-phase for the symmetric modes. For vanishingly small cylindrical sections, the low frequency modes of the spherical membrane are reproduced, starting with modes of order two in an expansion in spherical harmonics. In the membrane approximation, the mode structure exhibit finite jumps in the normal displacement across the seams joining the hemispherical caps to the cylindrical shell. The membrane results are expected to hold as long as the mode number $n$ is not so high that the difference $1 - \omega_n\, a/c$ becomes of order $\sqrt{\beta}$, where $\omega_n$ is the frequency, $a$ is the radius of the shell, $c$ is the longitudinal sound speed in the shell material, $\beta = h/a$, with $h$ describing the shell thickness. The membrane results have been checked against finite element calculations with excellent agreement.

The joint condition at the seam for the membrane approximation are derived by taking the effect of bending into account in a bending layer of width $(ha)^{1/2}$ at the seams. The finite jumps of the normal displacement are also smoothed over in the layer. The method of matched asymptotic expansions has been used. It should be possible to apply this method to more general vibrations as well as other thin shells with boundaries or with discontinuous physical or geometrical properties.

## REFERENCES

1. M. S. Tavakoli and R. Singh 1989 *Journal of Sound and Vibration* **100**, 97–123. Eigensolutions of joined/hermetic shell structures using the state space method.
2. A. E. H. Love 1888 *Proceedings of the Royal Society of London* (*A*) **179**, 491–546. The small free vibrations and deformation of thin elastic shells.
3. M. C. Junger and D. Feit 1972 *Sound*, *Structures*, *and Their Interactions*. Cambridge, Mass: M.I.T. Press.
4. W. Flügge 1960 *Stresses in Shells*. Berlin: Springer-Verlag.
5. J. Kevorkian and J. D. Cole 1981 *Perturbation Methods in Applied Mathematics*. New York: Springer-Verlag.

6. A. H. Nayfeh 1973 *Pertubation Methods*. New York: Interscience.
7. D. E. Killian, M. P. Kamat and A. H. Nayfeh 1983 *Journal of Sound and Vibrations* **86**, 9–22. Numerical perturbation solutions for the vibrations of prestressed, clamped cylindrical shells.
8. S. K. Wong and W. B. Bush 1993 *Journal of Sound and Vibration* **160**, 523–532. Axisymmetric vibrations of a clamped cylindrical shell using matched asymptotic expansions.
9. H. E. Williams 1992 *Journal of Sound and Vibration* **155**, 133–151. On the boundary conditions for the membrane equations of thin, cylindrical shells and resulting natural frequencies.
10. M. Abramowitz and I. A. Stegun 1968 *Handbook of Mathematical Functions*, *p*. 334. New York: Dover.
11. ABAQUS *Theory Manual*. *Version* 5.5. RI, U.S.A.: 1996. Hibbit, Karlson & Sorenson.