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LETTERS TO THE EDITOR

APPLICATION OF SHOHAT EXPANSION IN PERIODIC SOLUTION OF DIFFERENTIAL EQUATIONS

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Shohat [1] and Bellman [2] considered a general non-linear differential equation of the form

$$\ddot{y} + y = \varepsilon f(y, \dot{y}), \qquad 0 < \varepsilon \leqslant 1, \tag{1}$$

where ε is a small positive parameter, f is, in general, a polynomial function of its arguments and over-dots denote differentiation with respect to time t. Their perturbation method is based on using the new expansion variable ρ, where

$$\rho = \frac{\varepsilon}{1+\varepsilon} \quad \text{or} \quad \varepsilon = \frac{\rho}{1-\rho}, \tag{2}$$

and in assuming the angular frequency and the solution in the form

$$\varepsilon\omega(\varepsilon) = \rho + C_2\rho^2 + C_3\rho^3 + \cdots, \tag{3}$$

$$y(\theta, \rho) = y_0(\theta) + \rho y_1(\theta) + \rho^2 y_2(\theta) + \cdots, \tag{4}$$

$$\text{where} \quad \theta = \omega t. \tag{5}$$

The C_i's are determined in such a way that no secular term appears in the periodic solution. It is known that the series solution converges faster as the perturbation parameter becomes smaller. When ε is small, ρ is still smaller. Even if ε is large, $\rho < 1$. This may be the idea in introducting ρ. Using the above explained perturbation scheme, they examined the periodic solution of the van der Pol equation and found that their results were accurate for $0 < \varepsilon \leqslant 10$ and claimed that the Shohat expansion works well for all positive values of ε. Bellman [2] applied the Shohat expansion for the Duffing equation and presented a five-term solution but did not examine the accuracy of the solution for large values of ε. Mickens [3] has recently discussed the solution of the Duffing equation and found that the Shohat expansion offers no real advantages over the standard perturbation procedures and, in general, the angular frequency relation of equation (3) does not hold for large values of ε. It is of interest to understand the cause of such a discrepancy arising in the case of the Duffing equation. A careful examination of the van der Pol equation reveals that the angular frequency, ω, decreases with ε (softening type) whereas it increases with ε for the Duffing equation (hardening type). Now the angular frequency relation of equation (3) considered by them can be expressed in the form

$$\omega(\varepsilon) = (1-\rho)[1 + C_2\rho + C_3\rho^2 + \cdots] \tag{6}$$

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For large values of ε, $\rho \to 1$ which implies $\omega(\varepsilon) \to 0$. This explains the success of the Shohat expansion for the van der Pol equation. The Duffing equation,

$$\ddot{y} + y + \varepsilon y^3 = 0, \tag{7}$$

$$y(0) = A, \qquad \dot{y}(0) = 0, \tag{8}$$

considered by Mickens [3] is examined here for arbitrary positive values of ε. Introducing $y = \sqrt{\delta/\varepsilon}x$ and $\theta = \omega t$ in equations (7) and (8), one obtains

$$\omega^2 x'' + x + \delta x^3 = 0, \tag{9}$$

$$x(0) = \sqrt{\varepsilon/\delta}A, \qquad x'(0) = 0, \tag{10}$$

where δ is an arbitrarily small parameter and primes denote differentiation with respect to θ.

Defining

$$\rho = \frac{\delta}{1+\delta}, \qquad x = x_0 + \rho x_1, \qquad \omega^2 = 1 + \rho\omega_1, \tag{11, 12, 13}$$

and writing the initial conditions (10) for x_0 and x_1 as

$$x_0(0) = \sqrt{\varepsilon/\delta}A, \qquad x_0'(0) = 0 = x_1(0) = x_1'(0), \tag{14}$$

one can obtain the solution for equation (9) as

$$x = a\cos\theta + \frac{a^3}{32}(\cos 3\theta - \cos\theta)\rho, \qquad \omega^2 = 1 + 3/4a^2\rho, \tag{15, 16}$$

where $a = \sqrt{\varepsilon/\delta}A$.

The periodic solution of equations (7) and (8) can be written as

$$y(\theta) = \sqrt{\delta/\varepsilon}x = A\cos\theta + \frac{A^3}{32}\frac{\varepsilon}{\delta}(\cos 3\theta - \cos\theta)\rho$$

$$\cong \mathrm{A}\cos\theta + \frac{A^3}{32}\varepsilon(\cos 3\theta - \cos\theta) \tag{17}$$

$$\omega^2 = 1 + \frac{3}{4}\frac{\varepsilon}{1+\delta}A^2 \cong 1 + \frac{3}{4}\varepsilon A^2 \tag{18}$$

Since δ is arbitrarily small, $(1 + \delta)$ is assumed as 1, in the above equations (17) and (18).

TABLE 1

Comparison of angular frequency ω, for a Duffing equation

εA^2	Exact integration	Bellman [2]	Present study equation (18)
0·001	1·00037	1·00037	1·00037
0·01	1·00374	1·00374	1·00374
0·1	1·03672	1·03670	1·03682
1	1·31778	1·23905	1·32288
10	2·86665	0·59382	2·91548
100	8·53363	0·07837	8·71780
1000	26·81090	0·00808	27·40438

Mickens [3] uses the initial condition $y(0) = A$ as oppose to $y(0) = 1$ by Bellman [2]. Normalizing y by using the maximum amplitude A, the parameter in equation (7) becomes εA^2. This transformed equation is then in the form of Bellman's equation with εA^2 being replaced by ε. Though the solution of the Duffing equations (7) and (8) is a function of ε and A, the angular frequency ω depends on εA^2. The angular frequency ω, is computed for various values of εA^2 in Table 1 using Bellman's Shohat expansion and exact integration. The Shohat expansion used by Bellman for the Duffing equation requires a large number of terms for large values of ε to improve the accuracy in comparison with the exact integration, whereas the present solution agrees well with the exact integration with two terms because ρ becomes arbitrarily small as δ does so. The failure of the Shohat expansion with the parameter ε in the case of the Duffing equation, can be attributed mainly to the hardening type of behaviour of the solution i.e., ω increases with εA^2. It is successful in the case of the van der Pol equation because of the softening type of behaviour.

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