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EXACT SOLUTION OF IN-PLANE VIBRATIONS OF CIRCULAR ARCHES WITH ACCOUNT TAKEN OF AXIAL EXTENSION, TRANSVERSE SHEAR AND ROTATORY INERTIA EFFECTS

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Exact solution of free in-plane vibrations of circular arches of uniform cross-section is given by considering axial extension, transverse shear and rotatory inertia effects. In contrast with Kirchhoff's beam theory the restrictions of perpendicular cross-section and inextensible arc length are removed. The principal axes of the cross-section are assumed to coincide with the principal normal and binormal vectors of the centerline of the beam. A solution procedure is applied to obtain the fundamental matrix for various end conditions. Natural frequencies and mode shapes are given in figures and tables.

1. INTRODUCTION

The vibrational behavior of curved beams is of strong industrial interest and has received considerable attention in recent years. The elementary Bernoulli-Euler equation of motion of beams is derived on the assumption that the deflection of beams are due to bending only and that transverse shear, rotatory inertia and axial extension effects are negligible. This simplifies the analysis considerably and can be recognized as adequate for usual engineering problems. The classical governing equations of in-plane and out-of-plane vibrations of curved beams are given together with their solutions in the book by Love [1].

Afterwards, many other researchers calculated the natural frequencies of in-plane and out-of-plane vibrations of circular arches based on the classical beam theory in which the foregoing effects are not considered. Although the simple cases are treated by using the exact method, Ritz, Galerkin and finite element methods are employed extensively when complicated cases such as non-uniform cross-sections are of concern.

It is well-known that for beams having large cross-sectional dimensions in comparison to their lengths, and for beams in which high-frequency modes of vibration are required, the Timoshenko theory which takes into account the rotatory inertia and shear effects gives a better approximation to the actual beam behavior. Considerable research has been devoted to study the effects of rotatory inertia and shear on straight beam vibrations [2]. In the case of curved beams, Seidel and Erdelyi [3] found the frequency equation for a closed ring under the assumption of sinusoidal mode shapes. Rao and Sundararajan [4] extended the work of Seidel and Erdelyi to circular arches, free complete circular rings and periodically radially supported complete rings. Later, Bickford and Storm [5] used the transfer matrix technique to determine the small amplitude motion of a constant curvature

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prismatic bar. Further analytical treatments of in-plane vibration of curved beams with transverse shear and rotatory inertia effects included are reported by Wang and Guilbert [6] and by Rosettos and Perl [7].

In one of the earliest finite element treatments of Timoshenko beams, Kapur [8] develops a finite element based on the Timoshenko beam theory for straight beams. Subsequently, a variety of elements have been presented for dynamic analysis of straight and curved Timoshenko beams amongst which are those of references [9–20].

Kang *et al*. [21] apply the differential quadrature method in the computation of the eigenvalues of the equations of motion governing in-plane and out-of-plane vibrations of circular arches based on the Timoshenko beam theory.

All of the investigations reviewed above neglect the extension of the neutral fibre in order to simplify the analysis. The constraint of inextensionality may be justified in situations in which the extensional coupling arises only in higher frequencies. Lin and Soedel [22] have shown however, that extensional coupling effects can be significant in the case of thick rings. Chidamparam and Leissa [23] investigate the influence of centerline extensibility on the free vibration of loaded rings and arches. They find that centerline stretching during the vibratory motion causes a decrease in the vibration frequencies, and that this decrease may be quite large, especially for shallow arches. However, they ignore the rotatory inertia and shear effects. Issa *et al*. [24] analytically find the dynamic stiffness matrix for circular curved members, including the effects of shear deformation, rotatory inertia and extension of the centerline, for estimating the natural frequencies of continuous beams undergoing in-plane vibrations. Rossi and Laura [25] constitute a second group considering all of the aforementioned effects, to the authors' knowledge. They have studied the possibility of attaining dynamic stiffening of simply supported and clamped arches by employing the finite element method.

The foregoing review shows that only a few works have taken into account the complete effects namely transverse shear, rotatory inertia and centerline extension. All of them use numerical methods and give approximate results. The objective of the present study is to find an exact solution to the governing equations of motion of circular curved beams having uniform cross-sections with account taken of transverse shear, axial extension and rotatory inertia effects. It is still possible to ignore unimportant effects by vanishing the related terms in the coefficients matrix for particular cases. Owing to the exact theory presented herein, one will be able to avoid the errors arising from approximate methods and computational procedure.

2. ANALYSIS

2.1. GENERAL EQUATIONS OF BEAM THEORY

The in-plane behavior of elastic curved beams with account taken of axial and shear deformations is formulated by several authors as:

$$\mathrm{d}w/\mathrm{d}\phi = u + (\rho(\phi)/EA)R_t, \qquad \mathrm{d}u/\mathrm{d}\phi = -w + (\rho(\phi)/GA/k_n)R_n + \Omega_b,$$
$$\mathrm{d}\Omega_b/\mathrm{d}\phi = (\rho(\phi)/EI_b)\,M_b,$$
$$\mathrm{d}M_b/\mathrm{d}\phi = -\rho(\phi)R_n - \rho(\phi)m_b, \qquad \mathrm{d}R_t/\mathrm{d}\phi = R_n - \rho(\phi)p_t,$$
$$\mathrm{d}R_n/\mathrm{d}\phi = -R_t - \rho(\phi)p_n, \tag{1}$$

where R_n, R_t are normal and tangential components of internal force; M_b is the internal moment about the binormal axis; p_n, p_t are normal and tangential components of external distributed force; m_b is the external distributed moment about the binormal axis; u, w are normal and tangential displacements; Ω_b is the rotation angle about the binormal axis; E, G are Young's and shearing moduli; A is the cross-sectional area; I_b is the moment of inertia with respect to the binormal axis; k_n is the factor of shear distribution along the normal axis; ϕ is the angular co-ordinate; ρ is the radius of curvature of undeformed beam axis.

2.2. GOVERNING EQUATIONS OF MOTION

The equations of motion can be derived by means of d'Alembert principle. For free vibration, inertia forces and moments are considered as external effects. That is

$$p_n = -\mu\, \partial^2 u/\partial t^2, \qquad p_t = -\mu\, \partial^2 w/\partial t^2, \qquad m_b = -(\mu/A) I_b\, \partial^2 \Omega_b\, /\partial t^2, \tag{2}$$

where μ is the mass per unit length and t is the time. Substituting equations (2) into equations (1) and assuming that the motion is harmonic with angular frequency ω, the governing equations of motion are obtained to be

$$\mathrm{d}w/\mathrm{d}\phi = u + (\rho(\phi)/EA)R_t, \qquad \mathrm{d}u/\mathrm{d}\phi = -w + (\rho(\phi)/GA(\phi)/k_n)R_n + \rho(\phi)\Omega_b,$$

$$\mathrm{d}\Omega_b\,/\mathrm{d}\phi = (\rho(\phi)/EI_b\,(\phi))M_b, \qquad \mathrm{d}M_b\,/\mathrm{d}\phi = -\rho(\phi)R_n - \rho(\phi)\mu(\phi)\,(I_b\,(\phi)/A(\phi))\omega^2\Omega_b,$$

$$\mathrm{d}R_t\,/\mathrm{d}\phi = R_n - \rho(\phi)\mu(\phi)\omega^2 w, \qquad \frac{\mathrm{d}R_n}{\mathrm{d}\phi} = -R_t - \rho(\phi)\mu(\phi)\omega^2 u. \tag{3}$$

2.3. ANALYTICAL SOLUTION

Equations (3) are simultaneous linear differential equations of the first order with variable coefficients. They can be written in matrix form as

$$\mathrm{d}\vec{\mathbf{y}}/\mathrm{d}\phi = \mathbf{A}(\phi)\vec{\mathbf{y}}(\phi), \tag{4}$$

where

$$\vec{\mathbf{y}}(\phi) = \begin{bmatrix} w \\ u \\ \Omega_b \\ M_b \\ R_t \\ R_n \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \rho/EI_b & 0 \\ -1 & 0 & \rho & 0 & 0 & \rho k_n\,/GA \\ 0 & 0 & 0 & \rho/EI_b & 0 & 0 \\ 0 & 0 & \rho\mu(I_b/A)\omega^2 & 0 & 0 & -\rho \\ \rho\mu\omega^2 & 0 & 0 & 0 & 0 & 1 \\ 0 & \rho\mu\omega^2 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

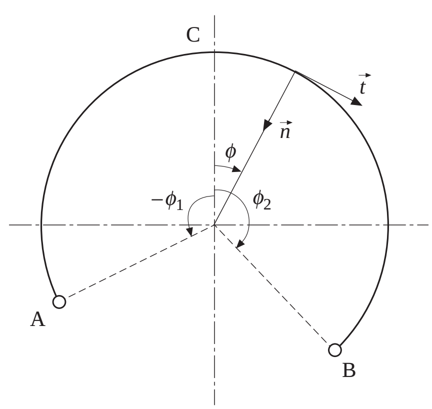

Figure 1. Circular curved beam of uniform cross-section.

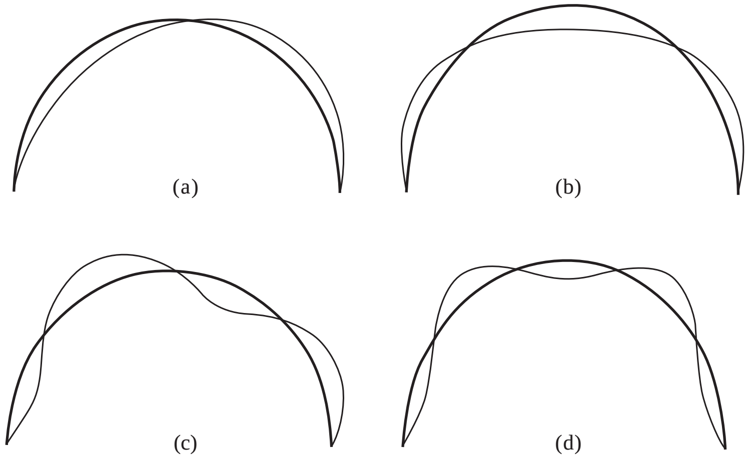

Figure 2. The lowest four modes of 180° circular arches with hinged ends.

For these homogeneous equations it is not generally possible to find an exact solution of the form $\vec{\mathbf{y}}(\phi) = \mathbf{Y}(\phi, \phi_0)\vec{\mathbf{y}}(\phi_0)$ where $\mathbf{Y}(\phi, \phi_0)$ is the fundamental matrix and $\vec{\mathbf{y}}(\phi_0)$ is the initial values vector in terms of the reference co-ordinate ϕ_0. But, an exact solution does exist for the particular case that all components of $\mathbf{A}$ are constant. This corresponds to circular curved beams of uniform cross-sections.

The solution of equation (4) for free in-plane vibrations of circular curved beams with uniform cross-sections is

$$\vec{\mathbf{y}}(\phi) = \mathrm{e}^{\mathbf{A}\phi}\vec{\mathbf{y}}(\phi_0), \tag{5}$$

provided the initial values vector $\vec{\mathbf{y}}(\phi_0)$ is known. $\mathbf{A}$ is a 6×6 matrix and the term $\mathrm{e}^{\mathbf{A}\phi}$ in equation (5) can be expressed exactly as

$$\mathrm{e}^{\mathbf{A}\phi} = \alpha_0\,\mathbf{I} + \alpha_1\,\mathbf{A} + \alpha_2\,\mathbf{A}^2 + \alpha_3\,\mathbf{A}^3 + \alpha_4\,\mathbf{A}^4 + \alpha_5\,\mathbf{A}^5, \tag{6}$$

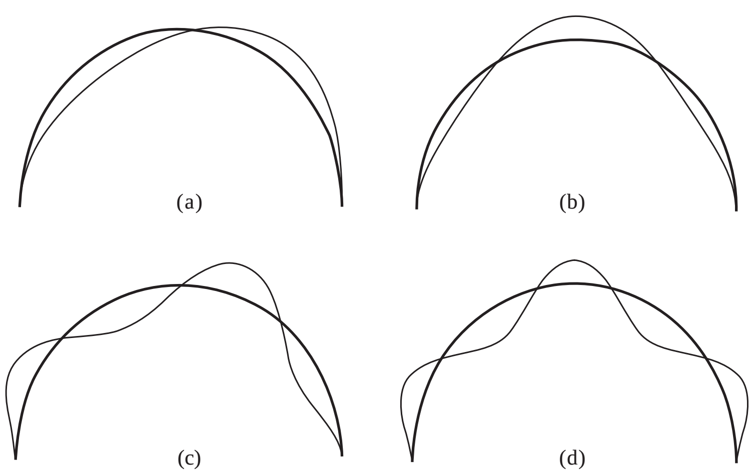

Figure 3. The lowest four modes of 180° circular arches with clamped ends.

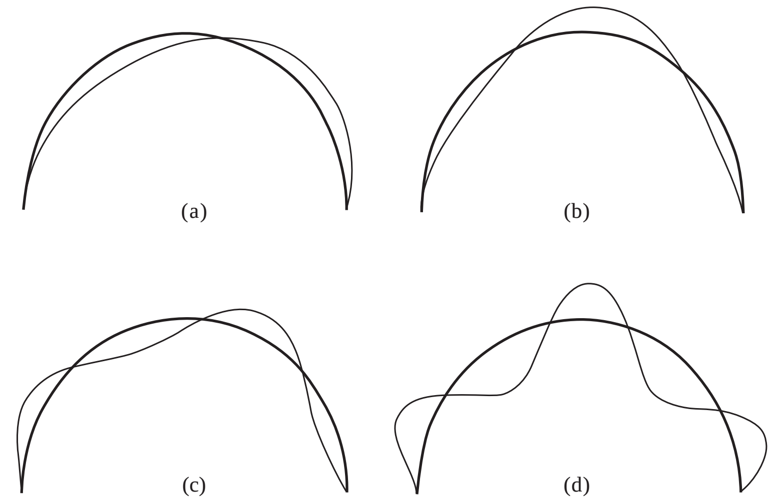

Figure 4. The lowest four modes of 180° circular arches with clamped and hinged ends.

where α_k $(k = 0, 1, \ldots, 5)$ are unknown coefficients and $\mathbf{I}$ is the unit matrix. The eigenvalues λ_i $(i = 1, 2, , \ldots, 6)$ of $\mathbf{A}$ also satisfy the same equation according to the Cayley–Hamilton theorem:

$$e^{\lambda_i \phi} = \alpha_0 + \alpha_1 \lambda_i + \alpha_2 \lambda_i^2 + \alpha_3 \lambda_i^3 + \alpha_4 \lambda_i^4 + \alpha_5 \lambda_i^5. \tag{7}$$

The last expression gives six simultaneous linear equations in terms of α_k. By solving these equations for α_k and by substituting them into equation (6), $e^{\mathbf{A}\phi}$ is found.

As known, the eigenvalues of $\mathbf{A}$ can be evaluated by setting $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. This leads to the polynomial characteristic equation

$$\lambda^6 + \left[2 + \beta\left(\frac{k_n}{GA} + \frac{2}{EA}\right)\right]\lambda^4 + \left[\left(1 - \frac{\beta k_n}{GA}\right)\left(1 - \frac{\beta}{EA}\right) + \frac{\beta}{EA}\left(2 + \frac{\beta k_n}{GA} + \frac{\beta}{EA}\right) - \frac{\beta\rho^2}{EI_b}\right]\lambda^2$$

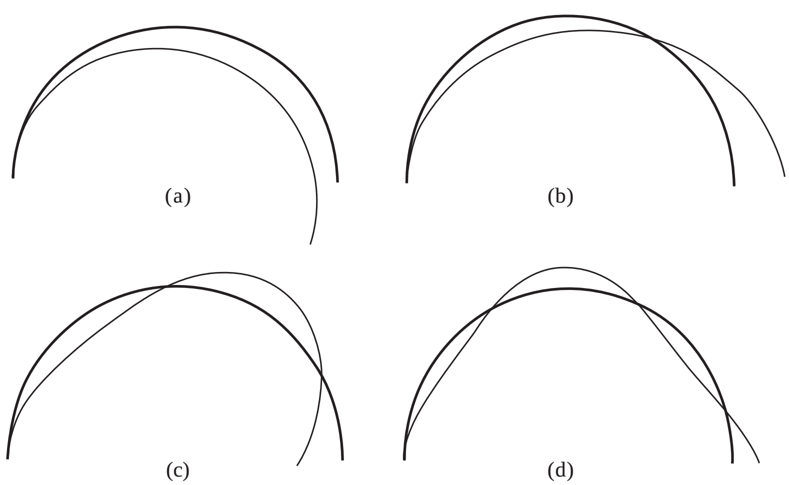

Figure 5. The lowest four modes of 180° circular arches with clamped and free ends.

TABLE 1

Frequency coefficient $c = \bar{\omega}R^2\theta^2\sqrt{\mu/EI}$ *for uniform fixed–fixed* 90° *circular arches*

R/i	Mode	Ref. [26]	(1)	(2)	(3)	(4)	(5)
100	1	55·74	55·73434	55·82526	55·3825	55·7845	55·3434
	2	103·6	103·5750	106·7298	102·5375	106·5514	102·3868
	3	191·9	191·8594	193·0367	189·0707	192·4118	188·4994
	4	220·3	220·3039	284·8169	219·4166	283·4113	219·1514
	5	305·1	305·1024	409·8742	300·4369	406·868	299·1958
75	1	55·66	55·66338	55·82526	55·04423	55·75308	55·97678
	2	100·3	100·3451	106·7298	98·73356	106·4131	98·5094
	3	175·7	175·6945	193·0358	175·1748	191·9284	174·9116
	4	190·9	190·8488	284·8349	186·0415	282·3079	185·1081
	5	296·2	296·1725	409·8074	286·8961	404·5627	284·7500
50	1	55·46	55·45927	55·82523	54·10526	55·6631	53·96596
	2	88·31	88·30372	106·7301	86·4506	106·0211	86·19077
	3	135·5	135·5172	193·0345	133·2986	190·5714	132·7272
	4	187·2	187·1628	284·8229	177·5005	279·2730	175·8392
	5	270·8	270·8472	409·824	269·6564	398·2587	265·8141

$$+\left(1-\frac{\beta}{EA}\right)\left[\rho^2+\left(1-\frac{\beta k_n}{GA}\right)\frac{I_n}{A}\right]\frac{\beta}{EI_b}=0. \tag{8}$$

where $\beta = \mu\omega^2\rho^2$. By introducing $\chi = \lambda^2$, equation (8) reduces to a cubic equation in terms of χ and the eigenvalues are found by means of Cardan formulas.

2.4. THE INITIAL VALUES AND BOUNDARY CONDITIONS

To specify the solution vector $\tilde{\mathbf{y}}(\phi)$ in equation (5), the initial values vector $\tilde{\mathbf{y}}(\phi_0)$ must be obtained as well as the term $e^{\mathbf{A}\phi}$. The six elements of it can be found by considering the following boundary conditions (e.g. for end B in Figure 1):

$$\text{Hinged end: } w(\phi_2)=0; \qquad u(\phi_2)=0; \qquad M_b(\phi_2)=0,$$

TABLE 2

Frequency coefficient $c = \bar{\omega}R^2\theta^2\sqrt{\mu/EI}$ *for uniform fixed–fixed* 120° *circular arches*

R/i	Mode	Ref. [26]	(1)	(2)	(3)	(4)	(5)
100	1	51·92	51·91417	51·96933	51·72205	51·9514	51·70454
	2	102·6	102·6437	103·5759	102·0216	103·4857	101·9366
	3	187·6	187·6238	188·3600	186·0336	188·0326	185·7236
	4	272·8	272·8042	281·3077	269·8413	280·5406	269·2141
	5	401·2	401·1827	404·9757	395·2590	403·2929	393·7767
75	1	51·87	51·87116	51·96936	51·5317	51·93761	51·50119
	2	101·9	101·8595	103·5761	100·7849	103·4153	100·6416
	3	187·0	187·0193	188·3598	184·2492	187·7782	183·7216
	4	257·9	257·9222	281·291	254·3053	279·9438	253·5605
	5	335·3	335·3554	404·9316	333·2775	402·0865	332·4988
50	1	51·75	51·74817	51·96935	50·99781	51·89786	50·93224
	2	99·31	99·30886	103·5760	97·12469	103·2155	96·85173
	3	185·1	185·1357	188·3591	179·2359	187·0569	178·1998
	4	199·7	199·7129	281·2906	198·4752	278·2554	198·0489
	5	296·0	296·0073	404·9476	285·3423	398·4590	282·9555

TABLE 3

Frequency coefficient $c = \bar{\omega} R^2 \theta^2 \sqrt{\mu/EI}$ *for uniform fixed–fixed* 150° *circular arches*

R/i	Mode	Ref. [26]	(1)	(2)	(3)	(4)	(5)
100	1	47·66	47·65777	47·69263	47·54103	47·68403	47·53256
	2	99·32	99·31750	99·71315	98·91841	99·66232	98·86906
	3	182·4	182·4143	182·9127	181·3956	182·7209	181·2108
	4	274·0	274·0332	276·8773	271·9680	276·4228	271·5375
	5	396·8	396·7676	399·2021	392·9392	398·2357	391·9823
75	1	47·63	47·63072	47·69263	47·42400	47·6773	47·40912
	2	99·0	99·00182	99·71302	98·30127	99·62241	98·21652
	3	182·4	182·0143	182·9129	180·2288	182·5711	179·9086
	4	274·0	271·1395	276·8839	267·6208	276·0648	266·9185
	5	394·6	394·5508	399·2365	387·9395	397·4276	386·3414
50	1	47·55	47·55337	47·69262	47·09346	47·65828	47·06123
	2	98·06	98·06049	99·71310	96·54452	99·51030	96·36843
	3	180·8	180·8237	182·9117	176·9458	182·1451	176·2864
	4	256·9	256·9427	276·8980	251·1634	275·0246	250·0629
	5	342·3	342·3105	399·2101	339·8953	395·2179	338·9705

$$\text{Clamped end: } w(\phi_2) = 0; \quad u(\phi_2) = 0; \quad \Omega_b(\phi_2) = 0,$$

$$\text{Free end: } M_b(\phi_2) = 0; \quad R_t(\phi_2) = 0; \quad R_n(\phi_2) = 0. \tag{9}$$

Substituting them (3 for each end) into equation (5) yields six simultaneous linear equations in terms of the initial values namely w_0, u_0, Ω_{b0}, M_{b0}, R_{t0} and R_{n0} at the reference co-ordinate ϕ_0. For a non-trivial solution, the determinant of the homogeneous system must vanish and this requirement will give the natural frequencies. Mode shapes are specified by substituting the normalized initial values into equation (5). Also one can apply the general method presented to some particular cases. That is, axial extension, shear or rotatory inertia effects can be disregarded by taking $\mathbf{A}(1,5) = 0$ or $\mathbf{A}(2,6) = 0$ or $\mathbf{A}(4,3) = 0$, respectively.

TABLE 4

Frequency coefficient $c = \bar{\omega} R^2 \theta^2 \sqrt{\mu/EI}$ *for uniform fixed–fixed* 180° *circular arches*

R/i	Mode	Ref. [26]	(1)	(2)	(3)	(4)	(5)
100	1	43·25	43·25054	43·27261	43·17525	43·26822	43·17091
	2	95·06	95·05805	95·26019	94·78901	95·23030	94·75567
	3	176·5	176·5296	176·8814	175·8285	176·7601	175·7111
	4	270·2	270·2342	271·6596	268·7898	271·3533	268·4875
	5	391·1	391·0999	392·7842	388·4248	392·1123	387·7377
75	1	43·24	43·23337	43·27257	43·09991	43·26476	43·09223
	2	95·90	94·89944	95·26026	94·41729	95·20651	94·36575
	3	176·3	176·2534	176·8816	175·0154	176·6657	174·8113
	4	269·0	269·0188	271·6630	266·4852	271·1114	265·9794
	5	389·7	389·6890	392·7656	384·9726	391·6286	383·9120
50	1	43·19	43·18431	43·27259	43·88654	43·25500	42·86968
	2	94·44	94·43909	95·26028	93·37788	95·13965	93·26808
	3	175·5	175·4451	176·8800	172·7292	176·3978	172·2951
	4	264·8	264·7849	271·6560	259·4751	270·4404	258·4766
	5	384·9	384·8703	392·7889	374·9177	390·1025	372·7893

TABLE 5

Frequency coefficient $c = \bar{\omega} R^2 \theta^2 \sqrt{\mu / EI}$ *for uniform hinged–hinged* 90° *circular arches*

R/i	Mode	Ref. [26]	(1)	(2)	(3)	(4)	(5)
100	1	33·92	33·91819	33·96056	33·85403	33·94027	33·83406
	2	79·16	79·15683	79·95256	78·83498	79·83553	78·72588
	3	151·5	151·4991	152·1705	150·4557	151·7202	150·0300
	4	216·5	216·4780	237·9761	215·2670	236·8765	214·8133
	5	261·7	261·79418	349·5550	260·5019	347·1565	259·7674
75	1	33·89	33·88529	33·96054	33·77174	33·92454	33·73672
	2	78·45	78·43807	79·95260	77·88450	79·74484	77·70250
	3	151·0	150·9628	152·1710	149·1410	151·3725	148·4183
	4	174·4	174·4427	237·9736	174·1365	236·0272	173·9414
	5	245·1	245·1234	349·5702	241·0977	345·2884	239·3448
50	1	33·79	33·79141	33·96053	33·53928	33·87970	33·46323
	2	75·74	75·73439	79·95263	74·66271	79·48746	74·34122
	3	122·0	122·0042	152·1706	121·7915	150·3921	121·4958
	4	149·3	149·3262	237·9724	145·4444	233·6575	144·0231
	5	238·9	238·8547	349·5635	229·7416	340·1951	226·3381

3. NUMERICAL EVALUATION AND CONCLUSIONS

The eigenvalues λ_i of the matrix **A** have been calculated by reducing equation (6) to Cardan form. In order to calculate the initial values, the reference co-ordinate ϕ_0 is set to be zero. That is $\phi_0 = \phi_C = 0$ (see Figure 1).

In Figures 2–5 are shown the lowest four modes of 180° circular arches with uniform rectangular cross-sections and having various end conditions.

Tables 1–8 list the values of the dimensionless frequency parameter $c = \bar{\omega} R^2 \theta^2 \sqrt{\mu / EI}$ for different slenderness ratios R/i where R is the radius of circular arch axis; i is the radius of gyration of the cross-sectional area; $\bar{\omega}$ is natural frequency and θ is the opening angle of the arch. In these tables, the values obtained by the present theory are compared to those obtained by Wolf [26] for five modes and for two different boundary

TABLE 6

Frequency coefficient $c = \bar{\omega} R^2 \theta^2 \sqrt{\mu / EI}$ *for uniform hinged–hinged* 120° *circular arches*

R/i	Mode	Ref. [26]	(1)	(2)	(3)	(4)	(5)
100	1	30·36	30·36033	30·38414	30·32607	30·37580	30·31780
	2	76·48	76·47634	76·74735	76·29374	76·68916	76·23734
	3	147·8	147·7448	148·1501	147·1565	147·9144	146·9290
	4	231·9	231·9183	234·5757	230·5152	233·9802	229·9762
	5	343·4	343·3509	345·5157	340·4258	344·1400	339·1900
75	1	30·34	30·34179	30·38417	30·28106	30·36934	30·26647
	2	76·27	76·25862	76·74730	75·93748	76·64407	75·83952
	3	147·4	147·4271	148·1494	146·3900	147·7319	145·9973
	4	228·5	228·4644	234·5747	226·1609	233·5163	225·3067
	5	322·7	322·7145	345·4731	322·3364	343·1240	321·9759
50	1	30·29	30·28895	30·38416	30·15343	30·35080	30·12124
	2	75·61	75·60040	76·74733	74·90006	76·51566	74·69487
	3	146·5	146·4927	148·1494	144·2301	147·2168	143·4124
	4	199·3	199·3012	234·5716	197·8177	232·2173	197·2652
	5	248·1	248·1072	345·4819	244·1621	340·2842	242·4045

TABLE 7

Frequency coefficient $c = \bar{\omega}R^2\theta^2\sqrt{\mu/EI}$ *for uniform hinged–hinged* 150° *circular arches*

R/i	Mode	Ref. [26]	(1)	(2)	(3)	(4)	(5)
100	1	26·43	26·43164	26·44557	26·41157	26·44185	26·40786
	2	72·71	72·70510	72·83194	72·5900	72·80021	72·55874
	3	143·1	143·1013	143·3676	142·7257	143·2317	142·5925
	4	229·2	229·1988	230·3020	228·2831	29·9415	227·9351
	5	339·2	339·1714	340·5330	337·2530	339·7403	336·4950
75	1	26·42	26·42079	26·44557	26·38523	26·43896	26·37866
	2	72·62	72·60569	72·83188	72·40197	72·77554	72·34726
	3	142·9	142·8942	143·3665	142·2300	143·1256	141·9974
	4	228·3	228·2147	230·3036	226·6178	229·6597	226·0291
	5	338·1	338·0243	340·5310	334·7154	339·1035	333·3854
50	1	26·39	26·38991	26·44555	26·31025	26·43072	26·29575
	2	72·33	72·31761	72·83196	71·86615	72·70526	71·74772
	3	142·3	142·2910	143·3665	140·8289	142·8241	140·3306
	4	224·5	224·4704	230·3005	221·1423	228·8664	219·9901
	5	333·9	333·9176	340·5504	327·1570	337·3432	324·5391

conditions. The numbered columns contain the frequency parameters obtained by the present theory for the following cases:

(1) Axial extension and rotatory inertia effects are considered as in [26].
(2) None of the effects are considered.
(3) Transverse shear and axial extension effects are considered.
(4) Only the rotatory inertia effect is considered.
(5) All of the effects are considered (primary case studied here).

Wolf has found good results by using the finite element method. Although the axial extension and rotatory inertia have been included, the shear deformation has been

TABLE 8

Frequency coefficient $c = \bar{\omega}R^2\theta^2\sqrt{\mu/EI}$ *for uniform hinged–hinged* 180° *circular arches*

R/i	Mode	Ref. [26]	(1)	(2)	(3)	(4)	(5)
100	1	22·37	22·36366	22·37183	22·35144	22·37009	22·34968
	2	68·27	68·26015	68·33013	68·18234	68·31198	68·16436
	3	137·8	137·7702	137·9536	137·5114	137·8689	137·4288
	4	224·6	224·6107	225·2239	223·9718	224·9902	223·7427
	5	334·0	333·9144	334·8891	332·6120	334·3664	332·0705
75	1	22·36	22·35735	22·37186	22·33560	22·36873	22·33250
	2	68·22	68·20545	68·33013	68·06770	68·29790	68·03600
	3	137·6	137·6271	137·9536	137·1696	137·8031	137·0236
	4	224·1	224·1099	225·2238	222·9893	224·8065	222·5925
	5	333·2	333·1781	334·9025	330·8272	333·9380	329·9577
50	1	22·34	22·33921	22·37183	22·29052	22·36482	22·28359
	2	68·06	68·04909	68·33021	67·74172	68·25748	67·67219
	3	137·2	137·2153	137·9534	136·2016	137·6155	135·8837
	4	222·6	222·5628	225·2190	220·1100	224·2858	219·2887
	5	330·9	330·8144	334·9015	325·7584	332·7567	323·9065

TABLE 9

Comparison of the results of [17] *and* [27] *with those obtained by the present theory*

	Ref. [27]	Ref. [17] (rad/s)				Present theory (rad/s)			
θ (°)	$\bar{\omega}_1$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$
20	5371·1	2830·2	5248·3	11642·0	20149·0	2827·48	5246·73	11587·2	19825·8
40	1312·3	1304·6	2309·3	3276·7	5430·1	1304·16	2299·65	3253·90	5245·07
60	561·76	560·24	1246·1	2456·7	2642·4	560·074	1226·61	2339·14	2627·75
90	229·95	229·77	553·63	1102·3	1786·7	229·591	538·332	1024·89	1583·23
120	115·74	115·64	307·39	630·97	1060·8	115·609	291·549	562·463	887·370
150	64·481	64·439	196·66	422·55	732·19	64·4161	177·256	348·716	559·208
180	37·885	37·865	139·94	318·18	568·26	37·8485	115·541	233·152	380·269
210	22·784	22·774	108·63	261·71	480·28	22·7629	78·7429	163·964	272·348
240	13·671	13·667	90·374	229·03	429·69	13·6592	55·2520	119·464	202·425
270	7·9259	7·9246	79·112	208·63	398·21	7·92040	39·5021	89·2883	154·649
300	4·1861	4·1855	71·681	194·78	376·82	4·18396	28·5533	67·9828	120·633
330	1·6923	1·6922	66·393	184·54	360·90	1·69177	20·7348	52·4561	95·6193

TABLE 10

Comparison of the results of [28] *with those obtained by the present theory*

Theoretical [28] (rad/s)	Experimental [28] (rad/s)	Present theory (rad/s)
44·3	41	40·71

neglected in [26]. The tables also give the results of the various combinations of the effects in case they are considered by the present theory.

It is readily seen that as R/i or θ decreases, the frequency parameters become progressively less accurate in the columns where some or all of the effects have been neglected. This is more important in higher modes. The tables also show that the results are slightly influenced by the boundary conditions and that the neglect of these effects is to be avoided specially for the beams with clamped ends.

Table 9 shows the natural frequencies obtained by Den Hartog [27] and Heppler [17] in comparison to those obtained by the present theory. Den Hartog's results are less accurate for small opening angles due to the neglect of transverse shear and rotatory inertia as explained in reference [17]. But, it is interesting that considerable differences arise between the results of Heppler and that of the present theory at large opening angles and in higher modes, although all the effects are taken into account in [17]. This can be attributed to taking a constant number (eight) of straight beam elements by Heppler. It is obvious that increasing angles result in longer elements when the number of elements are kept constant and this increases the errors specially in higher modes.

Tabarrok *et al*. [28] investigated a semi-circular cantilever arch with a uniform rectangular cross-section by employing a seven element model. The shear, stretching and rotatory inertia effects are taken into account. Table 10 shows their theoretical and experimental results for the first mode with those obtained by the present theory. As can be seen, the agreement between the computed and measured values is better when the present theory is of concern even for the first mode. It is expected that the discrepancy originated from seven element model will be worse in ascending modes.

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