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DETECTING STRUCTURAL DAMAGE USING FREQUENCY RESPONSE FUNCTIONS

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1. INTRODUCTION

There are many techniques under investigation to use measured vibration data to detect damage in structures [1–3]. These techniques need a large number of sensors to locate damage, and the structural response characteristics vary with small changes in boundary conditions and the environment. In this condensed paper, optimization of Frequency Response Functions (FRFs) is investigated as a means to diagnose damage using a minimum number of sensors.

2. DAMAGE DETECTION TECHNIQUE

The damage detection procedure assumes that a finite element model of the structure exists that has been correlated to match the frequency response of the actual undamaged structure in the lower frequency range. A minimum number of sensors will be located at critical points on the actual structure to measure the vibration response to a known low level excitation. The receptance FRF is then computed and assigned to the finite element model of the undamaged structure to locate the damage. The finite element model in the damaged condition will then be used to predict the remaining strength and safety of the structure.

The linear equations that describe the vibration of the undamaged structure subjected to a harmonic force input are

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathrm{Real}\,[\mathbf{p}_o \exp(i\omega t)], \tag{1}$$

where $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are the mass, damping, and stiffness matrices, initially of the undamaged model, $\mathbf{x}$ is the displacement vector, ω is the excitation frequency, $\mathbf{p}_o$ is a constant and possibly complex forcing vector, t is time, and $i = \sqrt{-1}$. Bold notation is used to denote matrices and vectors. The structural matrices are symmetric, and often $\mathbf{M}$ is diagonal. The $\mathbf{D}$ matrix is assumed proportional to $\mathbf{M}$ and $\mathbf{K}$. As a particular solution to equation (1), let

$$\mathbf{x}(t) = \mathrm{Real}\,[\mathbf{q}_o \exp(i\omega t)], \tag{2}$$

where $\mathbf{q}_o$ is a complex receptance frequency response vector. Substituting equation (2) into equation (1) gives

$$\mathrm{Real}\,[((\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})\mathbf{q}_o - \mathbf{p}_o)\exp(i\omega t)] = 0. \tag{3}$$

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A solution to equation (3) is

$$(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})\mathbf{q}_o - \mathbf{p}_o = 0. \tag{4}$$

In equation (4) it is assumed that the structural matrices are known approximately from a finite-element model, the force $\mathbf{p}_o$ is known, and that $\mathbf{q}_o$ is measured from the undamaged structure. Then equation (4) can be used in a reverse procedure to adjust or identify the system matrices to produce the measured FRFs in the healthy condition. After the model is correlated with the healthy response, we assume the actual structure sustains damage and measure the frequency response vector of the damaged structure, denoted $\mathbf{q}_d$. Using $\mathbf{q}_d$ in equation (4), which contains the identified healthy system matrices, then gives

$$\mathbf{d}(i\omega) = (\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})\mathbf{q}_d - \mathbf{p}_o, \tag{5}$$

where $\mathbf{d}(i\omega)$ is a receptance residual vector that identifies the d.o.f. at which damage occurs. If the full $\mathbf{q}_d$ vector is known at some frequency at which it is affected by the damage, then the non-zero entries of the vector $\mathbf{d}(i\omega)$ will identify the d.o.f. at which damage has occurred. The problem with this approach is that it is usually not possible to measure a full frequency response vector, that is, have a sensor at each d.o.f. in the model. An alternative approach is to perform a co-ordinate reduction on equation (5) and then make the resulting equation equal to zero by adjusting the $\mathbf{D}$ and $\mathbf{K}$ matrices to represent the damaged structure. In that way, the matrices of the healthy versus damaged structures can be compared to diagnose the damage.

This approach is performed by partitioning $\mathbf{q}_d$ as $\mathbf{q}_d = [\mathbf{q}_1 \;\; \mathbf{q}_2]^T$ where $\mathbf{q}_1$ denotes d.o.f. on the damaged structure where the frequency response is measured, and $\mathbf{q}_2$ denotes d.o.f. where the frequency response is not measured. The load vector is partitioned similarly. Define $\mathbf{H}$ as the frequency response matrix of the system. Then the system matrix $\mathbf{A}$ is defined as $\mathbf{A} = \mathbf{H}^{-1} = (\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D})$, which in partitioned form is

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} (\mathbf{K}_{11} - \omega^2\mathbf{M}_{11} + i\omega\mathbf{D}_{11}) & (\mathbf{K}_{12} - \omega^2\mathbf{M}_{12} + i\omega\mathbf{D}_{12}) \\ (\mathbf{K}_{21} - \omega^2\mathbf{M}_{21} + i\omega\mathbf{D}_{21}) & (\mathbf{K}_{22} - \omega^2\mathbf{M}_{22} + i\omega\mathbf{D}_{22}) \end{bmatrix}. \tag{6}$$

Rewriting equation (4) using $\mathbf{q}_d$ and equation (6) gives

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}. \tag{7}$$

Eliminating the $\mathbf{q}_2$ co-ordinates from equations (7) gives

$$(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})\mathbf{q}_1 - (\mathbf{p}_1 - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{p}_2) = \mathbf{0}. \tag{8}$$

Equation (8) will not equal zero unless the $\mathbf{A}$ matrices represent the damaged system, and thus the left hand side of equation (8) is a measure of the error between the system matrices of the undamaged and damaged systems. Therefore, the basis of the damage detection procedure is to update the $\mathbf{A}$ matrices in equation (8) to represent the damaged system by using an optimizer to adjust the properties of the finite elements. This approach is called frequency response function assignment, as the model parameters are being adjusted to assign the $\mathbf{q}_1$ vector as the frequency response of the system. It should be understood that although a dynamic reduction has been used to obtain equation (8), the $\mathbf{A}$ matrix is adjusted by the optimization procedure to exactly equal the damaged system. Thus there are no mathematical approximations in this procedure, assuming the analytical model represents the actual structural response. This is in contrast to damage detection methods that use dynamic expansion or Guyan reduction of the undamaged system. In reference

[4] it is shown that expansion or reduction of the undamaged system always introduces errors into the damage prediction whenever there is damage in the section of the model being reduced, i.e., the $\mathbf{A}_{22}$ matrix in equation (6).

Inverting the $\mathbf{A}_{22}$ matrix in equation (8) versus frequency in an optimization algorithm is computationally very expensive for large order systems. This limitation can be lessened by restricting the $\mathbf{D}$ matrix to have the same connectivity and symmetry as the $\mathbf{K}$ matrix. This restriction makes sense on a physical basis because material dependent damping will have the same connectivity as the stiffness matrix. From equation (6) we can write

$$\mathbf{A}_{22}(\omega) = (\mathbf{K}_{22} - \omega^2\mathbf{M}_{22} + i\omega\mathbf{D}_{22}). \tag{9}$$

Equation (9) shows that for a typical finite-element model the $\mathbf{A}_{22}$ matrix is sparse, symmetric, and has the same banding as the $\mathbf{K}$ matrix. Thus, its inverse can be found efficiently using a sparse matrix solver. A reduction in the number of design variables in the optimization can be obtained by assuming proportional damping, that is, $\mathbf{D} = (c_1\mathbf{K} + c_2\mathbf{M})$, where the c values are constants to be optimized. Computational requirements of the technique can also be reduced by carefully building the finite element model to minimize the bandwidth and number of non-zero entries in the stiffness matrix. When selecting the d.o.f. to measure the frequency response of the damaged structure, it is helpful to look for the columns of the stiffness matrix with the least number of zeros, that is, the greatest connectivity. This will provide the greatest accuracy in identifying the damage and place the maximum number of zero entries in the $\mathbf{A}_{22}$ matrix, which will speed-up computations.

An objective function that can be minimized to assign the $\mathbf{q}_1(\omega)$ frequency response vector from the damaged model to the undamaged system is formed using equation (8)

$$J = \sum_{r=1}^{nf} \mathbf{E}_r^{*\mathrm{T}}\mathbf{Q}\mathbf{E}_r, \tag{10}$$

where T denotes matrix transpose, * denotes complex conjugate, $\mathbf{Q}$ is a real diagonal weighting matrix, nf is the number of frequency points used to define the frequency response curves, and $\mathbf{E}_r$ is the error term given by

$$\mathbf{E}_r = [(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})\mathbf{q}_1 - (\mathbf{p}_1 - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{p}_2)]_{\omega=\omega_r}. \tag{11}$$

Typically, $\mathbf{p}_2$ is a zero vector, and $\mathbf{p}_1$ will contain ones and zeros, and this reduces computations.

When selecting the design variables to update the $\mathbf{A}$ matrices, the connectivity and symmetry of the FEM matrices can be maintained using an elemental proportional update technique. In this approach, scale factors are defined for each finite-element in the model, and the scale factors become the design variables in the optimization. The elemental matrices with updated scale factors are assembled to obtain the global matrices. This method exactly preserves the connectivity of the system matrices but assumes that proportional damage occurs to the elements, which means that bending stiffness and extensional stiffness of the element are dependent. Alternatively, separate factors could be used to represent the individual stiffness terms. The equations for the elemental damage update approach are

$$\bar{\mathbf{D}} = \sum_{r=1}^{L} d_r^d\mathbf{D}_r, \qquad \bar{\mathbf{K}} = \sum_{r=1}^{L} d_r^k\mathbf{K}_r, \tag{12a, b}$$

where the summation denotes matrix assembly of the elemental matrices $\mathbf{K}_r$ and $\mathbf{D}_r$, multiplied by the scale factors d_r^d and d_r^k, and where L is the total number of finite elements in the model. In this paper, changes in the mass matrix due to damage are assumed to be negligible and the mass matrix is not adjusted. Note that the scale factors for the healthy model are all equal to one. A design vector is set up with these scale factors, and the symmetry and connectivity of the model is always automatically retained. A computer algorithm to test the technique was developed using the MATLAB [5, 6] software system and the optimization step is performed using the CONSTR subroutine contained in the MATLAB optimization toolbox. The only constraints on the design variables are bounds on their magnitudes to ensure that stiffness and damping values cannot decrease below zero, and that stiffness values cannot increase from the undamaged model. Note that in some cases damping values could increase due to damage. The bounds on the design variables are used to help locate damage when a minimum number of sensors are used on the structure. The design variable bounds are

$$\xi_j^L \leqslant \xi_j \leqslant \xi_j^U, \qquad j = 1, 2, \ldots, ndv, \tag{13}$$

where L and U represent the lower and upper bounds, and ndv is the total number of design variables in the problem. No functional constraints are needed with this formulation. A final requirement to make the optimization computationally feasible is to derive a closed form gradient of the objective function. The gradient of equation (10) can be obtained exactly without any additional function evaluations or matrix inversions and is

$$\frac{\mathrm{d}J}{\mathrm{d}\xi_j} = \sum_{r=1}^{nf} 2\,\mathrm{Re}\,(s_r), \tag{14}$$

where

$$s_r = \mathbf{E}_r^{*\mathrm{T}} \mathbf{Q} \frac{\mathrm{d}\mathbf{E}_r}{\mathrm{d}\xi_j}, \qquad \frac{\mathrm{d}\mathbf{E}_r^{*\mathrm{T}}}{\mathrm{d}\xi_j} = \left(\frac{\mathrm{d}\mathbf{E}_r}{\mathrm{d}\xi_j}\right)^{*\mathrm{T}}.$$

The term $\mathrm{d}\mathbf{E}_r/\mathrm{d}\xi_j$ only involves matrix addition and multiplication to compute and is

$$\frac{\partial \mathbf{E}_r}{\partial \xi_i} = \left[\begin{array}{l} \left(\dfrac{\partial \mathbf{A}_{11}}{\partial \xi_j} - \dfrac{\partial \mathbf{A}_{12}}{\partial \xi_j} \mathbf{A}_{22}^{-1}\mathbf{A}_{21} - \mathbf{A}_{12}\left(\dfrac{\partial \mathbf{A}_{22}^{-1}}{\partial \xi_j} \mathbf{A}_{21} + \mathbf{A}_{22}^{-1} \dfrac{\partial \mathbf{A}_{21}}{\partial \xi_j}\right)\right)\mathbf{q}_1 + \\ \left(\dfrac{\partial \mathbf{A}_{12}}{\partial \xi_j} \mathbf{A}_{22}^{-1} + \mathbf{A}_{12} \dfrac{\partial \mathbf{A}_{22}^{-1}}{\partial \xi_j}\right)\mathbf{p}_2 \end{array}\right]_{\omega = \omega_r}, \tag{15}$$

where

$$\frac{\partial \mathbf{A}_{\alpha\beta}}{\partial \xi_j} = \frac{\partial(\mathbf{K}_{\alpha\beta} - \omega^2 \mathbf{M}_{\alpha\beta} + i\omega \mathbf{D}_{\alpha\beta})}{\partial \xi_j}, \qquad \alpha, \beta = 1 \text{ or } 2, \frac{\partial \mathbf{A}_{22}^{-1}}{\partial \xi_j} = -\mathbf{A}_{22}^{-1} \frac{\partial \mathbf{A}_{22}}{\partial \xi_j} \mathbf{A}_{22}^{-1}.$$

The gradients of the system matrices, $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{K}$, with respect to the design variables are simply matrices with all zeros, except for the elemental stiffness matrix assembled into the global stiffness matrix in the position corresponding to that particular finite-element. Moreover, the second order form of the system equations is used in which matrix sparsity is preserved, and the MATLAB sparse matrix functions take advantage of this to further streamline computations.

Damage is identified by changes in optimized scale factors that multiply the elemental matrices used to assemble the finite-element model. Scale factors different than one indicate damage to that particular finite-element. The accuracy in which damage can be located depends on: (1) the detail of the analytical model, (2) the repeatability of the frequency response data taken from the damaged structure, and (3) the success of the optimizer in finding the global minimum solution. The repeatability of FRF data is discussed in references [7, 8].

3. EXAMPLE PROBLEM: DAMAGE DIAGNOSIS ON A TRUSS BRIDGE

This example assigns a simulated measured frequency response to detect damage on the 18-member simply supported truss bridge shown in Figure 1. The node and element numbers are also given in Figure 1. A sinusoidal force $f(t) = 0{\cdot}01 \sin(\omega t)$ acts in the $+$X, $-$Y and $-$X, $-$Y axis at nodes 3 and 5, respectively (d.o.f. 5, 6, 9, 10 of the analytical model). These forces represent a Y (90°) pre-tensioned cable with a single force actuator to excite the bridge. The bridge is constructed of tension-compression truss elements. The vector of displacements is $\mathbf{x}(t) = [x1\ \ y1\ \ x2\ \ y2\ \ x3\ \cdots\ x16\ \ y16]^T$. The mass matrix contains non-structural mass at each bottom node of the bridge. Proportional damping ($\mathbf{D} = 0{\cdot}0005\mathbf{K}$) is assumed for both the healthy and damaged models. Sixteen scale factors d_r^k, that multiply the elemental stiffness matrices, $\mathbf{K}_r$, are the design variables. The elemental stiffness matrices are assembled into the global stiffness matrix according to equation (12). Rows and columns are switched in the system $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{K}$ matrices to put the assigned d.o.f. in the $\mathbf{A}_{11}$ position. The structural connectivity information used to assemble equation (12) is given in reference [9]. The first undamped natural frequency in the healthy model is 0·516 Hz. The first case is 50% damage (reduction in stiffness and damping) to element 6 and this reduces the first frequency of the bridge to 0·473 Hz or 8·2%. Various damage cases were run in which FRFs were assigned from various nodes in the X and/or Y directions, and the technique exactly corrected the stiffness matrix. A summary of all the cases studied is given in Table 1.

The damage analysis was run using nine frequency points spaced at 5 rad/s intervals starting at zero, or some variation of this (e.g., seven points at 6 rad/s intervals), until $J = 0$. Different frequency points were tried because sometimes the optimization did not go to zero. The damage diagnosis was assumed to be correct when the objective function equaled zero within the precision of the machine calculations. When this occurred the damage diagnosis for all the cases in Table 1 was correct. This result indicates that a more robust optimization method or improved objective function could make the procedure quicker. Note in Table 1 that damage was diagnosed in some cases using only one

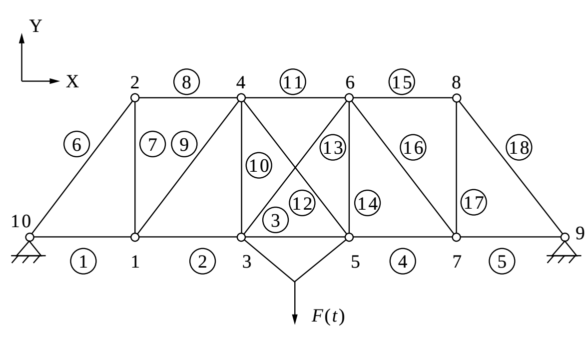

Figure 1. Finite element model of a truss bridge with excitation force.

TABLE 1

Summary of damage detection study with no noise in measurements

Single damage	d.o.f. assigned	Two damages	d.o.f. assigned
E1 = 50%	5, 6, 9, 10	**E**1 = 50%, **E**6 = 50%	3, 4, 5, 6, 9, 10, 15, 16
E1 = 50%	1, 2	**E**1 = 50%, **E**6 = 50%	4, 5, 6, 9, 10, 16
E1 = 50%	6, 10	**E**1 = 50%, **E**6 = 50%	5, 6, 9, 10
E6 = 50%	4, 5, 6, 9, 10, 16	**E**1 = 50%, **E**6 = 50%	4, 6, 10, 16
E6 = 50%	4, 6, 10, 16	**E**1 = 50%, **E**6 = 50%	6, 10
E6 = 50%	5, 6, 9, 10	**E**1 = 25%, **E**6 = 25%	5, 6, 9, 10
E6 = 50%	6, 10	**E**1 = 25%, **E**6 = 25%	6, 10
E6 = 50%	10	**E**1 = 1%, **E**6 = 1%	10
E6 = 5%	10	—	—
E11 = 50%	6, 10	—	—
E14 = 50%	6, 10	—	—

measurement d.o.f. However, in these cases, usually convergence of the optimization was slow and more than one trial was required. The FRFs for nodes 1, 3, 5, 7 in the Y-axis of the bridge are shown in Figure 2. Note that the healthy FRFs of nodes 1 and 7, and 3 and 5 are the same due to symmetry. However, the nonsymmetric damage destroys the symmetry in the damaged FRFs. From this example, it is concluded that the optimization

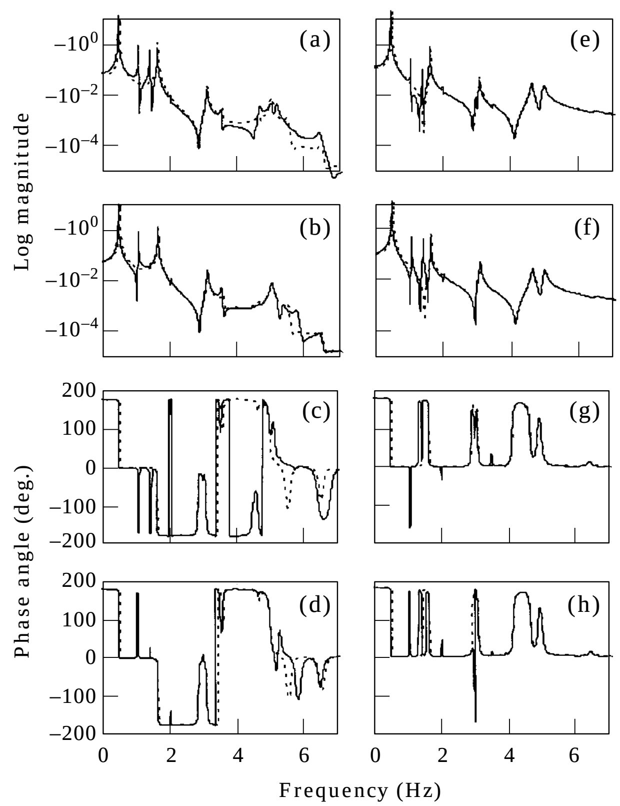

Figure 2. Frequency response magnitude and phase angle for bridge truss (healthy = dashed line, damaged (50% to element 6) = solid line) for (a) and (c) Node 1, (b) and (d) Node 7, (e) and (g) Node 3, (f) and (h) Node 5, all Y-axis.

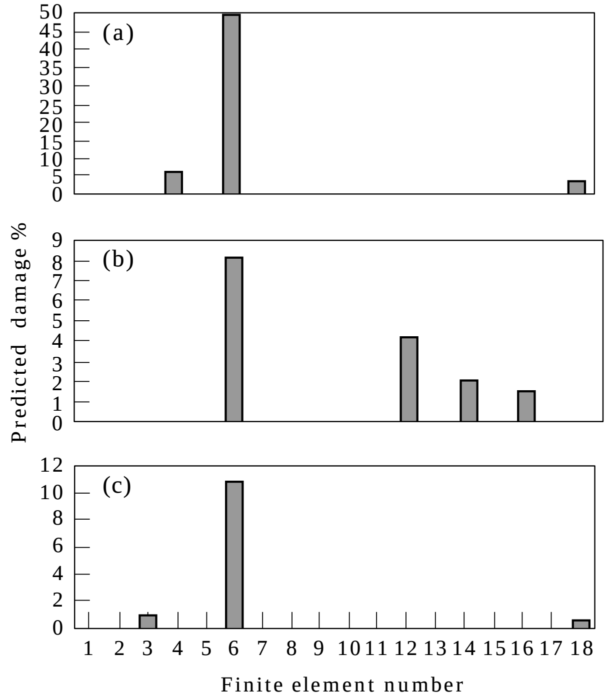

Figure 3. Damage prediction for bridge truss, damage to element 6, 5% noise (FRFs assigned at d.o.f. 6 and 10, at 23 frequency points); (a) 50% damage, (b) 10% damage, (c) 5% damage.

works most accurately by assigning as many d.o.f. as possible, using the smallest number of frequency points possible, and by concentrating on the areas where the FRFs have the greatest difference between the healthy and damaged cases. An algorithm could be developed to select the frequency points to be used in the optimization.

3.1. *Effect of measurement noise on damage detection*

A 5% random noise is added to the simulated FRFs to represent errors in measured data, and the damage detection problem was rerun for various cases of damage. The noise is uniformly distributed, with the mean = 0 and variance = 1, and is added directly to each FRF point as

$$\bar{q}_j(i\omega) = q_j(i\omega)\left(1 + \frac{0{\cdot}05}{n}\sum_{k=1}^{n} randn_k\right), \tag{16}$$

where $\bar{q}_j(i\omega)$ is the jth element of the FRF vector $\mathbf{q}_1$, and *randn* is the random noise generator function in MATLAB. Two noise conditions are considered; noise with no averaging of the data ($n = 1$), and noise with 5 averages of the simulated measured data ($n = 5$).

Initially three cases of 5% noise, no damage, and no averaging were run and the algorithm incorrectly predicted damage levels from 2 to 4%. In Figure 3, different levels of damage are applied to element 6 with 5% noise and no averaging and the algorithm closely predicted the damage in element 6, but put a small level of incorrect damage at other elements. In Figure 4, a damage level of 5% is applied to different elements with 5% noise and no averaging and the algorithm predicted the location and approximate

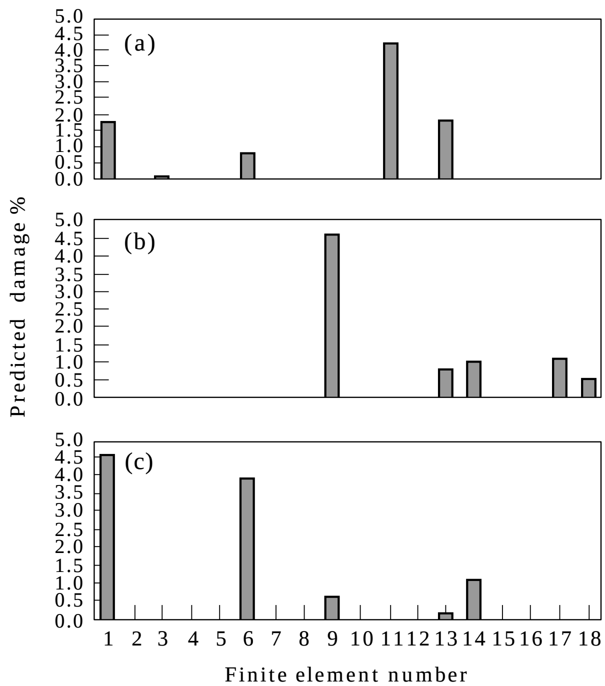

Figure 4. Damage prediction for bridge truss, damage to various elements, 5% noise (FRFs assigned at d.o.f. 6 and 10, at 23 frequency points); 5% damage to (a) element 11, (b) element 9, (c) elements 1 and 6.

damage level of the damage elements, but also put a small level of incorrect damage at other elements. In Figures 3 and 4, the best results (i.e., the result with the smallest J value) from a few trials are presented, where different random errors are used in each run. Finally, the experiment with no damage and 5% noise was repeated, except here five averages of FRF data are taken and then the damage detection algorithm is run. This simulation shows that the predicted damage due to noise is reduced from 2–4% to 0.3–1.5% by averaging.

In the above examples, the damage detection technique fairly well discriminates between noise and small levels (5%) of damage. With noise, the damage is still located but the accuracy of the damage diagnosis is reduced. The 5% damage produces only a small change in the FRFs that is mostly away from the resonant points. This damage would be difficult to detect using most damage detection methods. Overall, the results with noise show that using more frequency points improves accuracy, but makes convergence of the damage algorithm more difficult. Averaging measurements was the most effective way to reduce random measurement error.

4. CONCLUSIONS

The simulation example using FRF data and optimization diagnosed damage to a structure using only a minimum number of sensors on the structure. The technique does not use modal analysis or model reduction or a training step, and all the information contained in the FRF function is used, not just the information around the peaks as in modal analysis procedures. The analytical model, model connectivity, and bounds on structural stiffness values are also used to diagnose the damage.

The first limitation on the technique is that it requires a lot of computation which may restrict it to small models. The approach suggested to solve larger models is to "scan" the structural model by sections to detect and diagnose the damage. The scanning is based on the nodal connectivity patterns of the elements and runs a separate optimization using one or a group of elemental scale factors at a time as design variables. This brute force approach is actually more efficient than it might seem because there are only a small number of design variables in the optimization and convergence is very quick. Use of a laser vibrometer would provide more data to solve large problems, and an algorithm should be developed to automatically pick points from the healthy and damaged FRFs that would allow the optimizer to easily find the global minimum. Also, a more robust optimization algorithm should be used. A second limitation is that FRFs may not be exactly repeatable due to variations in temperature which changes the elastic modulus and causes boundary conditions to change. To overcome this, a technique that uses frequency shifting is being tested to remove global structural changes due to the environment.

Finally, only one excitation point is used in this example which gives the measured number of FRFs equal to the number of inputs multiplied by the number of measurements. Using more inputs would give more FRFs to assign, and may improve the damage diagnosis.

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