Journal of Sound and Vibration (1998) **210**(3), 307–327

POST-CRITICAL BEHAVIOUR OF ORTHOTROPIC CIRCULAR CYLINDRICAL SHELLS UNDER TIME DEPENDENT AXIAL COMPRESSION

R. Mao

Department of Engineering Mechanics, Shanghai Jiao Tong University, Shanghai 200030, *People's Republic of China*

AND

F. W. Williams

*Division of Structural Engineering, School of Engineering, The University of Wales Cardiff, Cardiff CF*2 3*TB, Wales*

(*Received* 15 *October* 1996, *and in final form* 10 *June* 1997)

A non-shallow non-linear shell theory is used to analyze the parametric resonance of orthotropic circular cylindrical shells under harmonically varying axial compression. As special cases, post-buckling and non-linear vibration problems are also studied. In the analysis the non-linear terms and the inertias contributed by both normal displacement w and circumferential displacement v are included. Therefore the final dynamic system includes two equations in w and v. The transverse shear deformation is taken into account by a first order theory. The spatial variables in the governing equations are eliminated by the Galerkin procedure. The final ordinary differential equations are solved by an asymptotic method. Numerical results show the dependence of the post-critical behaviour on the properties of material, geometry and excitation.

1. INTRODUCTION

Circular cylindrical shells under axial compression have long been of considerable interest because of their importance in structural applications. In the static case, when the compressive load P reaches a critical value P_{cr} the shell will buckle due to infinitesimal lateral disturbance, which inevitably exists. In the dynamic case, P is time dependent. If it is assumed to have the form

$$P(t) = P_0 + P_t \cos\theta(t)$$

where P_0 and P_t are positive constants, then

$$f = \mathrm{d}\theta/\mathrm{d}t$$

is the frequency of the load P. In this case, if $P_0 + P_t < P_{cr}$ the shell usually vibrates longitudinally only. However, when the frequency is equal or close to some special values, the shell will vibrate laterally if there is any lateral disturbance, even for $P_0 + P_t \ll P_{cr}$. This phenomenon is called parametric resonance.

Buckling studies determine the critical load. Post-buckling studies investigate how the shell behaves after buckling. Similarly, in the dynamic case, it is necessary to determine when the parametric resonance will occur and then to obtain the so-called stability

0022–460X/98/080307 + 21 $25.00/0/sv971151

boundary. The question as to how the shell will vibrate when resonance occurs is analogous to post-buckling in statics. Hence both post-buckling equilibrium and parametric resonance can be referred to as post-critical behaviours.

In contrast to isotropic shells, the post-critical behaviours of anisotropic shells have not been extensively investigated, especially for the dynamic case. Based on Donnell's shallow shell theory, Von Kármán's kinetic relations and a multi-mode model, Iu and Chia [1, 2] used the Galerkin procedure and the method of harmonic balance to study the non-linear vibration and, as a special case, the post-buckling equilibrium of unsymmetric cross-ply cylindrical shells with various boundary conditions and imperfections. Sun [3] analyzed the buckling and post-buckling behaviour of oval cylindrical shells by using a perturbation method. Reddy and Savoia [4] and Savoia and Reddy [5] treated laminated circular cylindrical shells with a layer-wise shell theory and obtained the post-buckling path in terms of load-shortening curves. Three-dimensional buckling analyses were presented by Kardomateas [6], Dong and Etitum [7] and Ye and Soldatos [8]. Although post-buckling is not involved, these three-dimensional studies are useful for checking the results from shell theories. The finite element method is also a powerful tool for post-buckling analysis as shown, for example, by Goldmanis and Riekstinsh [9].

For the dynamic case, there has been some recent work on the dynamic stability (parametric resonance) of cylindrical shells, but it is mostly limited to the calculation of the stability boundaries, not the resonance behaviour. Bert and Birman [10] used a thick shell theory to obtain the stability boundaries and to show their dependence on geometry and material, while Shaw *et al*. [11] did essentially the same thing for a thin shell theory. Liao and Cheng [12] studied the same problem using a degenerated shell element and a curved beam element, together with the infinite determinant and multi-scale methods. Kovtunov [13] investigated not only the stability boundary but also the resonance behaviour by using the finite element method for isotropic shells under axial compression, which changes with time according to the law of periodical trapezium impulse.

The present paper principally studies the post-critical behaviour of axially compressed cylindrical shells, including parametric resonance and, as its special cases, post-buckling equilibrium and non-linear vibration. As shown by Kardomateas [6], the non-linear term in circumferential displacement v has a significant influence on the buckling load. Therefore, in the present study of the post-critical behaviour, the non-linear terms in v as well as w are retained in the equations of motion. In addition, inertias associated with v and w are retained while those related to u, ψ and ϕ are neglected. As a result, the final dynamic system involves two unknowns, v and w. The material is assumed to be specially orthotropic. This mostly means laminates, for which the transverse shear should not be neglected. The present paper uses a kind of first order theory with a correction factor of 5/6 to account for the transverse shear. As usual, the governing partial differential equations are reduced to ordinary differential equations with respect to time by the Galerkin procedure. These ordinary differential equations, for which the non-linearity involves a third order polynomial in v and w, are solved by an asymptotic method.

2. BASIC EQUATIONS

Using a right-hand axis system, let x, y and z be the co-ordinates in the axial, circumferential and normal directions, respectively, for any point in the shell wall, with the origin at one end of the shell. The first order transverse-shear theory assumes the displacement relationships

$$u_1 = u + z\psi, \qquad u_2 = v + z\phi, \qquad u_3 = w, \tag{1}$$

where u_1, u_2 and u_3 are displacements in x, y and z directions, respectively, u, v and w are the displacements at the middle surface, and ψ and ϕ are the two rotations of the normal to the middle surface. For a cylindrical shell subjected to axial compression, it is only necessary to retain the non-linear terms from the derivatives of v and w in the strain expressions, and the non-linear terms involving the derivatives of u, ψ and ϕ can be neglected. In addition, the usual assumption of small strain is made. Thus

$$\varepsilon_1 = \varepsilon_1^0 + zk_1, \qquad \varepsilon_2 = \varepsilon_2^0 + zk_2, \qquad \varepsilon_6 = \varepsilon_6^0 + zk_6,$$
$$\varepsilon_4 = \varepsilon_4^0, \qquad \varepsilon_5 = \varepsilon_5^0, \tag{2}$$

where

$$\varepsilon_1^0 = \frac{\partial u}{\partial x} + \tfrac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right], \qquad \varepsilon_2^0 = \frac{\partial v}{\partial y} + \frac{w}{R} + \tfrac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{v}{R}\right)^2,$$

$$\varepsilon_6^0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial y} - \frac{v}{R}\right),$$

$$k_1 = \frac{\partial \psi}{\partial x}, \qquad k_2 = \frac{\partial \phi}{\partial y}, \qquad k_6 = \frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x}. \tag{3}$$

Here ε_1, ε_2 and ε_6 are in-plane strains, ε_4 and ε_5 are transverse shear strains, the superscript zero denotes the middle surface values of these strains, and k_1, k_2 and k_6 are the changes of curvature of the middle surface

From equations (1)–(3), the principle of virtual work gives the equations of equilibrium

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} + f_1 = 0, \tag{4}$$

$$\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{Q_2}{R} + \frac{\partial}{\partial x}\left(N_1 \frac{\partial v}{\partial x}\right) + \frac{N_2}{R}\left(\frac{\partial w}{\partial y} - \frac{v}{R}\right) + \frac{N_6}{R}\frac{\partial w}{\partial x} + f_2 = 0, \tag{5}$$

$$\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} - \frac{N_2}{R} + \frac{\partial}{\partial x}\left(N_1 \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left[N_2\left(\frac{\partial w}{\partial y} - \frac{v}{R}\right)\right]$$
$$+ \frac{\partial}{\partial x}\left[N_6\left(\frac{\partial w}{\partial y} - \frac{v}{R}\right)\right] + \frac{\partial}{\partial y}\left(N_6 \frac{\partial w}{\partial x}\right) + f_3 = 0, \tag{6}$$

$$\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - Q_1 + m_1 = 0, \qquad \frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} - Q_2 + m_2 = 0, \tag{7, 8}$$

and the natural boundary conditions at $x = 0$ and $x = L$:

$$N_1 + P(t) = 0 \quad \text{or} \quad u \text{ is given}, \tag{9}$$
$$N_6 = 0 \quad \text{or} \quad v \text{ is given}, \tag{10}$$
$$Q_1 = 0 \quad \text{or} \quad w \text{ is given}, \tag{11}$$
$$M_1 = 0 \quad \text{or} \quad \psi \text{ is given}, \tag{12}$$
$$M_6 = 0 \quad \text{or} \quad \phi \text{ is given}. \tag{13}$$

where R and L are, respectively, the radius and the length of the shell, and (f_1, f_2, f_3) and (m_1, m_2) are, respectively, the forces and the moments per unit area of the middle surface. In addition to equations (9)–(13), the natural boundary conditions also include the condition that N_2, N_6, Q_2, M_2, M_6, u, v, w, ψ and ϕ must be continuous and periodic functions of y with period $2\pi R$. The quantities N_i, M_i and Q_i are stress resultants, defined by

$$(N_i, M_i) = \int_{-T/2}^{T/2} \sigma_i\,(1, z)\,\mathrm{d}z, \qquad i = 1, 2, 6$$

$$Q_2 = \int_{-T/2}^{T/2} \sigma_4\,\mathrm{d}z, \qquad Q_1 = \int_{-T/2}^{T/2} \sigma_5\,\mathrm{d}z,$$

where T is the thickness of the shell and the σ_i are the stresses corresponding to the ε_i $(i = 1, 2, \ldots, 6)$.

In the present case, there are no distributed forces and moments except for the inertia forces. In addition, the inertia forces related to u, ψ and ϕ are assumed to be negligible in the present analysis. Thus

$$f_1 = 0, \qquad m_1 = m_2 = 0, \qquad f_2 = -\mu\ddot{v}, \qquad f_3 = -\mu\ddot{w}, \tag{14}$$

where μ is the mass per unit area of the middle surface and the superscript dots denote the second derivative with respect to time t. Furthermore, in the present problem the load is axial compression only, and therefore the non-linear terms related to N_2 and N_6 in equations (5) and (6) can be omitted. Thus equations (5) and (6) reduce to their final form

$$\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{Q_2}{R} + N_1\frac{\partial^2 v}{\partial x^2} = \mu\ddot{v}, \tag{15}$$

$$\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} - \frac{N_2}{R} + N_1\frac{\partial^2 w}{\partial x^2} = \mu\ddot{w}, \tag{16}$$

These two equations are the same as those obtained by Timoshenko and Gere [14] for isotropic shells, except only that the force N_1 here is no longer constant, and therefore the two dynamic equations (15) and (16) are non-linear.

Equation (4) can be satisfied by introducing a stress function F such that

$$N_1 = \frac{\partial^2 F}{\partial y^2}, \qquad N_6 = \frac{\partial^2 F}{\partial x\,\partial y}. \tag{17}$$

The constitutive equations for specially orthotropic materials can be written in the form

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix}, \qquad \begin{Bmatrix} Q_2 \\ Q_1 \end{Bmatrix} = \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4^0 \\ \varepsilon_5^0 \end{Bmatrix},$$

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} k_1 \\ k_2 \\ k_6 \end{Bmatrix}. \tag{18}$$

Using equations (18), (3) and (17), the force N_2 can be expressed in terms of v, w and F as

$$N_2 = \frac{1}{A_{22}^*}\left[\frac{\partial v}{\partial y} + \frac{w}{R} + \tfrac{1}{2}\left(\frac{\partial v}{\partial y}\right)^2 + \tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - A_{12}^*\frac{\partial^2 F}{\partial y^2}\right], \tag{19}$$

where the matrix $[A_{ij}^*]$ is the inverse of the matrix $[A_{ij}]$. Substituting equations (17) and (19) into (15) and (16) and making use of equations (18) and (3) gives

$$\frac{1}{A_{22}^*}\left[\frac{\partial^2 v}{\partial y^2} + \frac{1}{R}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2} - A_{12}^*\frac{\partial^3 F}{\partial y^3}\right] - \frac{\partial^3 F}{\partial x^2\,\partial y}$$

$$+ \frac{A_{44}}{R}\left(\frac{\partial w}{\partial y} + \phi - \frac{v}{R}\right) + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 v}{\partial x^2} = \mu\ddot{v}, \tag{20}$$

$$A_{55}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \psi}{\partial x}\right) + A_{44}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial \phi}{\partial y} - \frac{1}{R}\frac{\partial v}{\partial y}\right)$$

$$- \frac{1}{RA_{22}^*}\left[\frac{\partial v}{\partial y} + \frac{w}{R} + \tfrac{1}{2}\left(\frac{\partial v}{\partial y}\right)^2 + \tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - A_{12}^*\frac{\partial^2 F}{\partial y^2}\right] + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} = \mu\ddot{w}. \tag{21}$$

Equations (7) and (8) can also be expressed in terms of displacements and rotations, by using the constitutive and kinematic equations, to give

$$\frac{\partial}{\partial x}\left(D_{11}\frac{\partial \psi}{\partial x} + D_{12}\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial y}\left[D_{66}\left(\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x}\right)\right] = A_{55}\left(\frac{\partial w}{\partial x} + \psi\right), \tag{22}$$

$$\frac{\partial}{\partial x}\left[D_{66}\left(\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left(D_{12}\frac{\partial \psi}{\partial x} + D_{22}\frac{\partial \phi}{\partial y}\right) = A_{44}\left(\frac{\partial w}{\partial y} + \phi - \frac{v}{R}\right). \tag{23}$$

Since the strains are non-linear in the displacements, the compatibility equations for the stresses are also non-linear and have the form

$$\left(A_{11}^* - \frac{A_{12}^{*2}}{A_{22}^*}\right)\frac{\partial^4 F}{\partial y^4} + A_{66}^*\frac{\partial^4 F}{\partial x^2\,\partial y^2} + \frac{\partial^3 v}{\partial x^2\,\partial y} + \frac{A_{12}^*}{A_{22}^*}\left(\frac{\partial^3 v}{\partial y^3} + \frac{1}{R}\frac{\partial^2 w}{\partial y^2}\right)$$

$$+ \frac{\partial v}{\partial y}\frac{\partial^3 v}{\partial x^2\,\partial y} + \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x^2\,\partial y} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

$$+ \frac{A_{12}^*}{A_{22}^*}\left[\left(\frac{\partial^2 v}{\partial y^2}\right)^2 + \frac{\partial v}{\partial y}\frac{\partial^3 v}{\partial y^3} + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial y^3}\right] = 0. \tag{24}$$

To summarize, the five equations (20)–(24) for the five unknowns, v, w, ψ, ϕ and F are the basic equations for the present analysis.

3. REDUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

To solve the basic equations, F is expanded as the series

$$F = \sum_{m,n=1}^{N} f_{mn} \sin \alpha_m x \cos \beta_n y - P(t) y^2 \tag{25}$$

and the following single-wave modes are used for v, w, ψ and ϕ:

$$v = V(t) \sin \alpha_p x \sin \beta_q y, \qquad w = W(t) \sin \alpha_p x \cos \beta_q y,$$
$$\psi = \Psi(t) \cos \alpha_p x \cos \beta_q y, \qquad \phi = \Phi(t) \sin \alpha_p x \sin \beta_q y, \tag{26}$$

where

$$\alpha_i = i\pi/L, \qquad \beta_j = j/R, \qquad i = m \text{ or } p, j = n \text{ or } q.$$

Equations (25) and (26) satisfy the simply supported boundary conditions at $x = 0$ and $x = L$, namely

$$N_1 = -P(t), \qquad v = w = 0, \qquad M_1 = 0, \qquad \phi = 0 \tag{27}$$

and have the required continuity and periodicity with respect to y.

Substitution of equations (25) and (26) into equations (20), (21) and (24) yields, after application of the Galerkin procedure,

$$\mu \ddot{V} + \left[\frac{\beta_q^2}{A_{22}^*} + \frac{A_{44}}{R^2} - \alpha_p^2 P(t)\right] V + \frac{\beta_q}{R}\left(\frac{1}{A_{22}^*} + A_{44}\right) W - \frac{A_{44}}{R} \Phi$$
$$+ \sum_{m,n=1}^{N} f_{mn} \left[\left(\alpha_m^2 \beta_n + \beta_n^3 \frac{A_{12}^*}{A_{22}^*}\right) \delta_{mp} \delta_{nq} + V \frac{\alpha_p^2 \beta_n^2}{2} \frac{\eta_{m0} - \eta_{m,2p}}{2} \delta_{n,2q}\right] = 0, \tag{28}$$

$$\mu \ddot{W} + \frac{\beta_q}{R}\left(\frac{1}{A_{22}^*} + A_{44}\right) V + \left[\alpha_p^2 A_{55} + \beta_q^2 A_{44} + \frac{1}{R^2 A_{22}^*} - \alpha_p^2 P(t)\right] W + \alpha_p A_{55} \Psi - \beta_q A_{44} \Phi$$
$$+ \sum_{m,n=1}^{N} f_{mn} \left[\beta_n^2 \frac{A_{12}^*}{R A_{22}^*} \delta_{mp} \delta_{nq} - W \frac{\alpha_p^2 \beta_n^2}{2} \frac{\eta_{m0} - \eta_{m,2p}}{2} \delta_{n,2q}\right] = 0, \tag{29}$$

$$Z_{ij} f_{ij} = G_{pq}^v V + G_{pq}^w W + H_{pq}^v V^2 + H_{pq}^w W^2, \tag{30}$$

where δ is the Kronecker delta and

$$Z_{ij} = \beta_j^4 \left(A_{11}^* - \frac{A_{12}^{*2}}{A_{22}^*}\right) + \alpha_i^2 \beta_j^2 A_{66}^*$$

$$G_{pq}^v = \left(\alpha_p^2 \beta_q + \beta_q^3 \frac{A_{12}^*}{A_{22}^*}\right) \delta_{ip} \delta_{jq}, \qquad G_{pq}^w = \frac{\beta_q^2}{R} \frac{A_{12}^*}{A_{22}^*} \delta_{ip} \delta_{jq},$$

$$H^v_{pq} = -H^w_{pq} = \left(\alpha_p^2\,\beta_q^2 + \beta_q^4\,\frac{A^*_{12}}{A^*_{22}}\right)\frac{\eta_{i0} - \eta_{i,2p}}{2}\,\delta_{i,2q},$$

$$\eta_{ij} = \frac{2i[1-(-1)^{i+j}]}{(i^2-j^2)\pi}.$$

Substitution of equations (25) and (26) into equations (22) and (23) yields

$$(\alpha_p^2\,D_{11} + \beta_q^2\,D_{66} + A_{55})\Psi - \alpha_p\,\beta_q\,(D_{12}+D_{66})\Phi = -\alpha_p\,A_{55}\,W, \tag{31}$$

$$-\alpha_p\,\beta_q\,(D_{12}+D_{66})\Psi + (\alpha_p^2\,D_{66} + \beta_q^2\,D_{22} + A_{44})\Phi = A_{44}\left(\frac{V}{R} + \beta_q\,W\right), \tag{32}$$

directly, without the Galerkin procedure being used.

Equations (31) and (32) can be solved to give Ψ and Φ in terms of V and W. The form of the solution is

$$\Psi = e_{11}\,V + e_{12}\,W, \qquad \Phi = e_{12}\,V + e_{22}\,W. \tag{33, 34}$$

From equation (30) f_{ij} can be found as

$$f_{ij} = (G^v_{pq}\,V + G^w_{pq}\,W + H^v_{pq}\,V^2 + H^w_{pq}\,W^2)/Z_{ij}. \tag{35}$$

Substitution of equations (33)–(35) into equations (28) and (29) gives two ordinary differential equations in terms of V and W; i.e.,

$$\begin{aligned}\mu\ddot{V} &+ \left[\frac{\beta_q^2}{A^*_{22}} + \frac{A_{44}}{R^2} - \alpha_p^2\,P(t)\right]V + \frac{\beta_q}{R}\left(\frac{1}{A^*_{22}} + A_{44}\right)W - \frac{A_{44}}{R}\,\Phi \\ &+ \frac{1}{Z_{pq}}\,(G^v_{pq}\,V + G^w_{pq}\,W + H^v_{pq}\,V^2 + H^w_{pq}\,W^2)\left(\alpha_p^2\,\beta_q + \beta_q^3\,\frac{A^*_{12}}{A^*_{22}}\right) \\ &+ \sum_{m=1}^{N}\frac{V}{Z_{m,2q}}\,(G^v_{m,2q}\,V + G^w_{m,2q}\,W + H^v_{m,2q}\,V^2 + H^w_{m,2q}\,W^2)\,\frac{\alpha_p^2\,\beta_{2q}^2}{2}\,\frac{\eta_{m0}-\eta_{m,2p}}{2} = 0,\end{aligned} \tag{36}$$

$$\begin{aligned}\mu\ddot{W} &+ \frac{\beta_q}{R}\left(\frac{1}{A^*_{22}} + A_{44}\right)V + \left[\alpha_p^2\,A_{55} + \beta_q^2\,A_{44} + \frac{1}{R^2 A^*_{22}} - \alpha_p^2\,P(t)\right]W + \alpha_p\,A_{55}\,\Psi - \beta_q\,A_{44}\,\Phi \\ &+ \frac{1}{Z_{pq}}\,(G^v_{pq}\,V + G^w_{pq}\,W + H^v_{pq}\,V^2 + H^w_{pq}\,W^2)\beta_q^2\,\frac{A^*_{12}}{RA^*_{22}} \\ &- \sum_{m=1}^{N}\frac{W}{Z_{m,2q}}\,(G^v_{m,2q}\,V + G^w_{m,2q}\,W + H^v_{m,2q}\,V^2 + H^w_{m,2q}\,W^2)\,\frac{\alpha_p^2\,\beta_{2q}^2}{2}\,\frac{\eta_{m0}-\eta_{m,2p}}{2} = 0.\end{aligned} \tag{37}$$

4. INITIAL POST-BUCKLING

Post-buckling behaviour can be obtained from equations (36) and (37) as a special case. Instead of starting directly from equations (36) and (37), the more general non-linear dynamic system,

$$\mu\ddot{u}_i + \sum_{j=1}^{N} [d_{ij}\,\dot{u}_j + (a_{ij} - \alpha^2 P(t)\delta_{ij})u_j] + \sum_{j,k=1}^{N} b_{ijk}\,u_j\,u_k + \sum_{j,k,l=1}^{N} c_{ijkl}\,u_j\,u_k\,u_l = 0,$$

$$i = 1, 2, \dots, N, \tag{38}$$

is considered, where u_i are the generalized displacements of the system, $[a_{ij}]$ is a symmetric and postive definite matrix, and $d_{ij}\,\dot{u}_j$ is the equivalent viscous damping. The coefficients μ, α, d_{ij}, a_{ij}, b_{ijk} and c_{ijkl} are all constant. It is obvious that equations (36) and (37) are a special case of equation (38) for $N = 2$ and $d_{ij} = 0$.

Let the matrix $[a_{ij}]$ have distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_N$, for which the normalized eigenvectors are

$$\{x_{1j}, x_{2j}, \dots, x_{Nj}\}^{\mathrm{T}}, \qquad j = 1, 2, \dots, N.$$

By introducing a new set of generalized displacements w_j such that

$$u_i = \sum_{j=1}^{N} x_{ij}\,w_j, \tag{39}$$

equations (38) can be transformed into

$$\mu\ddot{w}_m + (\lambda_m - \alpha^2 P(t))w_m + \sum_{n=1}^{N}\left(\sum_{i,j=1}^{N} x_{im}\,x_{jn}\,d_{ij}\right)\dot{w}_n$$

$$+ \sum_{n,s=1}^{N}\left(\sum_{i,j,k=1}^{N} x_{im}\,x_{jn}\,x_{ks}\,b_{ijk}\right)w_n\,w_s + \sum_{n,s,t=1}^{N}\left(\sum_{i,j,k,l=1}^{N} x_{im}\,x_{jn}\,x_{ks}\,x_{lt}\,c_{ijkl}\right)w_n\,w_s\,w_t = 0,$$

$$m = 1, 2, \dots, N. \tag{40}$$

(Note that w_i is not related to the physical displacement w in the z direction).

Two special cases are now considered:

(1) *Linear static equilibrium*. In this case equations (40) reduce to the system of equations of equilibrium

$$(\lambda_m - \alpha^2 P)w_m = 0, \qquad m = 1, 2, \dots, N, \tag{41}$$

where P is a constant. From equations (41) the buckling load can be obtained as

$$P_{cr} = \lambda_1/\alpha^2, \tag{42}$$

from which a dimensionless load $\bar{P}$ can be defined as

$$\bar{P}(t) = \frac{P(t)}{P_{cr}} = \frac{P_0}{P_{cr}} + \frac{P_t}{P_{cr}}\cos\theta(t) = \bar{P}_0 + \bar{P}_t\cos\theta(t). \tag{43}$$

(2) *Linear free vibration*. In this case, equations (40) become

$$\mu\ddot{w}_m + \lambda_m\,w_m = 0, \qquad m = 1, 2, \dots, N, \tag{44}$$

which give the natural frequencies ω_m as

$$\omega_m^2 = \lambda_m/\mu, \qquad m = 1, 2, \dots, N. \tag{45}$$

By using the lowest frequency ω_1, a dimensionless time $\bar{t}$ can be defined as

$$\bar{t} = \omega_1\,t. \tag{46}$$

Introducing $\bar{P}(t)$, $\bar{t}$ and dimensionless displacements $v_m = w_m/T$ into equations (40) gives

$$v''_m + \Omega_m^2 v_m = \bar{P}_t \cos\theta(t) v_m - \sum_{n=1}^{N} \bar{d}_{mn} v'_n - \sum_{n,s=1}^{N} \bar{b}_{mns} v_n v_s - \sum_{n,s,t=1}^{N} \bar{c}_{mnst} v_n v_s v_t,$$

$$m = 1, 2, \ldots, N, \tag{47}$$

where the prime denotes the derivative with respect to $\bar{t}$, and Ω_m^2, $\bar{d}_{mn}$, $\bar{b}_{mns}$ and $\bar{c}_{mnst}$ are the dimensionless quantities

$$\Omega_m^2 = \frac{\lambda_m}{\lambda_1} - \bar{P}_0, \qquad \bar{d}_{mn} = \frac{T}{\lambda_1} \sum_{i,j=1}^{N} x_{im} x_{jn} d_{ij},$$

$$\bar{b}_{mns} = \frac{T^2}{\lambda_1} \sum_{i,j,k=1}^{N} x_{im} x_{jn} x_{ks} b_{ijk}, \qquad \bar{c}_{mnst} = \frac{T^3}{\lambda_1} \sum_{i,j,k,l=1}^{N} x_{im} x_{jn} x_{ks} x_{lt} c_{ijkl}.$$

(Note that v_m is not related to the physical displacement v in the y direction.)

For post-buckling the dynamic terms in equations (47) should be omitted to give

$$\left(\frac{\lambda_m}{\lambda_1} - \bar{P}_0\right) v_m + \sum_{n,s=1}^{N} \bar{b}_{mns} v_n v_s + \sum_{n,s,t=1}^{N} \bar{c}_{mnst} v_n v_s v_t = 0, \qquad m = 1, 2, \ldots, N. \tag{48}$$

The method of Rik [15] is now used to study equations (48). Consider a more general case in which the equilibrium is governed by equations of the form

$$f_m(\bar{P}_0, \vec{v}) = 0, \qquad m = 1, 2, \ldots, N, \tag{49}$$

with the condition

$$f_{m,n} = f_{n,m},$$

where $\vec{v}$ denotes $(v_1, v_2, \ldots, v_N)$, and $f_{m,n} = \partial f_m/\partial v_n$. Equations (49) determine the path of equilibrium in the $(N+1)$-dimensional space S composed of $\bar{P}_0$ and $\vec{v}$. The theorem of existence of implicit functions states that if

$$\det|f_{m,n}(\bar{P}_0, \vec{v})| \neq 0 \tag{50}$$

at a point in the space S, then equations (49) uniquely determine v_n $(n = 1, 2, \ldots, N)$ as single-valued functions of $\bar{P}_0$ in the neighbourhood of that point. A point at which

$$\det|f_{m,n}(\bar{P}_0, \vec{v})| = 0 \tag{51}$$

is called a point of singularity. A point of singularity may be a turning point (limit point), where the tangent to the equilibrium path is perpendicular to the direction of $\bar{P}_0$, or a bifurcating point, where the path bifurcates and has more than one tangent. The bifurcation is of the most interest.

For equations (48), the straight line $v_m = 0$ $(m = 1, 2, \ldots, N)$, i.e., the $\bar{P}_0$-axis, is obviously a path of equilibrium, called the fundamental path. The points $\bar{P}_0 = \lambda_m/\lambda_1$ $(m = 1, 2, \ldots, N)$ on the $\bar{P}_0$-axis are bifurcations. The bifurcation corresponding to the smallest λ_m/λ_1 is called the critical point, denoted by $(\bar{P}_0^c, \vec{v}^c)$. The solution to the eigenvalue problem at the critical point is called the buckling mode, denoted by $\vec{v}^*$. At each bifurcation, the path bifurcates into two branches, of which one is the fundamental path and the other is the post-buckling path. Here the tangent to the post-buckling path at the critical point, called the initial post-buckling slope, is the quantity required.

It is obvious from equations (49) that the direction vector $(\mathrm{d}\overline{P}_0/\mathrm{d}s, \mathrm{d}\vec{v}/\mathrm{d}s)$, of a tangent to the post-buckling path should satisfy

$$\frac{\mathrm{d}}{\mathrm{d}s} f_m(\overline{P}_0, \vec{v}) = \frac{\partial f_m}{\partial \overline{P}_0}\frac{\mathrm{d}\overline{P}_0}{\mathrm{d}s} + \sum_{i=1}^{N} f_{m,i}\frac{\mathrm{d}v_i}{\mathrm{d}s}, \tag{52}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} f_m(\overline{P}_0, \vec{v}) = \frac{\partial^2 f_m}{\partial s \partial \overline{P}_0}\frac{\mathrm{d}\overline{P}_0}{\mathrm{d}s} + \frac{\partial f_m}{\partial \overline{P}_0}\frac{\mathrm{d}^2\overline{P}_0}{\mathrm{d}s^2} + \sum_{i=1}^{N}\left(\frac{\partial f_{m,i}}{\partial s}\frac{\mathrm{d}v_i}{\mathrm{d}s} + f_{m,i}\frac{\mathrm{d}^2 v_i}{\mathrm{d}s^2}\right) = 0,$$

$$m = 1, 2, \ldots, N, \tag{53}$$

where s is the length of the post-buckling path. Using equations (51)–(53), Rik [15] gives the relationship between the tangential vectors to the fundamental path, $(\mathrm{d}\overline{P}_0/\mathrm{d}s, \mathrm{d}\vec{v}/\mathrm{d}s)_1$, and to the post-buckling path, $(\mathrm{d}\overline{P}_0/\mathrm{d}s, \mathrm{d}\vec{v}/\mathrm{d}s)_2$, at the bifurcation as

$$(\mathrm{d}\overline{P}_0/\mathrm{d}s)_2 = \alpha\beta(\mathrm{d}\overline{P}_0/\mathrm{d}s)_1, \qquad (\mathrm{d}\vec{v}/\mathrm{d}s)_2 = \alpha(\beta(\mathrm{d}\vec{v}/\mathrm{d}s)_1 + \vec{v}^*), \tag{54}$$

where

$$\beta = -\tfrac{1}{2}\left(\sum_{i,j,k=1}^{N} f_{i,jk}\, v_i^*\, v_j^*\, v_k^*\right)\Bigg/\left(\sum_{i,j,k=1}^{N} f_{i,jk}\, v_i^*\, v_j^*\left(\frac{\mathrm{d}v_k}{\mathrm{d}s}\right)_1 + \sum_{i,j=1}^{N}\frac{\partial f_{i,j}}{\partial \overline{P}_0}\frac{\mathrm{d}\overline{P}_0}{\mathrm{d}s}\, v_i^*\, v_j^*\right),$$

$$\alpha = \left(\beta^2 + 2\beta\sum_{i=1}^{N} v_i^*\,(\mathrm{d}v_i/\mathrm{d}s)_1 + 1\right)^{-0\cdot 5}. \tag{55}$$

For equations (48),

$$(\overline{P}_0^c, \vec{v}^c) = (1, 0, \ldots, 0), \quad \vec{v}^* = (1, 0, \ldots, 0), \quad ((\mathrm{d}\overline{P}_0/\mathrm{d}s)_1, (\mathrm{d}\vec{v}/\mathrm{d}s)_1) = (1, 0, \ldots, 0) \tag{56}$$

and therefore

$$\beta = \frac{f_{1,11}}{2} = \overline{b}_{111}, \qquad \alpha = \frac{1}{\sqrt{1+\beta^2}} = \frac{1}{\sqrt{1+\overline{b}_{111}^2}},$$

$$\left(\frac{\mathrm{d}\overline{P}_0}{\mathrm{d}s}, \frac{\mathrm{d}\vec{v}}{\mathrm{d}s}\right)_2 = \frac{1}{\sqrt{1+\overline{b}_{111}^2}}(\overline{b}_{111}, 1, 0, \ldots, 0). \tag{57}$$

From the last of these equations, it can be concluded that the tangent to the post-buckling path of the system of equations (48) at the critical point is in the plane of $(\overline{P}_0, v_1)$ and has slope

$$\partial\overline{P}_0/\partial v_1 = \overline{b}_{111}. \tag{58}$$

For many structures, $b_{ijk} = 0$ for all i, j and k. Then the second order derivatives $\mathrm{d}^2\overline{P}_0/\mathrm{d}s^2$ and $\mathrm{d}^2\vec{v}/\mathrm{d}s^2$ are needed to describe the initial post-buckling behaviour. These second order derivatives can be obtained from equations (53) under the condition of normalization,

$$(\mathrm{d}\overline{P}_0/\mathrm{d}s)^2 + \sum_{i=1}^{N}(\mathrm{d}v_i/\mathrm{d}s)^2 = 1,$$

that can be differentiated to give

$$\frac{\mathrm{d}\bar{P}_0}{\mathrm{d}s}\frac{\mathrm{d}^2\bar{P}_0}{\mathrm{d}s^2} + \sum_{i=1}^{N} \frac{\mathrm{d}v_i}{\mathrm{d}s}\frac{\mathrm{d}^2 v_i}{\mathrm{d}s^2} = 0. \tag{59}$$

Equations (53) and (59) are valid at all points on the post-buckling path, including the critical point at which, due to equation (56) and the condition $b_{ijk} = 0$,

$$\left(\frac{\mathrm{d}\bar{P}_0}{\mathrm{d}s}, \frac{\mathrm{d}\vec{v}}{\mathrm{d}s}\right) = \left(\frac{\mathrm{d}\bar{P}_0}{\mathrm{d}s}, \frac{\mathrm{d}\vec{v}}{\mathrm{d}s}\right)_2 = (0, 1, 0, \ldots, 0). \tag{60}$$

Substitution of equation (60) into equation (59) yields

$$\mathrm{d}^2 v_1/\mathrm{d}s^2 = 0. \tag{61}$$

Then, for $m = 1$, equation (53) gives

$$\frac{\mathrm{d}^2\bar{P}_0}{\mathrm{d}s^2} = 6\bar{c}_{1111}, \tag{62}$$

and for the other values of m equations (53) give

$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}s^2} = 0, \qquad i = 2, 3, \ldots, N. \tag{63}$$

By making use of equations (60) and (61) and the definition of $\vec{v}$ beneath equations (49), it can be shown that

$$\frac{\mathrm{d}^2\bar{P}_0}{\mathrm{d}s^2} = \frac{\partial^2\bar{P}_0}{\partial v_1^2}.$$

Hence equation (62) yields

$$\frac{\partial^2\bar{P}_0}{\partial v_1^2} = 6\bar{c}_{1111}. \tag{64}$$

Equations (58) and (64) completely describe the initial post-buckling behaviour of the system of equations (48).

5. PARAMETRIC RESONANCE

The asymptotic method of Evan-Iwanowski [16] is now used to solve the dynamic system of equations (47) for parametric resonance. The detailed derivation of reference [16] is not repeated here. Instead, the relevant results are cited without much derivation.

Let $F_m(\theta, \vec{v}, \vec{v}')$ denote the right sides of equations (47). Consider the following equations, in which ε is a small parameter:

$$v_m'' + \Omega_m^2 v_m = \varepsilon F_m(\theta, \vec{v}, \vec{v}'), \qquad m = 1, 2, \ldots, N. \tag{65}$$

Once equations (65) are solved, setting $\varepsilon = 1$ gives the solution to equations (47). The asymptotic method seeks a solution to equations (65) with the form, for $m = 1, 2, \ldots, N$.

$$v_m = a_m(\tau)\cos\psi_m(\tau) + \sum_{i=1}^{M} \varepsilon^i U_{mi}(\theta, \vec{a}, \vec{\psi}), \tag{66}$$

$$a'_m(\tau) = \sum_{i=1}^{M} \varepsilon^i A_{mi}(\theta, \vec{a}, \vec{\psi}), \tag{67}$$

$$\psi'_m(\tau) = \Omega_m + \sum_{i=1}^{M} \varepsilon^i B_{mi}(\theta, \vec{a}, \vec{\Psi}), \tag{68}$$

where $\tau = \varepsilon \bar{t}$ is called slow time and $\vec{a}$ and $\vec{\psi}$ denote $(a_1, a_2 \cdots, a_N)$ and $(\psi_1, \psi_2 \cdots, \psi_N)$, respectively. θ, $\vec{a}$ and $\vec{\psi}$ are all functions of time τ or $\bar{t}$, which is omitted on many occasions for simplicity of presentation. The functions U_{mi}, A_{mi} and B_{mi} can be determined by substituting equations (66)–(68) into (65), equating similar terms at the two sides of each equation and eliminating secular terms. In the following attention is confined to the case of $M = 1$, to obtain a first order asymptotic solution.

For the first order asymptotic solution of equations (47), $v_m = a_m \cos\psi_m$ is substituted into their right sides and the square and cubic terms are expanded to obtain

$$\begin{aligned} v''_m + \Omega_m^2 v_m = \frac{\bar{P}_t}{2} a_m [\cos(\theta + \psi_m) + \cos(\theta - \psi_m) + \sum_{n=1}^{N} \bar{d}_{mn} a_n \Omega_n \sin\psi_n \\ - \tfrac{1}{2} \sum_{n,s=1}^{N} \bar{b}_{mns} a_n a_s [\cos(\psi_n + \psi_s) + \cos(\psi_n - \psi_s)] \\ - \tfrac{1}{4} \sum_{n,s,t=1}^{N} \bar{c}_{mnst} a_n a_s a_t [\cos(\psi_n + \psi_s + \psi_t) + \cos(\psi_n + \psi_s - \psi_t) \\ + \cos(\psi_n - \psi_s + \psi_t) + \cos(\psi_n - \psi_s - \psi_t)], \\ m = 1, 2, \ldots, N. \end{aligned} \tag{69}$$

Then, the coefficients A_{m1}, B_{m1} and U_{m1} needed by the first order solution to equations (66)–(68) can be obtained from the coefficients of equation (69) by using the formulas given by Evan-Iwanowski [16]. This involves two different cases, which are dealt with separately in the next two subsections.

5.1. NON-RESONANCE CASE

For simplicity of presentation, the first equation (with $m = 1$) of (47) is taken as a sample. By using equations (2.23) and (2.24) of reference [16] for equation (69), it is found that

$$A_{11} = -\tfrac{1}{2} \bar{d}_{11} \tag{70}$$

$$\begin{aligned} B_{11} = \frac{3a_1^2}{8\Omega_1} \bar{c}_{1111} + \frac{1}{4\Omega_1} [(\bar{c}_{1122} + \bar{c}_{1221} + \bar{c}_{1212}) a_2^2 + (\bar{c}_{1133} + \bar{c}_{1331} + \bar{c}_{1313}) a_3^2 \\ + \cdots + (\bar{c}_{11NN} + \bar{c}_{1NN1} + \bar{c}_{1N1N}) a_N^2], \end{aligned} \tag{71}$$

$$\begin{aligned} U_{11} = \tfrac{1}{2} \bar{P}_t a_1 \left[\frac{\cos(\theta + \psi_1)}{\bar{f}(\bar{f} + 2\Omega_1)} + \frac{\cos(\theta - \psi_1)}{\bar{f}(\bar{f} - 2\Omega_1)} \right] + \sum_j \bar{d}_{1j} \frac{\Omega_j a_j \sin\psi_j}{\Omega_1^2 - \Omega_j^2} \\ - \tfrac{1}{2} \sum_{j,k} \bar{b}_{1jk} a_j a_k \left[\frac{\cos(\psi_j + \psi_k)}{\Omega_1^2 - (\Omega_j + \Omega_k)^2} + \frac{\cos(\psi_j - \psi_k)}{\Omega_1^2 - (\Omega_j - \Omega_k)^2} \right] \end{aligned}$$

$$-\tfrac{1}{4}\sum_{j,k,l}\bar{c}_{1jkl}\,a_j\,a_k\,a_l\left[\frac{\cos(\psi_j+\psi_k+\psi_l)}{\Omega_1^2-(\Omega_j+\Omega_k+\Omega_l)^2}+\frac{\cos(\psi_j+\psi_k-\psi_l)}{\Omega_1^2-(\Omega_j+\Omega_k-\Omega_l)^2}\right.$$

$$\left.+\frac{\cos(\psi_j-\psi_k+\psi_l)}{\Omega_1^2-(\Omega_j-\Omega_k+\Omega_l)^2}+\frac{\cos(\psi_j-\psi_k-\psi_l)}{\Omega_1^2-(\Omega_j-\Omega_k-\Omega_l)^2}\right], \tag{72}$$

where $\bar{f}=\mathrm{d}\theta/\mathrm{d}\bar{t}=f/\omega_1$ is the dimensionless frequency of the excitation, and the subscripts j, k and l in the summations cover 1 through N, except that those which may cause a denominator to be identical to zero are omitted. Substitution of equations (70) and (71) into (67) and (68) with $m=1$ and $\varepsilon=1$ yields

$$a_1' = -\tfrac{1}{2}\bar{d}_{11}\,a_1, \qquad \psi_1' = \Omega_1 + B_{11}. \tag{73, 74}$$

Equation (73) can be solved independently to give

$$a_1 = a_{10}\exp(-\tfrac{1}{2}\bar{d}_{11}\,\bar{t}), \tag{75}$$

where a_{10} is a constant determined by the initial conditions. Equation (74) is coupled with the vibrations of other generalized displacements, and can be solved together with its counterparts for $v_2, v_3, \ldots, v_N$ without any difficulty.

As proved by Evan-Iwanowski [16], in the non-resonance case, U_{11} in equation (66) is negligible compared with $a_1\cos\psi_1$. After U_{11} is omitted, the first order solution becomes

$$v_1 = a_{10}\exp(-\tfrac{1}{2}\bar{d}_{11}\,\bar{t})\cos\psi_1 \tag{76}$$

which is called the zeroth order solution. The exciting force $\bar{P}_t$ does not appear in equation (76) because it contributes only to the small term U_{11}. Therefore, equation (76) is also a zeroth order solution for the non-linear free vibration of a cylindrical shell subjected to constant axial compression, and equation (74) represents its non-linear frequency. From equations (74) and (71) it can be seen that the non-linear vibrations of the different generalized displacements v_m of system (47) are coupled with each other through their non-linear frequencies.

Now suppose that only v_1 vibrates in the free vibration. This can be achieved, for example, by setting the initial conditions such that

$$a_{10}\neq 0, \quad a_{m0}=0 \qquad \text{for} \quad m=2,3,\ldots,N.$$

For such a mode, equations (74) and (71) reduce to the simple equation for non-linear frequency

$$\frac{\psi_1'}{\Omega_1} = 1 + \tfrac{3}{8}\,\bar{c}_{1111}\,\frac{a_1^2}{\Omega_1^2}, \tag{77}$$

which is similar to that given by Atluri [17].

5.2. RESONANCE CASE

For some structures and loads, some of the denominators in equation (72) may be equal or close to zero. Then U_{11} is no longer negligible and may become infinite, so that resonance occurs. For example, when $\bar{f}=2\Omega_1$, the second denominator in equation (72) becomes zero, and the resulting resonance is called the main resonance and is a most important one.

For resonance, the above solution procedure breaks down, and a different procedure is given by equations (2.28) of reference [16] as

$$v_1 = a_1 \cos \psi_1$$

$$a_1' = A_{11} + \frac{\overline{P}_t\, a_1}{2\overline{f}} \sin(\theta - 2\psi_1), \qquad \psi_1' = B_{11} - \frac{\overline{P}_t}{2\overline{f}} \cos(\theta - 2\psi_1).$$

Usually ψ_1 is replaced by a new variable ϕ_1 such that

$$\psi_1 = \frac{\theta}{2} + \phi_1,$$

so that the above equations become

$$v_1 = a_1 \cos\left(\frac{\theta}{2} + \phi_1\right), \tag{78}$$

$$a_1' = A_{11} - \frac{\overline{P}_t\, a_1}{2\overline{f}} \sin 2\phi_1, \qquad \phi_1' = \Omega_1 - \frac{\overline{f}}{2} + B_{11} - \frac{\overline{P}_t}{2\overline{f}} \cos 2\phi_1. \tag{79, 80}$$

In the following, discussion is confined to the stationary case, for which

$$\overline{f} = \text{constant}, \qquad a_1' = 0, \qquad \phi_1' = 0. \tag{81}$$

Substituting equations (81) into (79) and (80), and using equations (70) and (71) with $a_2, a_3, \ldots, a_N$ having died out, gives a_1 as a function of $\overline{f}$ and other parameters, i.e.,

$$a_1 = \left[\frac{8\Omega_1}{3\bar{c}_{1111}}\left(\frac{\overline{f}}{2} - \Omega_1 \pm \tfrac{1}{2}\left(\frac{\overline{P}_t^2}{\overline{f}^2} - \overline{d}_{11}\right)^{1/2}\right)\right]^{1/2}, \tag{82}$$

which is an important description of the behaviour of the main resonance.

Inspection of equation (82) shows that, for given Ω_1, $\overline{P}_t$ and $\overline{d}_{11}$, there are certain frequencies $\overline{f}_1$, $\overline{f}_2$ and $\overline{f}_3$ on the $\overline{f}$-axis such that one of (A), (B) and (C) below occurs.

(A) For $\overline{f} \leqslant \overline{f}_1$ or $\overline{f} \geqslant \overline{f}_3$, a_1 is zero or imaginary, so that resonance does not occur and the system is stable; i.e., any disturbance to the system will die out exponentially.

(B) For $\overline{f}_1 < \overline{f} \leqslant \overline{f}_2$, a_1 has one non-zero real root, so that resonance occurs and the system is unstable; i.e., any disturbance of the system will develop into a periodic vibration with amplitude equal to a_1.

(C) For $\overline{f}_2 < \overline{f} < \overline{f}_3$, a_1 has two non-zero real roots, and calculation shows that the system is unstable for large disturbances; i.e., a large disturbance will develop into a periodic vibration with its amplitude given by the larger root, whereas small disturbances will die out exponentially. Therefore this case is called conditionally stable.

It can be concluded that $\overline{f}_1$, $\overline{f}_2$ and $\overline{f}_3$ represent the boundaries between stability and instability and are of great interest. In the next section, these boundaries are considered in more detail by examining examples.

In subsections 5.1 and 5.2 we have considered only the vibration of v_1. Similar results can be obtained for the vibrations of $v_2, v_3, \ldots, v_N$.

6. NUMERICAL RESULTS

6.1. POST-BUCKLING

To check the analysis and the numerical procedures developed in the previous sections, buckling loads were calculated for an example given in reference [6] which consists of a cylinder with length L, outer radius R_2, inner radius R_1 and $L/R_2 = 5$. The material is orthotropic with moduli (in GPa) of $E_{22} = 57$, $E_{11} = E_{33} = 14$, $G_{31} = 5{\cdot}0$ and $G_{12} = G_{23} = 5{\cdot}7$, and with Poisson ratios $\upsilon_{23} = 0{\cdot}277$ and $\upsilon_{31} = 0{\cdot}4$. The normalized buckling load used in reference [6] is

$$\hat{P} = \frac{Q_{cr}}{\pi(R_2^2 - R_1^2)} \frac{R_2}{E_{33}\, T},$$

where $T = R_2 - R_1$ and Q_{cr} is the buckling load. The two smallest [6] values of R_2/R_1 have been selected for comparison. The first of these is $R_2/R_1 = 1{\cdot}05$, for which $\hat{P}_p = 0{\cdot}6920$, $\hat{P}_e = 0{\cdot}6764$, $\hat{P}_D = 0{\cdot}7904$, $(p, q)_p = (1, 2)$, $(p, q)_e = (1, 2)$ and $(p, q)_D = (9, 4)$. The second one is $R_2/R_1 = 1{\cdot}10$, for which $\hat{P}_p = 0{\cdot}6602$, $\hat{P}_e = 0{\cdot}6641$, $\hat{P}_D = 0{\cdot}7883$, $(p, q)_p = (2, 2)$, $(p, q)_e = (2, 2)$ and $(p, q)_D = (6, 3)$. Here the subscripts p, e and D denote the present theory, elasticity theory and Donnell's shallow shell theory, respectively, with the results for the last two theories being taken from reference [6]. p and q are the wavenumbers along the length and round the circumference of the cylinder, respectively. The comparisons help to check the correctness of the present theory and numerical procedures, and also show that the non-shallow shell theory gives better accuracy than the shallow shell theory.

Another check is with the results of Mao and Ling [18] for a laminated cylindrical shell, with $R = 0{\cdot}075$ m, $T = 5 \times 0{\cdot}0015$ m and $L = 1{\cdot}0$ m. Each ply is a carbon-fibre/epoxy–resin composite with ply properties $E_{11} = 168{\cdot}0$ GPA, $E_{22} = 14{\cdot}0$ GPa, $G_{12} = 8{\cdot}4$ GPa, $\upsilon_{12} = 0{\cdot}3$ and $\upsilon_{23} = 0{\cdot}4$.

Mao and Ling [18] used a thin-walled beam theory to obtain buckling loads of $P_{cr} = 167$ kN/cm for a $(0°/0°/0°/0°/0°)$ laminate and $P_{cr} = 107$ kN/cm for a $(0°/90°/0°/90°/0°)$ laminate. Because the present non-shallow shell theory is very good for calculating buckling of a long cylindrical shell as a thin-walled beam, it gave the same results as above to within the three-digit accuracy shown. In fact, since the shell theory is more accurate than the thin-walled beam theory of Mao and Ling [18], the above comparison can be considered as a check of the correctness and accuracy of their theory. In addition, the calculation also found those shell-type buckling loads that are lower than the beam-type buckling loads listed above; i.e., for this shell–beam, shell-type buckling occurs earlier than beam-type buckling.

The present theory is now used to study post-buckling behaviour. Unsurprisingly, the $\bar{b}_{mns}$ become zero, so that $\bar{c}_{1111}$ and $\bar{c}_{2222}$ govern the post-buckling behaviour. Furthermore, since $\lambda_1 < \lambda_2$, the coefficient $\bar{c}_{1111}$ is then the parameter governing post-buckling.

Figure 1 is for the cylindrical shell defined in its caption, for which E_{11}/E_{22} is varied, and dimensionless plots of P^*_{cr} versus E_{11}/E_{22}, and of the scattered results for $\bar{c}_{1111}$, are shown. Each star denotes the value of the dimensionless post-buckling parameter $\bar{c}_{1111}$ corresponding to the buckling load denoted by the dot on the curve which has the same abscissa as the star. The wavenumbers of the buckling mode are in parentheses. To avoid congestion, when successive stars share the same wavenumbers only the first and last ones include parenthesized wavenumbers. The results show some abrupt jumps of $\bar{c}_{1111}$ where p changes, and also it can be seen that $\bar{c}_{1111}$ can become negative as E_{11}/E_{22} is reduced towards unity; i.e., when the material becomes less anisotropic.

Figure 2 is for the isotropic cylindrical shell defined in its caption, and it can be seen that the parameter $\bar{c}_{1111}$ is negative and jumps whenever p changes. The buckling mode is

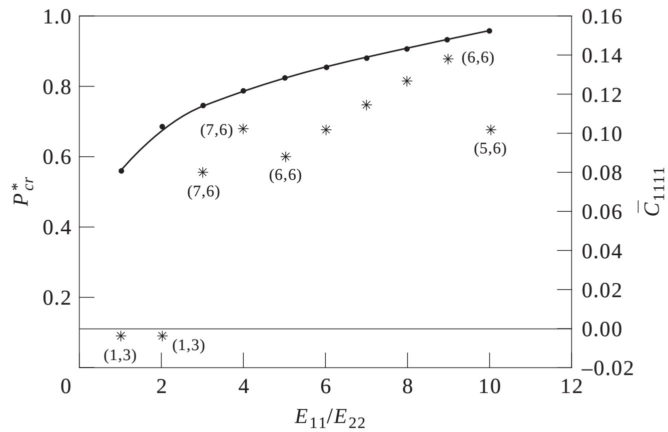

Figure 1. The dimensionless buckling load and post-buckling parameter for different values of E_{11}/E_{22} of a cylindrical shell with $R/T = 50$, $L/R = 5$, $\nu_{12} = \nu_{23} = 0{\cdot}3$ and $G_{12}/E_{22} = 0{\cdot}385$. The curve is for $P^*_{cr} = P_{cr}\,R/E_{22}\,T^2$ and the stars are for $\bar{c}_{1111}$. The numbers in parentheses are the wavenumbers p and q.

composed of V and W components, and calculation shows that the ratios of their absolute values, i.e., $|V|/|W|$, lies in the range between 0·255 and 0·340. This means that even though the W components dominates, the V component is not negligible. A mode for which the W component dominates is called a W mode, and otherwise the mode is called a V mode. The buckling mode of an axially compressed cylindrical shell is always a W mode, except that in the limit it buckles as a column with $|V|/|W| = 1$.

Figure 3 is for the (0°/90°/0°/90°/0°) laminated shell defined in its caption. The curve shows that the buckling load is almost invariant as L/R changes. Closer inspection shows that the circumferential wavenumbers are the same for all the calculated buckling loads and that the longitudinal wavenumber increases progressively as L/R increases, such that the longitudinal wavelength is almost unchanged. In other words, all the buckling loads shown by the curve have almost the same wave size. This explains why the buckling load does not vary significantly along the curve. However, if the shell is very long, column-type

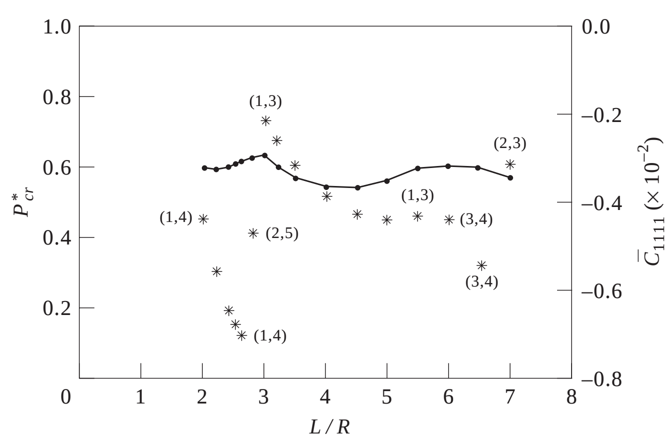

Figure 2. The dimensionless buckling load and post-buckling parameter for an isotropic shell with $R/T = 50$ and $\nu = 0{\cdot}3$. The curve is for $P^*_{cr} = P_{cr}\,R/E_{22}\,T^2$ and the stars are for $\bar{c}_{1111}$. The numbers in parentheses are the wavenumbers p and q.

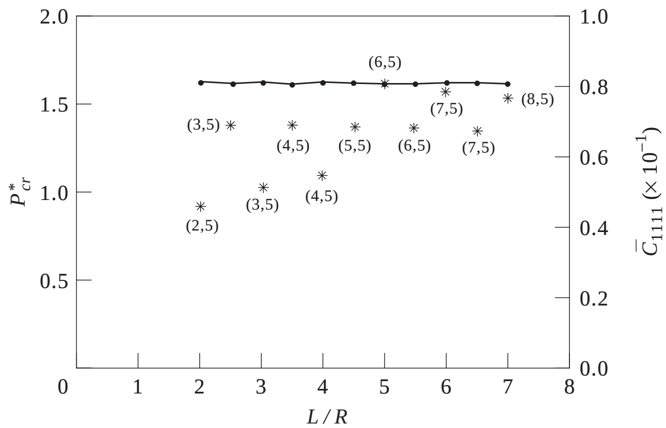

Figure 3. The dimensionless buckling load and post-buckling parameter for different values of L/R for a cylindrical shell with $R/T = 50$, $\nu_{12} = 0{\cdot}28$, $\nu_{23} = 0{\cdot}39$, $E_{11}/E_{22} = 15$ and $G_{12}/E_{22} = 0{\cdot}57$. The curve is for $P^*_{cr} = P_{cr}\,R/E_{22}\,T^2$ and the stars are for $\bar{c}_{1111}$. The numbers in parentheses are the wavenumbers p and q.

buckling will become critical. For example, for a shell with $L/R = 24$, a column-type buckling mode was obtained with $(p, q) = (1, 1)$, $|V|/|W| = 1$ and $\bar{c}_{1111} = -0{\cdot}104 \times 10^{-10}$, although these results are not shown in Figure 3. The almost zero value of $\bar{c}_{1111}$ confirms the usual conclusion of the classical theory of stability; namely, that the initial post-buckling equilibrium of a compressed column is neutral.

Figures 1–3 lead to the conclusion that the initial post-buckling behaviour of specially orthotropic cylindrical shells may be very different from that of isotropic ones, because the initial post-buckling equilibrium of an orthotropic shell may be stable, while that of isotropic ones is always unstable.

6.2. NON-LINEAR FREQUENCY

The non-linear frequencies are no longer constant but are functions of amplitude, and they are of great interest in the theory of non-linear vibration. Two numerical examples are now used to examine this phenomenon.

The first example is the isotropic cylindrical shell defined in the caption of Figure 4. As before, the vibration of v_1 is considered as a sample. The wavenumbers p and q were chosen such that the lowest (fundamental) frequency ω_1 is achieved. The values were found to be

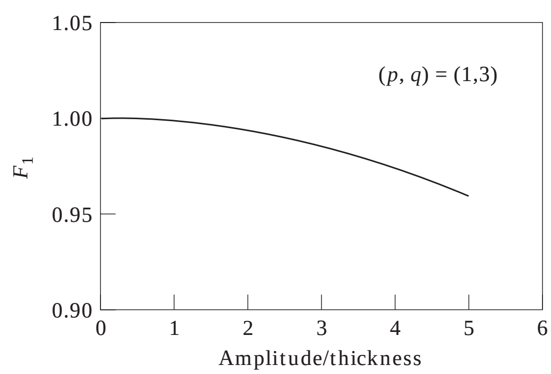

Figure 4. The dimensionless nonlinear fundamental frequency F_1 of an isotropic shell with $R/T = 50$, $L/R = 5$ and $\nu = 0{\cdot}3$.

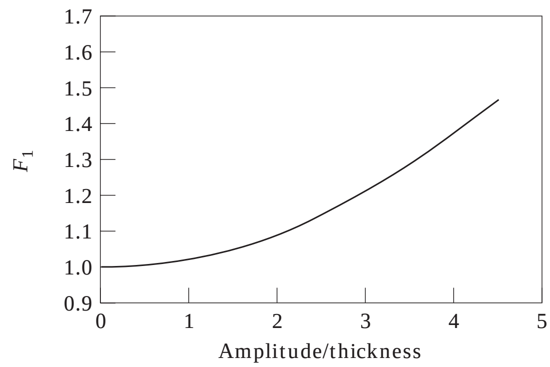

Figure 5. The dimensionless non-linear fundamental frequency F_1 (for which $p = 1$ and $q = 9$) of an orthotropic shell with $R/T = 100$, $L/R = 0{\cdot}5$, $E_{11}/E_{22} = 3$, $G_{12}/E_{22} = 0{\cdot}5$ and $\nu_{12} = \nu_{23} = 0{\cdot}25$.

$p = 1$ and $q = 3$. To avoid confusion, the meanings of some symbols involved in the following discussion are first explained in the next two paragraphs.

The frequency ψ_1' in equation (74) can be written as

$$\psi_1' = \frac{\mathrm{d}\psi_1}{\mathrm{d}\bar{t}} = \frac{\mathrm{d}\psi_1}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\bar{t}} = \frac{1}{\omega_1}\frac{\mathrm{d}\psi_1}{\mathrm{d}t}.$$

Since $\mathrm{d}\psi_1/\mathrm{d}t$ is the real physical non-linear frequency, ψ_1' is a dimensionless form of the non-linear frequency, which is denoted by F_1 below; i.e., $F_1 = \psi_1'$.

The frequency Ω_1 is the linear fundamental frequency of any system described by equations (47). For the present case $\Omega_1^2 = 1 - \bar{P}_0$, where $\bar{P}_0 = P_0/P_{cr}$. For free vibration, $\bar{P}_0 = 0$ and so $\Omega_1 = 1$.

In Figure 4 is shown the dependence of F_1 on a_1, and it can be seen that the non-linearity causes a weak softening effect. This result is due to the negative $\bar{c}_{1111}$ shown in Figure 2, which is in accordance with the well known fact that axially compressed isotropic cylindrical shells are imperfection sensitive. However, Atluri [17] obtained a strong hardening effect. This discrepancy is partly due to the constraint on u in the boundary conditions used by Atluri [17], but the inclusion of the non-linearity and inertia of v in the present theory is also a cause of the discrepancy.

The second example is the specially orthotropic cylindrical shell defined in the caption of Figure 5. The F_1 versus a_1 curve in Figure 5 shows the hardening effect of the non-linearity for the shell studied, which is due to $\bar{c}_{1111}$ being positive. (In contrast to isotropic shells, an orthotropic shell may have positive $\bar{c}_{1111}$. This result can also be found in the literature; see e.g., Sun [3]).

6.3. PARAMETRIC RESONANCE

The first example is the (0°/90°/0°/90°/0°) laminated cylindrical shell defined in the caption of Figure 6, which gives the amplitude–frequency plot of the shell. The intersections of the upper curve and the lower curve with the $\bar{f}$-axis at $\bar{f}_1$ and $\bar{f}_2$, respectively, and the abscissa of the intersection of the two curves is $\bar{f}_3$. The meanings of $\bar{f}_1, \bar{f}_2$ and $\bar{f}_3$ were defined at the end of section 5 above. As shown by Lu *et al*. [19], when $\bar{f}$ is in the interval of conditional stability $(\bar{f}_2, \bar{f}_3)$, only disturbances with magnitude so large as to be close to the lower curve can cause resonance, which occurs with amplitude defined by the upper curve. Therefore, the interval $(\bar{f}_1, \bar{f}_2)$, is more important and is called the interval of instability or the interval of resonance.

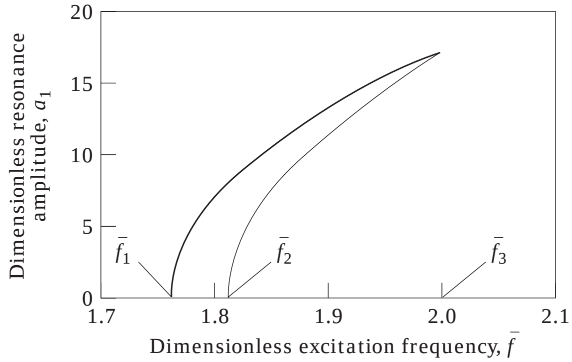

Figure 6. The stationary parametric main resonance (for which $p = 1$ and $q = 3$) of a laminated cylindrical shell with $R/T = 50$, $L/R = 5$, $E_{11}/E_{22} = 15$, $G_{12}/E_{22} = 0{\cdot}57$ $\nu_{12} = 0{\cdot}28$, $\nu_{23} = 0{\cdot}39$, $\bar{P}_0 = 0{\cdot}2$, $\bar{P}_t = 0{\cdot}1$ and $\bar{d}_{11} = 0{\cdot}05$ (critical damping $(\bar{d}_{11})_{cr} = 0{\cdot}0559$).

From equation (82), it can be seen that the amplitude of resonance depends on the parameter of non-linearity, $\bar{c}_{1111}$. Strong non-linearity, i.e., large $\bar{c}_{1111}$, can suppress the amplitude of resonance. The length of the interval $(\bar{f}_1, \bar{f}_2)$ depends on the excitation $\bar{P}_t$ and on the damping $\bar{d}_{11}$. This dependence is shown in Figure 7, on which the stability boundaries are the curves *AB* and *BC* and the straight line *BD*. They are loci of $\bar{f}_1, \bar{f}_2$ and $\bar{f}_3$, respectively, as $\bar{P}_t$ varies. Therefore, the region between *BA* and *BC* is the instability region, the region between *BC* and *BD* is the region of conditional stability and the system is stable elsewhere. The point *B* represents a ‘‘threshold’’ for resonance determined by the damping. The line *BD′* is not needed yet.

The resonance behaviour of the shell studied as the first example of section 6.2 is, as shown in Figure 8, quite different from that of Figure 6 above. In Figure 8, there is no positive $\bar{f}_3$. The intervals $\bar{f} < \bar{f}_1, \bar{f}_1 < \bar{f} < \bar{f}_2$ and $\bar{f} > \bar{f}_2$ are conditionally stable, unstable and stable, respectively. A figure similar to Figure 7 could be drawn for this problem. However, there is no need to do so because it is almost identical to Figure 7, as follows. The loci of $\bar{f}_1$ and $\bar{f}_2$ are identically the same curves as *AB* and *BC* in Figure 7, since these curves do not depend on $\bar{c}_{1111}$. However, the line *BD* is now replaced by the line *BD′* which starts from the point *B* and goes down through the point (0·09443, 0) for isotropic shells. Hence the region between *BC* and *BA* is in the instability region, the region between *BA* and *BD′* is the region of conditional stability and the system is stable elsewhere.

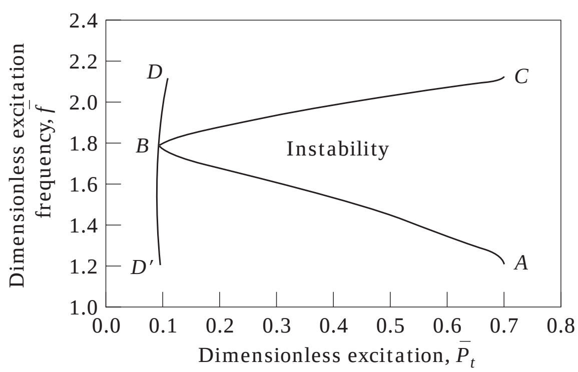

Figure 7. Stability boundaries for the parametric main resonance of the laminated shell defined in the caption of Figure 6 when $\bar{P}_t$ is varied from the value of 0·1 used in Figure 6.

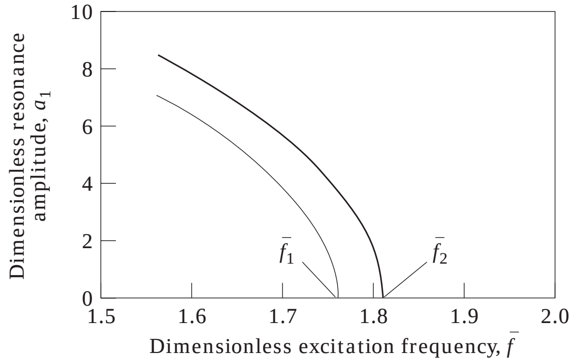

Figure 8. The stationary parametric main resonance (for which $p = 1$ and $q = 3$) of an isotropic shell with $R/T = 50$, $L/R = 5$, $\nu = 0{\cdot}3$, $\bar{P}_0 = 0{\cdot}2$, $\bar{P}_t = 0{\cdot}1$ and $\bar{d}_{11} = 0{\cdot}05$ (critical damping $(\bar{d}_{11})_{cr} = 0{\cdot}0559$).

7. CONCLUDING REMARKS

A non-linear theory for post-critical behaviour of orthotropic cylindrical shells under harmonically varying axial compression is presented in this paper. With this theory, the parametric resonance is analyzed plus, as special cases, the non-linear vibration, buckling and post-buckling behaviours. The theory is based on a non-shallow shell theory. Moreover, the non-linearities and the inertias from both the normal displacement w and the circumferential displacement v are included. Therefore, as numerical examples show, not only does the theory give better results than do theories based on Donnell's shallow shell theory, but also it can be used for the analysis of long shells which buckle as columns.

ACKNOWLEDGMENTS

This work was supported by the Cardiff Advanced Chinese Engineering Centre.

REFERENCES

1. Y. P. Iu and C. Y. Chia 1988 *International Journal of Solids and Structures* **24**, 195–210. Nonlinear vibration and postbuckling of unsymmetric cross-ply circular cylindrical shells.
2. Y. P. Iu and C. Y. Chia 1988 *International Journal of Mechanical Sciences* **30**, 705–718. Effect of transverse shear on nonlinear vibration and postbuckling of anti-symmetric cross-ply imperfect cylindrical shells.
3. G. Sun 1991 *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **58**, 848–851. Buckling and initial postbuckling behavior of laminated oval cylindrical shells under axial compression.
4. J. N. Reddy and M. Sovoia 1992 *American Institute of Aeronautics and Astronautics Journal* **30**, 2148–2154. Layer-wise shell theory for postbuckling of laminated circular cylindrical shells.
5. M. Sovoia and J. N. Reddy 1994 *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **61**, 998–1000. Postbuckling behaviour of stiffened cross-ply cylindrical shells.
6. G. A. Kardomateas 1995 *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **62**, 43–52. Bifurcation of equilibrium of thick orthotropic cylindrical shells under axial compression.
7. S. B. Dong and P. Etitum 1995 *International Journal of Solids and Structures* **32**, 1211–1230. Three-dimensional stability analysis of laminated anisotropic circular cylinders.
8. J.-Q. Ye and K. P. Soldatos 1995 *International Journal of Solids and Structures* **32**, 1949–1995. Three-dimensional buckling analysis of laminated hollow cylinders and cylindrical panels.
9. M. Goldmanis and A. Riekstinsh 1994 *Composite Structures* **29**, 457–462. Postbuckling FE analysis of composite cylindrical panels in axial compression.

10. C. W. Bert and V. Birman 1988 *Acta Mechanica* **71**, 61–76. Parametric instability of thick, orthotropic, circular cylindrical shells.
11. D. Shaw, Y. L. Shen and P. Tsai 1993 *Composites & Structures* **48**, 467–472. Dynamic buckling of an imperfect composite circular cylindrical shell.
12. C.-L. Liao and C.-R. Cheng 1994 *Journal of Sound and Vibration* **174**, 335–351. Dynamic stability of stiffened laminated composite plates and shells subjected to in-plane pulsating forces.
13. V. B. Kovtunov 1993 *Computers & Structures* **46**, 149–156. Dynamic stability and non-linear parametric vibration of cylindrical shells.
14. S. P. Timoshenko and J. M. Gere 1961 *Theory of Elastic Stability*. New York: McGraw-Hill.
15. E. Rik 1979 *International Journal of Solids and Structures* **15**, 529–551. An incremental approach to the solution of snapping and buckling problems.
16. R. M. Evan-Iwanowski 1976 *Resonance Oscillations in Mechanical Systems*. Amsterdam: Elsevier Science.
17. S. Atluri 1972 *International Journal of Solids and Structures* **8**, 549–569. A perturbation analysis of nonlinear free flexural vibrations of a circular cylindrical shell.
18. R. Mao and F. H. Ling 1994 *Thin-Walled Structures* **18**, 291–316. Post-critical behaviour of thin-walled composite beams.
19. C.-H. Liu, R. Mao and R. M. Ivan-Iwanowski 1995 *Journal of Sound and Vibration* **184**, 329–342. Nonstationary parametric responses of shear deformable orthotropic plates.