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NORMAL MODES AND NEAR-RESONANCE RESPONSE OF BEAMS WITH NON-LINEAR EFFECTS

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Non-linear normal modes and the associated frequencies of a uniform beam with simply-supported or clamped conditions at both ends have been derived. Some restricted orthogonality conditions have been pointed out. The effects of the longitudinal inertia on the non-linear transverse motion are shown to be extremely small. The efficacy of using the non-linear normal modes towards computation of near-resonance response has been clearly brought out.

1. INTRODUCTION

The analysis of both free and forced vibrations of a non-linear, multi-degrees-of-freedom system gets complicated due to modal coupling. A concept of non-linear normal modes, following that of the linear ones, was first introduced by Rosenberg [1] using a geometric method. The idea of the non-linear normal modes is based upon the observation that there exist periodic solutions to the equations of free vibration in which all the co-ordinates cross their equilibrium positions simultaneously and also attain their maximum values simultaneously. However, the linear normal modes, unlike the non-linear modes, are orthogonal to one another. Further, the non-linear mode shapes, as well as the associated (natural) frequencies, are dependent on the amplitude of motion.

The concept of non-linear normal modes has been elaborated by several researchers like Stupnicka [2], Rand [3], Shaw and Pierre [4], King and Vakakis [5], Nayfeh and Nayfeh [6]. Different methods, namely, harmonic balance [2], invariant manifold theory [4], energy method [5] and multiple time scale method [6] have been used for conservative systems having multiple or infinite degrees of freedom. The efficacy of the simple harmonic balance technique for obtaining the non-linear modes of a multi-degrees-of-freedom system has been clearly brought out in reference [2]. Numerical computation of near-resonance response of such systems, using the non-linear normal modes, is also amply demonstrated in this reference. The non-linear equation of a beam has also been studied using Galerkin's technique with the spatial distribution represented by a combination of linear modes and the observed (linear) modal coupling has also been verified experimentally [7].

In the present work, the non-linear normal modes of a uniform beam having weak non-linearities and linear boundary conditions are obtained. Towards this end, instead of using a numerical approach following harmonic balance [2], a combination of harmonic balance and perturbation technique has been used. Thus the effects of non-linearities can be very easily tracked to different orders of approximation. Finally, the non-linear normal modes are obtained as a combination of certain linear normal modes. The contributions

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of different linear modes are found to be amplitude dependent. The transverse vibration of the beam is analysed by both including and neglecting longitudinal inertia.

Results are explicitly obtained for only two symmetric boundary conditions, namely, simply-supported and clamped at both ends. The non-linear normal modes are then used to obtain the steady state, near-resonance response for both symmetric and asymmetric locations of a point harmonic load. Thus the contributions of symmetric and antisymmetric modes are clearly seen. The negligible effect of the longitudinal inertia towards both the non-linear normal modes and the near-resonance response is demonstrated. It is clearly shown that the use of non-linear, rather than linear, modes considerably reduces the computational effort. The numerical results show good agreement with available experimental results.

2. THEORETICAL ANALYSIS

2.1. EQUATIONS OF MOTION

The equations of motion for planar vibration of a uniform beam, including the non-linear terms are given by [8]

$$\rho A \frac{\partial^2 u^*}{\partial t^2} - EA \frac{\partial^2 u^*}{\partial \xi^2} = EA \frac{\partial w^*}{\partial \xi} \frac{\partial^2 w^*}{\partial \xi^2} \tag{1a}$$

$$\rho A \frac{\partial^2 w^*}{\partial t^2} + EI \frac{\partial^4 w^*}{\partial \xi^4} = EA \frac{\partial}{\partial \xi} \left[\frac{\partial u^*}{\partial \xi} \frac{\partial w^*}{\partial \xi} + \tfrac{1}{2} \left(\frac{\partial w^*}{\partial \xi} \right)^3 \right] \tag{1b}$$

where the symbols are listed in Appendix A.

By neglecting the longitudinal inertia and making the usual assumption [8] one derives the following non-linear differential equation governing the transverse motion of the beam:

$$\rho A \frac{\partial^2 w^*}{\partial t^2} + EI \frac{\partial^4 w^*}{\partial \xi^4} = EA \left[\frac{1}{2l} \int_0^l \left(\frac{\partial w^*}{\partial \xi} \right)^2 \mathrm{d}\xi \right] \frac{\partial^2 w^*}{\partial \xi^2}. \tag{2}$$

Using the following non-dimensional parameters

$$w = \frac{w^*}{l\gamma^2}, \qquad x = \frac{\xi}{l}, \qquad u = \frac{u^*}{l} \qquad \text{and} \qquad \tau = \frac{1}{l} \left(\frac{E}{\rho} \right)^{1/2} \gamma t$$

and introducing a small parameter $\varepsilon = \gamma^2/2$ equations (1a), (1b) and (2) become, respectively,

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{1}{2\varepsilon} \frac{\partial^2 u}{\partial x^2} = 2\varepsilon \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \tag{3a}$$

$$\frac{\partial^2 w}{\partial \tau^2} + \frac{\partial^4 w}{\partial x^4} = \frac{\partial}{\partial x} \left[\frac{1}{2\varepsilon} \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} + \varepsilon \left(\frac{\partial w}{\partial x} \right)^3 \right] \tag{3b}$$

$$\frac{\partial^2 w}{\partial \tau^2} + \frac{\partial^4 w}{\partial x^4} = \varepsilon \left[\int_0^1 \left(\frac{\partial w}{\partial x} \right)^2 \mathrm{d}x \right] \frac{\partial^2 w}{\partial x^2}. \tag{4}$$

In what follows, the non-linear normal modes with reference to equations (3a) and (3b) are first discussed and then equation (4) is considered (i.e., neglecting the longitudinal inertia).

2.2. NON-LINEAR NORMAL MODES WITH LONGITUDINAL INERTIA

In this section, the beam is taken to be fixed at both ends so far as the longitudinal motion is concerned. The linear normal modes of a beam can be obtained from equation (3a) after substituting $\varepsilon = 0$. Let $\varphi_n(x)$ and ω_n^l, ($n = 1, 2, 3, \ldots$) be the nth linear normal mode and the corresponding linear natural frequency. To obtain the non-linear normal modes of transverse vibration from equations (3a) and (3b), for $\varepsilon \neq 0$, $w(x, \tau)$ is assumed as

$$w(x, \tau) = a\psi_n \cos \omega_n \tau. \tag{5}$$

Substituting equation (5) into equation (3a) one obtains

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{1}{2\varepsilon}\frac{\partial^2 u}{\partial x^2} = \varepsilon a^2 \frac{\mathrm{d}\psi_n}{\mathrm{d}x}\frac{\mathrm{d}^2\psi_n}{\mathrm{d}x^2} + \varepsilon a^2 \frac{\mathrm{d}\psi_n}{\mathrm{d}x}\frac{\mathrm{d}^2\psi_n}{\mathrm{d}x^2}\cos 2\omega_n \tau. \tag{6}$$

The steady state solution to equation (6) is assumed in the form:

$$u(x, \tau) = u_1(x) + u_2(x, \tau) \tag{7}$$

where $u_1(x)$ and $u_2(x, \tau)$ are the solutions with only the first and the second term, respectively, on the right hand side of equation (6), and

$$\frac{\mathrm{d}u_1}{\mathrm{d}x} = -\varepsilon^2 a^2\left(\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)^2 + f \tag{8}$$

with f as the constant of integration. Equation (8), on integration and after using the boundary conditions that the longitudinal displacement is zero at both ends, i.e., at $x = 0$ and $x = 1$, yields

$$f = \varepsilon^2 a^2 \int_0^1 \left(\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)^2 \mathrm{d}x. \tag{9}$$

Next, the time-dependent part of u, $u_2(x, \tau)$ is easily obtained as

$$u_2(x, \tau) = \varepsilon a^2 \sum_{i=1}^{\infty} \frac{C_i \Gamma_i(x)}{\left(\dfrac{1}{2\varepsilon}\nu_i^2 - 4\omega_n^2\right)} \cos 2\omega_n \tau \tag{10}$$

where

$$C_i = \frac{\int_0^1 \dfrac{\mathrm{d}\psi_n}{\mathrm{d}x}\dfrac{\mathrm{d}^2\psi_n}{\mathrm{d}x^2}\Gamma_i \,\mathrm{d}x}{\int_0^1 \Gamma_i^2 \,\mathrm{d}x} \tag{11}$$

and Γ_i's and ν_i's are the linear mode shapes and natural frequencies for the longitudinal vibration, i.e.,

$$\Gamma_i(x) = \sin i\pi x \quad \text{and} \quad \nu_i = i\pi, \qquad i = 1, 2, 3, \ldots. \tag{12}$$

Now for small non-linearity, ($\varepsilon \ll 1$), the nth non-linear mode $\psi_n(x)$ and associated frequency ω_n are assumed (close to the linear ones) as

$$\omega_n^2 = (\omega_n^l)^2 + \varepsilon\beta_n^{(1)} + \varepsilon^2\beta_n^{(2)} + \cdots, \qquad \psi_n = \varphi_n + \varepsilon\delta_n^{(1)} + \varepsilon^2\delta_n^{(2)} + \cdots. \tag{13, 14}$$

Substituting equations (5), (7–12) into equation (3b) and by harmonic balance, one gets

$$-a\omega_n^2\psi_n + a\frac{\mathrm{d}^4\psi_n}{\mathrm{d}x^4} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\varepsilon}{4}a^3\left(\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)^3 + \frac{\varepsilon}{2}a^3\sum_{i=1}^{\infty}\frac{C_i\left(\frac{\mathrm{d}\Gamma_i}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)}{(v_i^2 - 8\varepsilon\omega_n^2)}\right]$$

$$+\tfrac{1}{2}\varepsilon a^3\left[\int_0^1\left(\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)^2\right]\frac{\mathrm{d}^2\psi_n}{\mathrm{d}x^2}. \tag{15}$$

Using expansions (13) and (14) in equations (15) and (11), one can equate the coefficients of like powers of ε in both sides to get

$$\varepsilon^0: \qquad -(\omega_n^l)^2\varphi_n + \frac{\mathrm{d}^4\varphi_n}{\mathrm{d}x^4} = 0 \tag{16a}$$

$$\varepsilon^1: \qquad -(\omega_n^l)^2\delta_n^{(1)} + \frac{\mathrm{d}^4\delta_n^{(1)}}{\mathrm{d}x^4} = \beta_n^{(1)}\varphi_n + a^2\frac{\mathrm{d}}{\mathrm{d}x}[\Lambda] + \tfrac{1}{2}a^2\left[\int_0^1\left(\frac{\mathrm{d}\varphi_n}{\mathrm{d}x}\right)^2\mathrm{d}x\right]\frac{\mathrm{d}^2\varphi_n}{\mathrm{d}x^2} \tag{16b}$$

where

$$\Lambda = \left[\tfrac{1}{4}\left(\frac{\mathrm{d}\varphi_n}{\mathrm{d}x}\right)^3 + \tfrac{1}{2}\sum_{i=1}^{\infty}\frac{C_i^l\left(\frac{\mathrm{d}\Gamma_i}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}\varphi_n}{\mathrm{d}x}\right)}{v_i^2}\right] \tag{17a}$$

$$C_i^l = \frac{\int_0^1\frac{\mathrm{d}\varphi_n}{\mathrm{d}x}\frac{\mathrm{d}^2\varphi_n}{\mathrm{d}x^2}\Gamma_i\,\mathrm{d}x}{\int_0^1\Gamma_i^2\,\mathrm{d}x}. \tag{17b}$$

Equation (16a) is trivially satisfied. To solve equation (16b), we assume

$$\delta_n^{(1)} = \sum_{i=1}^{\infty}\Delta_i^{(1)}\varphi_i. \tag{18}$$

Substituting equation (18) into equation (16b) one can rewrite the right hand side of the same as

$$\beta_n^{(1)}\varphi_n + a^2\sum_{i=1}^{\infty}\left(\lambda_i^{(1)} + \frac{\alpha_i^{(1)}}{2}\right)\varphi_i \tag{19}$$

where

$$\lambda_i^{(1)} = \frac{\left(\int_0^1 \frac{\mathrm{d}\Lambda}{\mathrm{d}x}\,\varphi_i\,\mathrm{d}x\right)}{\left(\int_0^1 \varphi_i^2\,\mathrm{d}x\right)} \tag{20a}$$

$$\alpha_i^{(1)} = \frac{\left(\int_0^1 \left(\frac{\mathrm{d}\varphi_n}{\mathrm{d}x}\right)^2 \mathrm{d}x\right)\left(\int_0^1 \frac{\mathrm{d}^2\varphi_n}{\mathrm{d}x^2}\,\varphi_i\,\mathrm{d}x\right)}{\left(\int_0^1 \varphi_i^2\,\mathrm{d}x\right)}. \tag{20b}$$

Now substituting equations (18–20) into equation (16b) and equating the coefficients of φ_i's from both sides of this equation one obtains

$$\beta_n^{(1)} = -a^2\left(\lambda_n^{(1)} + \frac{\alpha_n^{(1)}}{2}\right), \qquad \delta_n^{(1)} = a^2 \sum_{i \neq n} \frac{\left(\lambda_i^{(1)} + \frac{\alpha_i^{(1)}}{2}\right)\varphi_i}{[(\omega_i^l)^2 - (\omega_n^l)^2]}. \tag{21a, b}$$

Hence, the non-linear normal modes and the associated frequencies are obtained as

$$\psi_n = \varphi_n + \varepsilon a^2 \sum_{i \neq n} \frac{\left(\lambda_i^{(1)} + \frac{\alpha_i^{(1)}}{2}\right)\varphi_i}{[(\omega_i^l)^2 - (\omega_n^l)^2]} \tag{22a}$$

$$\omega_n^2 = (\omega_n^l)^2 - \varepsilon a^2\left(\lambda_n^{(1)} + \frac{\alpha_n^{(1)}}{2}\right). \tag{22b}$$

The amplitude dependence of the non-linear mode shape and frequency is clearly exhibited by equations (22a) and (22b). The results for simply-supported and clamped–clamped boundary conditions are explicitly obtained as given below.

2.2.1. *Simply-supported beam*

For the simply-supported non-linear beam, the boundary conditions are

$$w(0, \tau) = w(1, \tau) = 0, \qquad \frac{\partial^2 w(0, \tau)}{\partial x^2} = \frac{\partial^2 w(1, \tau)}{\partial x^2} = 0. \tag{23}$$

In this case

$$\varphi_n(x) = \sin(n\pi x), \qquad (\omega_n^l)^2 = (n\pi)^4. \tag{24a, b}$$

Substituting equations (24a, b) into equations (22a, b) one gets to a second order approximation (details are given in Appendix B),

$$\psi_n = \sin n\pi x + \tfrac{3}{2}\,\varepsilon^2\,\frac{a^2(n\pi)^2}{(3^4 - 1)}\sin 3n\pi x$$

$$\omega_n^2 = (n\pi)^4[1 + \tfrac{3}{8}\,\varepsilon a^2] - \tfrac{1}{2}\,\varepsilon^2 a^2 (n\pi)^6.$$

It is seen that the coupling of the linear modes exists in the non-linear mode shape only in the second order approximation.

2.2.2. *Clamped–clamped beam*

Now the boundary conditions are

$$w(0,\tau)=w(1,\tau)=0, \qquad \frac{\partial w(0,\tau)}{\partial x}=\frac{\partial w(1,\tau)}{\partial x}=0. \tag{25}$$

The linear modes and the natural frequencies are given by

$$\varphi_n=\left[(\sin\mu_n x-\sinh\mu_n x)-\frac{(\sin\mu_n-\sinh\mu_n)}{(\cos\mu_n-\cosh\mu_n)}(\cos\mu_n x-\cosh\mu_n x)\right] \tag{26a}$$

$$(\omega_n^l)^2=\mu_n^4 \tag{26b}$$

where μ_n's are the roots of the transcendental equation $\cos\mu_n\cosh\mu_n=1$.

For such a beam, the first three non-linear modes and the corresponding frequencies are obtained as

$$\psi_1=\varphi_1+0{\cdot}006598\varepsilon a^2\varphi_3, \qquad \omega_1^2=(\omega_1^l)^2[1+0{\cdot}2351\varepsilon a^2] \tag{27a}$$

$$\psi_2=\varphi_2, \qquad \omega_2^2=(\omega_2^l)^2[1+0{\cdot}4169\varepsilon a^2] \tag{27b}$$

$$\psi_3=\varphi_3-0{\cdot}05062\varepsilon a^2\varphi_1, \qquad \omega_3^2=(\omega_3^l)^2[1+0{\cdot}50225\varepsilon a^2]. \tag{27c}$$

2.3. NON-LINEAR NORMAL MODES WITHOUT LONGITUDINAL INERTIA

To get the non-linear normal mode of a beam when the longitudinal inertia is neglected, equation (5) can be substituted into equation (4). Following harmonic balance, one gets

$$-a\omega_n^2\psi_n+a\frac{\mathrm{d}^4\psi_n}{\mathrm{d}x^4}=\tfrac{3}{4}\varepsilon a^3\left[\int_0^1\left(\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)^2\mathrm{d}x\right]\frac{\mathrm{d}^2\psi_n}{\mathrm{d}x^2}. \tag{28}$$

For small non-linearities, the non-linear mode ψ_n and the associated frequency ω_n are expanded as in equations (13) and (14), respectively. Then substituting these into equation (28) and subsequently equating the coefficients of the like powers of ε from both sides of this equation one finds

$$\varepsilon^0: \qquad -(\omega_n^l)^2\varphi_n+\frac{\mathrm{d}^4\varphi_n}{\mathrm{d}x^4}=0 \tag{29a}$$

$$\varepsilon^1: \qquad -(\omega_n^l)^2\delta_n^{(1)}+\frac{\mathrm{d}^4\delta_n^{(1)}}{\mathrm{d}x^4}=\beta_n^{(1)}\varphi_n+\tfrac{3}{4}a^2\left[\int_0^1\left(\frac{\mathrm{d}\varphi_n}{\mathrm{d}x}\right)^2\mathrm{d}x\right]\frac{\mathrm{d}^2\varphi_n}{\mathrm{d}x^2}. \tag{29b}$$

The solutions to the above equations are carried out by the method described in section 2.2. The final results are obtained as

$$\psi_n=\varphi_n+\tfrac{3}{4}\varepsilon a^2\sum_{i\neq n}\frac{\alpha_i^{(1)}\varphi_i}{[(\omega_i^l)^2-(\omega_n^l)^2]} \tag{30}$$

$$\omega_n^2=(\omega_n^l)^2-\tfrac{3}{4}\varepsilon a^2\alpha_n^{(1)}. \tag{31}$$

Now we recalculate the non-linear normal modes and the associated frequencies from equations (30) and (31) for simply-supported and clamped–clamped end conditions to demonstrate the effect of the longitudinal inertia.

2.3.1. *Simply-supported end conditions*

Substitution of equations (24a) and (24b), i.e., the linear normal mode shape and frequency, into equations (30) and (31), yields

$$\omega_n^2 = (n\pi)^4[1 + \tfrac{3}{8}\,\varepsilon a^2], \qquad \psi_n\,(x) = \varphi_n\,(x) \tag{32, 33}$$

since $\alpha_i^{(1)} = 0$ if $i \neq n$ and $\alpha_n^{(1)} = -\frac{1}{2}\,(n\pi)^4$.

Equation (33) shows that for a simply-supported beam, the non-linear normal modes are the same as the linear normal modes and hence also orthogonal to each other. The above result is well established in the literature [6].

2.3.2. *Clamped–clamped end conditions*

Using the nth linear normal mode φ_n and frequency ω_n^l as in section 2.2.2, and noting that

$$\int_0^1 \frac{\mathrm{d}^2\varphi_n}{\mathrm{d}x^2}\,\varphi_i\,\mathrm{d}x = 0 \qquad \text{if } i + n = \text{odd},$$

one obtains

$$\psi_n = \varphi_n + \tfrac{3}{4}\,\varepsilon a^2 \sum_{i+n=\text{even}, i\neq n} \frac{\alpha_i^{(1)}\varphi_i}{[(\omega_i^l)^2 - (\omega_n^l)^2]} \tag{34}$$

where $\alpha_i^{(1)}$'s are obtained from equation (20b).

From equation (34) one can see that the odd order non-linear normal modes comprise only odd order linear normal modes and even order non-linear normal modes are given by a combination of only even order linear normal modes. Hence, the odd and even order non-linear normal modes are orthogonal to each other.

The non-linear normal modes for the clamped–clamped beam and the corresponding frequencies, taking only the first three linear normal modes into consideration, are obtained, up to the first order, as

$$\psi_1 = \varphi_1 + 0{\cdot}006675\varepsilon a^2\varphi_3, \qquad \omega_1^2 = (\omega_1^l)^2[1 + 0{\cdot}2351\varepsilon a^2] \tag{35a}$$

$$\psi_2 = \varphi_2, \qquad \omega_2^2 = (\omega_2^l)^2[1 + 0{\cdot}41672\varepsilon a^2] \tag{35b}$$

$$\psi_3 = \varphi_3 - 0{\cdot}05017\varepsilon a^2\varphi_1, \qquad \omega_3^2 = (\omega_3^l)^2[1 + 0{\cdot}50226\varepsilon a^2]. \tag{35c}$$

Comparing equations (35a–c) with equations (27a–c), we can conclude that the effect of the longitudinal inertia on the non-linear normal modes and frequencies of transverse vibration is negligible.

2.4. NEAR-RESONANCE RESPONSE OF NON-LINEAR BEAM (NEGLECTING LONGITUDINAL INERTIA)

The equation of motion of planar, transverse vibration of a harmonically forced slender beam, neglecting the longitudinal inertia, is given by

$$\rho A\,\frac{\partial^2 w^*}{\partial t^2} + EI\,\frac{\partial^4 w^*}{\partial \xi^4} - EA\left[\frac{1}{2l}\int_0^l \left(\frac{\partial w^*}{\partial \xi}\right)^2 \mathrm{d}\xi\right]\frac{\partial^2 w^*}{\partial \xi^2} = p^*(x)\cos\Theta t. \tag{36}$$

Using the non-dimensional parameters used so far together with

$$p = \frac{p^* l}{EA\gamma^4} \qquad \text{and} \qquad \Theta = \frac{1}{l}\left(\frac{E}{\rho}\right)^{1/2} \gamma\Omega,$$

equation (36) becomes

$$\frac{\partial^2 w}{\partial \tau^2} + \frac{\partial^4 w}{\partial x^4} - \varepsilon\left[\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\right]\frac{\partial^2 w}{\partial x^2} = p \cos \Omega\tau. \tag{37}$$

In the absence of any secondary and internal resonance, the response can be written as

$$w(x, \tau) = \sum_i a_i \, \psi_i \,(x) \cos \Omega\tau \tag{38}$$

where ψ_i is the ith non-linear normal mode. One should not assume that the non-linear normal modes are orthogonal to each other. When the forcing frequency is close to a linear natural frequency i.e., ω_n^l, then the participation of the corresponding non-linear normal mode is largest whereas the other modes participate only weakly. Mathematically this can be written in the following form:

when $$\Omega = \omega_n^l + \varepsilon\Omega_1, \quad a_i = \varepsilon b_i, \qquad \text{for } i \neq n. \tag{39}$$

Substituting equations (38) and (39) into equation (37) and neglecting terms $o(\varepsilon^2)$, one obtains

$$-a_n \, \Omega^2 \psi_n \cos \Omega\tau + a_n \frac{\mathrm{d}^4\psi_n}{\mathrm{d}x^4} \cos \Omega\tau - \varepsilon a_n^3 \left[\int_0^1 \left(\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)^2 \mathrm{d}x\right] \frac{\mathrm{d}^2\psi_n}{\mathrm{d}x^2} \cos^3 \Omega\tau$$

$$+ \varepsilon \sum_{i \neq n} \left[-b_i \, \Omega^2 \psi_i \cos \Omega\tau + b_i \frac{\mathrm{d}^4\psi_i}{\mathrm{d}x^4} \cos \Omega\tau\right] = p \cos \Omega\tau. \tag{40}$$

Applying the harmonic balance technique to equation (40) and substituting equation (28), one obtains

$$a_n \, \psi_n \,(\omega_n^2 - \Omega^2) + \varepsilon \sum_{i \neq n} \left[-b_i \, \Omega^2 \psi_i + b_i \frac{\mathrm{d}^4\psi_i}{\mathrm{d}x^4}\right] = p. \tag{41}$$

From equation (30) one notes that

$$\int_0^1 \psi_n \, \psi_i \, \mathrm{d}x = o(\varepsilon), \qquad i \neq n. \tag{42}$$

Hence, multiplying equation (41) by ψ_n and integrating over x, while neglecting terms $o(\varepsilon^2)$, one obtains

$$a_n = \frac{\int_0^1 p\psi_n \, \mathrm{d}x}{(\omega_n^2 - \Omega^2) \int_0^1 \psi_n^2 \, \mathrm{d}x}. \tag{43a}$$

Use of equation (31) in equation (43a) yields a cubic equation for the determination of a_n.

Since the non-linear frequency is close to the linear natural frequency and the nth order linear mode of the beam is resonantly excited, we use $\omega_n^2 - \Omega^2 = o(\varepsilon)$. Now multiplying equation (41) by ψ_m, $(m \neq n)$, and integrating over x, one gets the following result by using equations (42) and (39) while neglecting terms of $o(\varepsilon^2)$:

$$a_m = \frac{\int_0^1 p\varphi_m \,\mathrm{d}x}{((\omega_m^l)^2 - \Omega^2)\int_0^1 \varphi_m^2 \,\mathrm{d}x}, \qquad (m \neq n). \tag{43b}$$

From equation (43b) it is observed that a_m's $(m \neq n)$ are nothing but the participation of the mth linear mode from the linear theory. Substituting equations (42a) and (43b) into equation (38), one finally obtains the near resonance response.

Now a special situation when some non-linear normal modes are orthogonal to each other is discussed. This happens, for example, for all the non-linear modes of a simply-supported beam (see section 2.2.1) and the non-linear modes of even and odd order for a clamped–clamped beam. In such cases, when two orthogonal non-linear normal modes, say of order n and m are excited, the response can be written as

$$w(x, \tau) = a_n \psi_n \cos \Omega\tau + a_m \psi_m \cos \Omega\tau + \varepsilon \sum_{i \neq n,m} b_i \psi_i \cos \Omega\tau \tag{44}$$

with

$$\int_0^1 \psi_n \psi_m \,\mathrm{d}x = 0, \quad \int_0^1 \psi_n \psi_i \,\mathrm{d}x = o(\varepsilon), \quad \int_0^1 \psi_m \psi_i \,\mathrm{d}x = o(\varepsilon), \qquad \text{when } i \neq n, m. \tag{45}$$

Substituting equation (44) into equation (37) and by balancing of harmonics, one obtains

$$a_n \psi_n (\omega_n^2 - \Omega^2) + a_m \psi_m (\omega_m^2 - \Omega^2) - \varepsilon a_n a_m^2 \left[\int_0^1 \left(\frac{\mathrm{d}\psi_m}{\mathrm{d}x}\right)^2 \mathrm{d}x\right] \frac{\mathrm{d}^2\psi_n}{\mathrm{d}x^2}$$

$$- \varepsilon a_m a_n^2 \left[\int_0^1 \left(\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)^2 \mathrm{d}x\right] \frac{\mathrm{d}^2\psi_m}{\mathrm{d}x^2} + \varepsilon \sum_{i \neq n,m} \left[-b_i \Omega^2 \psi_i + b_i \frac{\mathrm{d}^4\psi_i}{\mathrm{d}x^4}\right] = p. \tag{46}$$

Multiplying equation (46) by ψ_n and ψ_m separately and integrating over x, the following equations are obtained by retaining terms up to $o(\varepsilon)$:

$$a_n (\omega_n^2 - \Omega^2) + \varepsilon a_n a_m^2 \frac{\left[\int_0^1 \left(\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)^2 \mathrm{d}x\right]\left[\int_0^1 \left(\frac{\mathrm{d}\psi_m}{\mathrm{d}x}\right)^2 \mathrm{d}x\right]}{\int_0^1 \psi_n^2 \,\mathrm{d}x} = \frac{\int_0^1 p\psi_n \,\mathrm{d}x}{\int_0^1 \psi_n^2 \,\mathrm{d}x} \tag{47a}$$

$$a_m (\omega_m^2 - \Omega^2) + \varepsilon a_m a_n^2 \frac{\left[\int_0^1 \left(\frac{\mathrm{d}\psi_m}{\mathrm{d}x}\right)^2 \mathrm{d}x\right]\left[\int_0^1 \left(\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)^2 \mathrm{d}x\right]}{\int_0^1 \psi_m^2 \,\mathrm{d}x} = \frac{\int_0^1 p\psi_m \,\mathrm{d}x}{\int_0^1 \psi_m^2 \,\mathrm{d}x}. \tag{47b}$$

Now equations (47a) and (47b) can be solved simultaneously to obtain a_n and a_m. It may be noted from equation (43b) that even the linear theory can be used to calculate a_i's, $(i \neq m, n)$.

2.4.1. *Numerical results and discussions*

In reference [7] the non-linear response of a harmonically excited, clamped–clamped beam was obtained using the linear modes. The response was assumed to be of the following form:

$$w(x, \tau) = \sum_{i=1}^{3} A_i^* \varphi_i(x) \cos \Omega\tau$$

and A_i^*'s were obtained by solving three simultaneous non-linear algebraic equations resulting from the usual Galerkin's technique.

From the analysis presented in section 2.4 one can see the efficacy of using non-linear normal modes for calculating the response of a non-linear system. While the harmonic balance method presented in reference [7] requires solving three simultaneous non-linear algebraic equations, the present method reduces the computation to merely solving one cubic equation. Furthermore, in the ordinary harmonic balance method, the number of equations increases proportionately with the number of linear modes considered. But in the method presented in this work, all the modal participation can be obtained from a single equation.

To illustrate further that both methods give the same results, we show below the results of a clamped–clamped beam, harmonically excited by a point load. The beam parameters are taken to be the same as those in reference [7]. To compare the results with those reported in the same reference, we collect the coefficients of the first three linear normal modes (denoted by, say, A_i^*, with $i = 1, 2$ and 3) from equations (38), (43a) and (43b). When the nth linear mode is excited, the participation of different linear modes are as follows:

$$A_n^* = a_n, \quad A_m^* = a_m + \varepsilon \Delta_m^{(1)}, \qquad (m \neq n).$$

For the dimensions of the beam treated in reference [7], the non-dimensional parameters are obtained as $\gamma^2 = 2{\cdot}12 \times 10^{-6}$ and $p = 55624{\cdot}7775$. When the load is symmetrical about the mid-point of the beam and ψ_n is antisymmetric, the response amplitude is very small (since $a_n = 0$) and can be calculated from the linear theory. The situation is exactly similar when the load is antisymmetric and ψ_n is symmetric. For the simply-supported and clamped–clamped beams, it was seen that ψ_n is symmetric or antisymmetric according to whether n is odd or even, respectively. Then from the above argument the non-linear analysis is required only if: n is odd and the load is symmetric about the mid-point i.e., $x = 1/2$; n is even and the load is antisymmetric about $x = 1/2$; or the load is asymmetric.

Figures 1(a) and (b) show the participation of the first and third linear modes when the first linear mode is resonantly excited by a load at $x = 1/2$. The factor (γ^2) in the ordinate is used to compare the results of reference [7] and arises because of the difference in the non-dimensionalisation procedure. The results of the present analysis are in very good agreement with those of reference [7]. Both the results show marked deviation from the linear theory. It is also to be noted that the experimental data obtained in reference [7] support the theoretical results. Further, like all cases of cubic non-linearity, when multiple solutions for the periodic response appear, the intermediate one is unstable.

When the first mode is resonantly excited by a point harmonic loading applied at $x = 1/4$, both the first and second non-linear normal modes, which are orthogonal to each other, get excited. Their response amplitudes are obtained by solving equations (48a) and (48b) and are shown in Figures 2(a) and (b). The results again show excellent agreement with those reported in reference [7].

When the third linear mode is excited by a point loading at $x = 1/2$, the first mode is also excited owing to the modal coupling. The modal participation of the first and third linear modes are shown in Figures 3(a) and (b), respectively. The results also show reasonable agreement with those of reference [7].

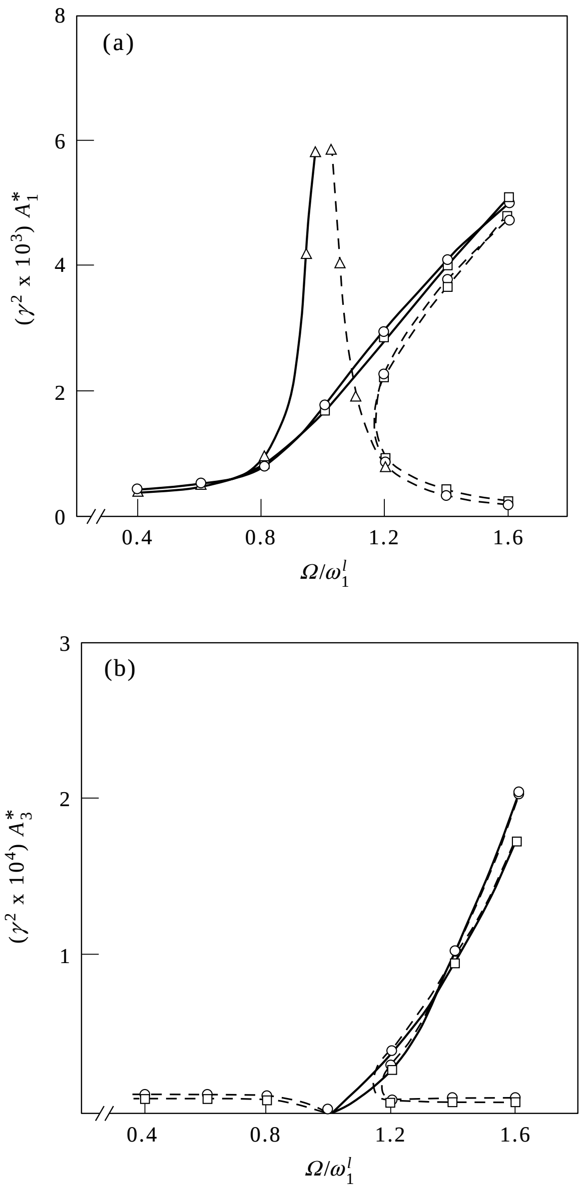

Figure 1. (a) Participation of the first mode in the response. (b) Participation of the third mode in the response. □, reference [7]; ○, present method; ——, in phase; – – –, out of phase.

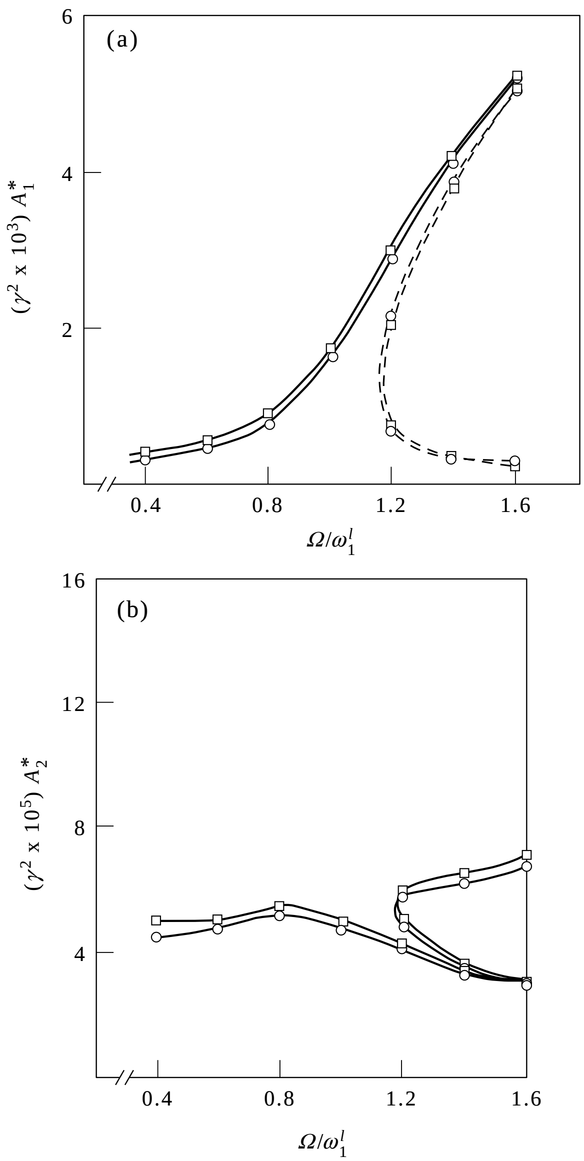

Figure 2. (a) Participation of the first mode in the response. (b) Participation of the second mode in the response. □, reference [7]; ○, present method; ——, in phase; – – –, out of phase.

2.5. NEAR-RESONANCE RESPONSE OF NON-LINEAR BEAM (WITH LONGITUDINAL INERTIA)

The non-dimensional equations of motion for coupled longitudinal and transverse vibration of a harmonically excited, slender beam are given by

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{1}{2\varepsilon}\frac{\partial^2 u}{\partial x^2} = 2\varepsilon \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}, \tag{48a}$$

$$\frac{\partial^2 w}{\partial \tau^2} + \frac{\partial^4 w}{\partial x^4} = \frac{\partial}{\partial x}\left[\frac{1}{2\varepsilon}\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + \varepsilon\left(\frac{\partial w}{\partial x}\right)^3\right] + p\cos\Omega\tau. \tag{48b}$$

When $\Omega = \omega_n^l + \varepsilon\Omega_1$, the response, as in section 2.4, is written in the form:

$$w(x, \tau) = a_n\,\psi_n \cos \Omega\tau + \varepsilon \sum_{i \neq n} b_i\,\psi_i \cos \Omega\tau. \tag{49}$$

Equation (49) is substituted into equation (48a) and solved for $u(x, \tau)$, which is then replaced in equation (48b). Thereafter balancing the harmonics and omitting terms of

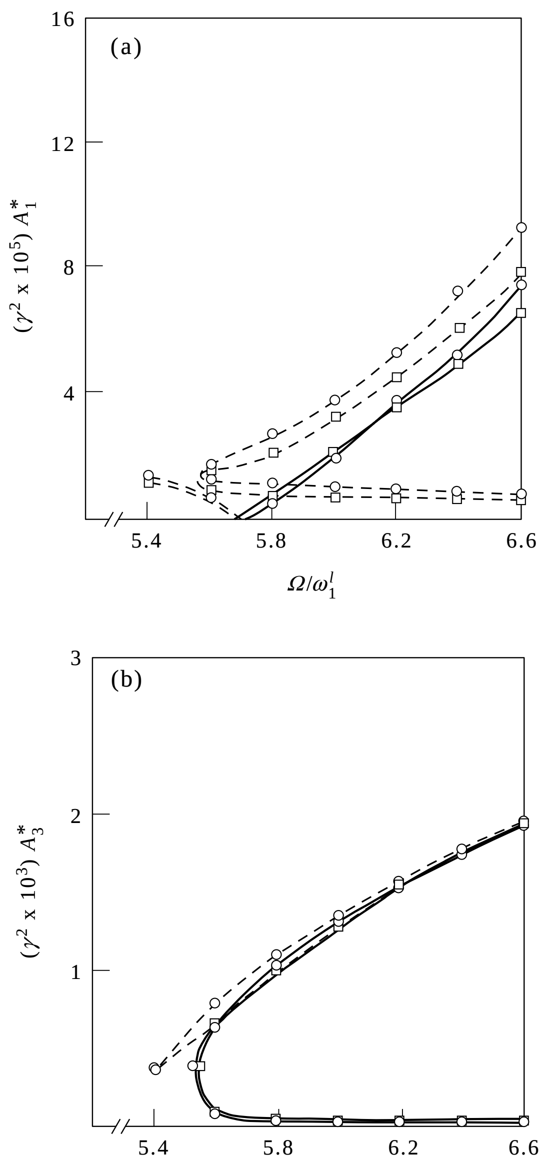

Figure 3. (a) Participation of the first mode in the response. (b) Participation of the third mode in the response. □, reference [7]; ○, present method; ——, in phase; – – –, out of phase.

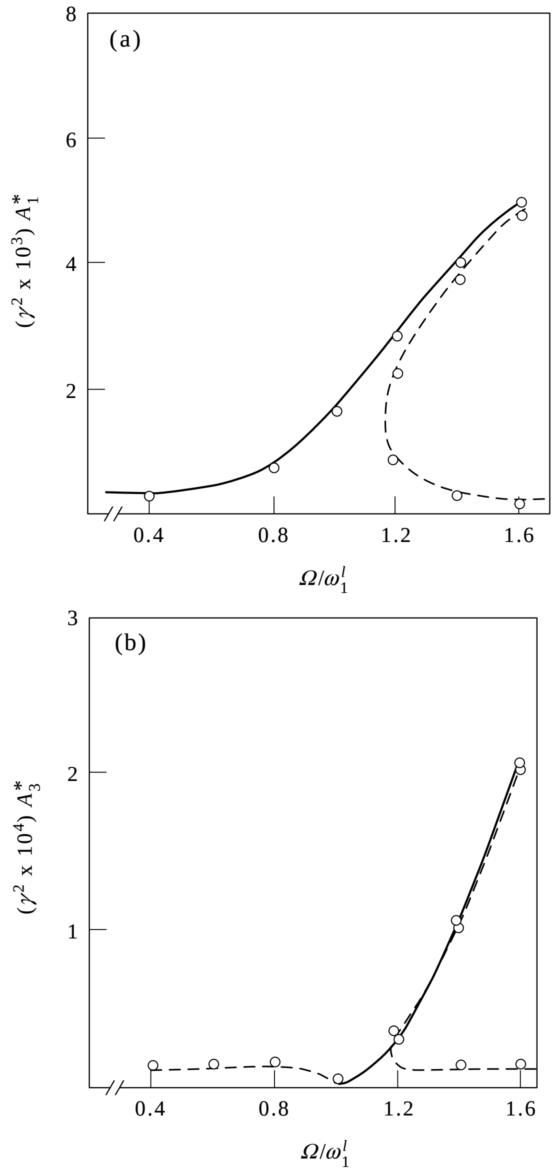

Figure 4. (a) Participation of the first mode in the response. (b) Participation of the third mode in the response. ——, in phase response neglecting longitudinal inertia; – – –, out of phase response neglecting longitudinal inertia; ○, including longitudinal inertia.

order $o(\varepsilon^2)$ and higher, the following equation resulted:

$$- a_n \Omega^2 \psi_n + a_n \frac{\mathrm{d}^4\psi_n}{\mathrm{d}x^4} - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\varepsilon}{4} a_n^3\left(\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)^3 + \frac{\varepsilon}{2} a_n^3 \sum_{i=1}^{\infty} \frac{C_i^l\left(\dfrac{\mathrm{d}\Gamma_i}{\mathrm{d}x}\right)\left(\dfrac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)}{\nu_i^2}\right]$$

$$- \tfrac{1}{2}\varepsilon a_n^3\left[\int_0^1 \left(\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)^2\right]\frac{\mathrm{d}^2\psi_n}{\mathrm{d}x^2} + \varepsilon \sum_{i \neq n}\left[-\Omega^2 b_i\, \psi_i + b_i \frac{\mathrm{d}^4\psi_n}{\mathrm{d}x^4}\right] = p. \tag{50}$$

Substituting equation (15) into equation (50) one obtains

$$a_n \psi_n (\omega_n^2 - \Omega^2) + \varepsilon \sum_{i \neq n} \left[-b_i \Omega^2 \psi_i + b_i \frac{\mathrm{d}^4 \psi_i}{\mathrm{d}x^4} \right] = p. \tag{51}$$

Using the result given by equation (42) one finally obtains from equation (51)

$$a_n = \frac{\int_0^1 p \psi_n \, \mathrm{d}x}{(\omega_n^2 - \Omega^2) \int_0^1 \psi_n^2 \, \mathrm{d}x}. \tag{52}$$

As noted in section 2.4, b_i's are obtained from linear analysis with $i \neq n$.

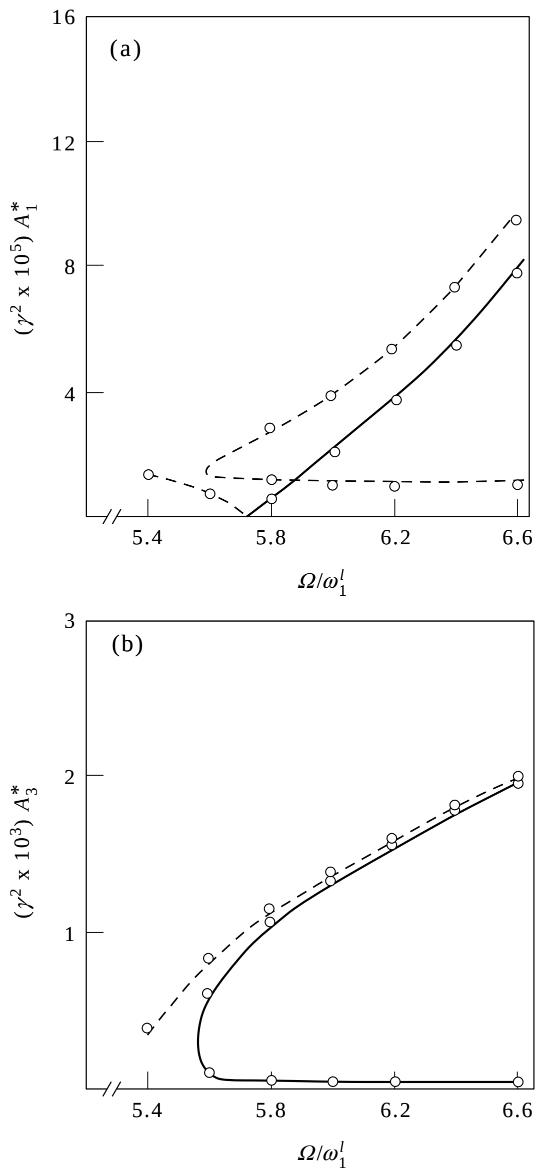

Figure 5. (a) Participation of the first mode in the response. (b) Participation of the third mode in the response. ——, in phase response neglecting longitudinal inertia; – – –, out of phase response neglecting longitudinal inertia; ○, including longitudinal inertia.

2.5.1. *Numerical results and discussions*

Here we recalculate the results reported in section 2.4.1, the only difference being that here we take the longitudinal inertia into consideration. The results are then compared with those of section 2.4.1.

Figures 4(a) and (b) show the participation of the first and the third linear modes when the first linear mode is excited by a point load at $x = 1/2$. The effect of longitudinal inertia is seen to be extremely small.

Figures 5(a) and (b) show the participation of the first and the third linear mode when the third linear mode is excited by the same loading. Here also the effect of longitudinal inertia is negligibly small.

3. CONCLUSIONS

A very simple method for obtaining the non-linear normal modes of a uniform beam is presented. The non-linear normal modes are derived for simply-supported and clamped–clamped end conditions by both retaining and neglecting the longitudinal inertia. It is shown that the longitudinal inertia has negligible effect on the non-linear normal modes and associated frequencies in the first approximation. For a simply-supported beam, when the longitudinal inertia is neglected, the non-linear modes are the same as the linear ones. However, these differ only in the second approximation if the longitudinal inertia is taken into consideration.

For a clamped–clamped beam, it is shown that the even order non-linear modes comprise a combination of even order linear normal modes. Similarly, the odd order non-linear normal modes consist of a combination of odd order linear normal modes. This implies orthogonality between the even and odd order non-linear normal modes.

The non-linear normal modes can be used profitably to determine the near-resonance response of the beam. The results show excellent agreement with those obtained by Galerkin's technique with the linear normal modes. The method using the linear normal modes requires simultaneous solution of non-linear algebraic equations. The number of equations to be handled equals the number of the linear normal modes considered. The method presented in this work reduces the computation to merely solving one cubic equation.

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APPENDIX A: LIST OF SYMBOLS

w^* transverse displacement of the beam
u^* longitudinal displacement of the beam
ρ density of the beam material
E Young's modulus of the beam material
A area of cross-section of the beam
l length of the beam
I second moment of area of cross-section about the neutral axis
r radius of gyration of the beam cross-section $= \sqrt{I/A}$
γ slenderness ratio, $r/l \ll 1$
$\varepsilon = \gamma^2/2$
ξ longitudinal distance of a point on the beam from left support
t time
x non-dimensional distance, ξ/l
τ non-dimensional time, $(1/l)(E/\rho)^{1/2}\gamma t$
w non-dimensional transverse displacement
u non-dimensional longitudinal displacement
φ_n nth linear normal mode
ω_n^l linear natural frequency of the nth linear mode
ψ_n nth non-linear normal mode
ω_n frequency corresponding to the nth non-linear normal mode
p^* transverse force per unit length
p non-dimensional transverse force
Θ frequency of excitation
Ω non-dimensional frequency of excitation
a_i participation of the ith non-linear normal mode
A_i^* participation of the ith linear normal mode
$\Gamma_i = \sin i\pi x, \quad i = 1, 2, 3, \ldots$
$\nu_i = i\pi, \quad i = 1, 2, 3, \ldots$

APPENDIX B: NON-LINEAR NORMAL MODES FOR SIMPLY-SUPPORTED SLENDER BEAM WITH LONGITUDINAL INERTIA INCLUDING SECOND ORDER TERMS

The non-linear normal modes are obtained by solving equation (15), which is reproduced below:

$$- a\omega_n^2\psi_n + a\,\frac{d^4\psi_n}{dx^4} = \frac{d}{dx}\left[\frac{\varepsilon}{4}\,a^3\left(\frac{d\psi_n}{dx}\right)^3 + \frac{\varepsilon}{2}\,a^3\sum_{i=1}^{\infty}\frac{C_i\left(\dfrac{d\Gamma_i}{dx}\right)\left(\dfrac{d\psi_n}{dx}\right)}{(\nu_i^2 - 8\varepsilon\omega_n^2)}\right]$$

$$+ \tfrac{1}{2}\,\varepsilon a^3\left[\int_0^1\left(\frac{d\psi_n}{dx}\right)^2\right]\frac{d^2\psi_n}{dx^2}. \tag{B1}$$

One can expand ω_n^2 and ψ_n as in equations (13) and (14) and C_i as

$$C_i = C_i^{(0)} + \varepsilon C_i^{(1)} + \varepsilon^2 C_i^{(2)} \tag{B2}$$

where

$$C_i^{(0)} = \frac{\int_0^1 \dfrac{d\varphi_n}{dx}\,\dfrac{d^2\varphi_n}{dx^2}\,\Gamma_i\,dx}{\int_0^1 \Gamma_i^2\,dx}$$

$$C_i^{(1)} = \frac{\int_0^1 \left[\frac{\mathrm{d}\delta_n^{(1)}}{\mathrm{d}x} \frac{\mathrm{d}^2\varphi_n}{\mathrm{d}x^2} + \frac{\mathrm{d}\varphi_n}{\mathrm{d}x} \frac{\mathrm{d}^2\delta_n^{(1)}}{\mathrm{d}x^2} \right] \Gamma_i \, \mathrm{d}x}{\int_0^1 \Gamma_i^2 \, \mathrm{d}x}$$

$$C_i^{(2)} = \frac{\int_0^1 \left[\frac{\mathrm{d}\delta_n^{(2)}}{\mathrm{d}x} \frac{\mathrm{d}^2\varphi_n}{\mathrm{d}x^2} + \frac{\mathrm{d}\varphi_n}{\mathrm{d}x} \frac{\mathrm{d}^2\delta_n^{(2)}}{\mathrm{d}x^2} + \frac{\mathrm{d}\delta_n^{(1)}}{\mathrm{d}x} \frac{\mathrm{d}^2\delta_n^{(1)}}{\mathrm{d}x^2} \right] \Gamma_i \, \mathrm{d}x}{\int_0^1 \Gamma_i^2 \, \mathrm{d}x}.$$

Now, substituting ω_n^2, ψ_n and C_i into equation (B1) and equating the coefficients of ε^0 and ε^1 from both sides one obtains equations (16a) and (16b) respectively. Similarly equating the coefficients of ε^2 from both sides one gets

$$\begin{aligned} -(\omega_n^l)^2\delta_n^{(2)} + \frac{\mathrm{d}^4\delta_n^{(2)}}{\mathrm{d}x^4} &= \beta_n^{(2)}\varphi_n + \beta_n^{(1)}\delta_n^{(1)} + \tfrac{1}{2} a^2 \left[\int_0^1 \left(\frac{\mathrm{d}\varphi_n}{\mathrm{d}x} \right)^2 \mathrm{d}x \right] \frac{\mathrm{d}^2\delta_n^{(1)}}{\mathrm{d}x^2} \\ &+ a^2 \left[\int_0^1 \left(\frac{\mathrm{d}\delta_n^{(1)}}{\mathrm{d}x} \frac{\mathrm{d}\varphi_n}{\mathrm{d}x} \right) \mathrm{d}x \right] \frac{\mathrm{d}^2\varphi_n}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x} \left[\tfrac{3}{4} a^2 \left(\frac{\mathrm{d}\varphi_n}{\mathrm{d}x} \right)^2 \frac{\mathrm{d}\delta_n^{(1)}}{\mathrm{d}x} \right. \\ &\left. + \tfrac{1}{2} a^2 \sum_{i=1}^{\infty} \left(\frac{8C_1^{(0)}(\omega_n^l)^2}{v_i^4} \frac{\mathrm{d}\Gamma_i}{\mathrm{d}x} \frac{\mathrm{d}\varphi_n}{\mathrm{d}x} + \frac{C_i^{(1)}}{v_i^2} \frac{\mathrm{d}\Gamma_i}{\mathrm{d}x} \frac{\mathrm{d}\varphi_n}{\mathrm{d}x} + \frac{C_i^{(0)}}{v_i^2} \frac{\mathrm{d}\delta_n^{(1)}}{\mathrm{d}x} \frac{\mathrm{d}\Gamma_i}{\mathrm{d}x} \right) \right]. \end{aligned} \tag{B3}$$

For a simply-supported beam, with $\phi_n = \sin n\pi x$ and $\Gamma_n = \sin n\pi x$ one finally obtains

$$\psi_n = \sin n\pi x = \varepsilon^2 \tfrac{3}{160} a^2 (n\pi)^2 \sin 3n\pi x \tag{B4}$$

$$\omega_n^2 = (n\pi)^4 \left[1 = \tfrac{3}{8} \varepsilon a^2 - \tfrac{1}{2} \varepsilon^2 a^2 (n\pi)^2\right]. \tag{B5}$$

Similar solutions can also be obtained for a simple-supported beam after neglecting the longitudinal inertia. In this case, solving equation (28) up to the second order terms one gets

$$\psi_n = \sin n\pi x + o(\varepsilon^3), \quad \omega_n^2 = (n\pi)^4 [1 + \tfrac{3}{8} \varepsilon a^2 + o(\varepsilon^3)]. \tag{B6, B7}$$