*Journal of Sound and Vibration* (1998) **210**(3), 403–406

# LETTERS TO THE EDITOR

## AN APPROXIMATE METHOD FOR ANALYZING VIBRATING, SIMPLY SUPPORTED CIRCULAR PLATES OF RECTANGULAR ORTHOTROPY

P. A. A. Laura, S. A. Vera, D. A. Vega and M. D. Sanchez
*Institute of Applied Mechanics* (*CONICET*)
*and*
*Departments of Engineering and Physics*, *Universidad Nacional del Sur*, 8000—*Bahía Blanca*, *Argentina*

(*Received* 14 *August* 1997)

### 1. INTRODUCTION

The free vibrations of circular, isotropic plates have been of practical and academic interest for over a century and a half [1]. Extensive information on eigenvalues and mode shapes is available when the plate boundary is either clamped, free or simply supported.

A rather limited amount of information is available in the case of vibrating circular plates of polar orthotropy [2–4] and it is very scarce when dealing with circular plates of rectangular orthotropy, the problem being of considerable technological importance.

A recent study dealt with clamped circular plates of rectangular orthotropy, where it is a rather simple task to construct polynomial co-ordinate functions that satisfy identically the essential boundary conditions [5]. The optimized Rayleigh–Ritz method was employed to obtain the fundamental frequency coefficient.

The present study deals with the determination of the fundamental frequency of transverse vibration of: (1) a solid, simply supported circular plate of rectangular orthotropy (Figure 1(a)); and (2) a circular annular plate the material of which obeys the same constitutive relations, simply supported at the outer boundary and free at the inner contour (Figure 1(b)).

The same polynomial co-ordinate functions that satisfy identically the outer, essential boundary condition are used for both problems. Clearly, in the case of the doubly connected plate, the energy functional is evaluated between the inner and outer boundaries [5].

The fundamental eigenvalue is determined by means of the optimized Rayleigh–Ritz method. Good engineering accuracy is achieved in the case of an isotropic plate.

### 2. APPROXIMATE ANALYTICAL SOLUTION

Using Lekhnitskii's classical notation [6], one expresses the governing functional in the form

$$J[W] = \tfrac{1}{2}\iint\left[D_1\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + 2D_1 v_2\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + D_2\left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 4D_k\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y - \frac{\rho h}{2}\omega^2\iint W^2\,\mathrm{d}x\,\mathrm{d}y, \quad (1)$$

where $W(x, y)$ is the amplitude of transverse vibration.

0022–460X/98/080403 + 04 $25.00/0/sv971271 

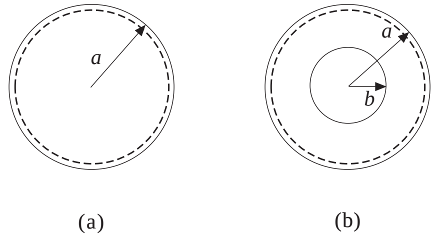


Figure 1. The vibrating structural systems under study: (a) solid circular plate simply supported, (b) annular circular plate simply supported at the outer boundary and free at the inner.

As shown in previous studies [7], one is able to approximate the fundamental mode of vibration of isotropic circular plates by means of the polynomial co-ordinate function

$$W \simeq W_a(x, y) = A_0(\alpha_0 r^{\gamma} + \beta_0 r^2 + 1), \tag{2a}$$

where $r = \sqrt{x^2 + y^2}$; $\gamma$ = Rayleigh's optimization parameter; and $\alpha_0$ and $\beta_0$ are parameters that are determined substituting equation (2a) in the boundary conditions. For the isotropic plate, they are

$$W(a) = 0, \qquad M_r(a) = -D\left(\frac{\mathrm{d}^2 W}{\mathrm{d}r^2} + \frac{v}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\right)\bigg|_{r=a} = 0. \tag{3a, b}$$

Expression (3b) constitutes the natural boundary condition: the radial bending moment at $r = a$ is equal to zero, $v$ being the Poisson ratio.

Clearly, in the case of a circular plate of rectangular orthotropy it will be a rather complicated task to construct a polynomial co-ordinate function that satisfies the natural boundary condition at the outer boundary. It appears reasonable then to determine $\alpha_0$ and $\beta_0$ in the case of circular plates of rectangular orthotropy from the conditions

$$W(a) = 0, \qquad \frac{\mathrm{d}^2 W}{\mathrm{d}r^2} + \frac{v_2}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\bigg|_{r=a} = 0 \tag{4a, b}$$

since in the case of isotropic circular plates, $v_2 = v$, the approximating function (2a) yields excellent accuracy when the plate is simply connected.

The accuracy of the results can be improved by taking additional co-ordinate functions. One has then

$$W_\alpha(x, y) = \sum_{n=0}^{N} A_n(\alpha_n r^{\gamma} + \beta_n r^2 + 1) r^{2n}. \tag{5}$$

In the present study numerical values of the fundamental frequency coefficient $\Omega_1 = \sqrt{\rho h/D_1}\,\omega_1 a^2$ have been obtained for $N = 0$.

Substituting equation (5) in equation (1) and requiring that

$$\frac{\partial J}{\partial A_n} = 0, \qquad n = 0, 1, \ldots, N, \tag{6}$$

one obtains a homogeneous linear system of equations in the $A_n$'s. The non-triviality condition yields a secular determinant, the lowest root of which constitutes the fundamental frequency coefficient.

TABLE 1

*Values of $\Omega_1$ in the case of solid ($b/a = 0$) and annular, isotropic plates*

| $b/a$ | Present study, $\nu = 1/3$ | Results available in reference [8] $\nu = 1/3$ | Results available in reference [8] $\nu = 0{\cdot}3$ |
|---|---|---|---|
| 0 | 4·98 | — | — |
| 0·1 | 4·99 | 4·933 | 4·86 |
| 0·2 | 4·88 | 4·726 | — |
| 0·3 | 4·74 | 4·654 | 4·66 |
| 0·4 | 4·78 | 4·752 | — |
| 0·5 | 5·06 | 5·040 | 5·07 |
| 0·6 | 5·67 | 5·664 | — |
| 0·7 | 6·86 | 6·864 | 6·93 |
| 0·8 | 9·45 | 9·431 | — |
| 0·9 | 17·51 | 17·81 | 17·7 |

Since $\Omega_1$ is an upper bound with respect to the exact result of the eigenvalue, by minimizing it with respect to $\gamma$ one is able to optimize $\Omega_1$ [7].

Admittedly, in the case of rectangular orthotropic plates the mode shapes are also functions of the azimuthal co-ordinate $\theta$ but, as a first order approximation, it seems reasonable to disregard this variation when determining the fundamental frequency parameter.

## 3. NUMERICAL RESULTS

In Table 1 one depicted values of $\Omega_1$ for the isotropic plate for which

$$D_1 = D_2 = D_3 = D, \qquad D_3 = D_1 \nu_2 + 2D_k. \tag{5}$$

The values of $\Omega_1 = \sqrt{\rho h/D}\,\omega_1 a^2$ are determined for $\nu_2 = \nu = 1/3$ in order to compare with the exact results available in reference [8].†

For $b/a = 0{\cdot}3$, the present result is 2% higher than the value reported in reference [8]. For other values of $b/a$, the agreement is quite good from an engineering viewpoint, and for $b/a = 0{\cdot}9$ the fundamental eigenvalue determined using the present approach, 17·51, is lower than the value presented in reference [8]: 17·81. Since the frequency coefficients calculated in this study are upper bounds one can conclude that, for this case, the value of $\Omega_1$ obtained using the optimized Rayleigh–Ritz method is more accurate than the one available in reference [8].

It is interesting to notice that in the case of a solid, simply supported circular plate the exact value of $\Omega_1$ determined in reference [1] is 4·93515 for $\nu = 0{\cdot}30$.

The present approach yields $\Omega_1 = 4{\cdot}9361$ which is in remarkably good agreement with the exact fundamental frequency coefficient.

In Table 2 are shown values of $\Omega_1 = \sqrt{\rho h/D_1}\,\omega_1 a^2$ for orthotropic circular plates for which $D_2/D_1 = 1/2$, $D_k/D_1 = 1/3$ and $\nu_2 = 1/3$.

In view of the good engineering agreement observed in the case of the results depicted in Table 1, one may expect that the eigenvalues contained in Table 2 will possess sufficient accuracy from a practical viewpoint. The results can be improved by adding more

† In Table 1 are depicted the exact results contained in [8] for $\nu = 1/3$ and 0·3. This was done for illustration purposes.

TABLE 2

*Values of $\Omega_1$ in the case of solid ($b/a = 0$) and annular circular plates of rectangular orthotropy ($D_2/D_1 = 1/2$, $D_k/D_1 = 1/3$, $\nu_2 = 1/3$)*

| $b/a$ | $\Omega_1$ |
|---|---|
| 0 | 4·4933 |
| 0·1 | 4·5000 |
| 0·2 | 4·4075 |
| 0·3 | 4·2795 |
| 0·4 | 4·3155 |
| 0·5 | 4·5652 |
| 0·6 | 5·1139 |
| 0·7 | 6·1918 |
| 0·8 | 8·5244 |
| 0·9 | 15·7845 |

co-ordinate functions in the radial and also the azimuthal variables since, due to the orthotropic characteristics of the plate material, even the fundamental mode will depict variations with respect to the angular co-ordinate.

From the analysis of the results shown in Table 1, it becomes apparent that more accurate eigenvalues are needed in the case of circular, annular plates. Most of the data available has been determined over three decades ago with limited computational capabilities.

## ACKNOWLEDGMENTS

M. D. Sanchez, D. A. Vega and S. A. Vera have been supported by CONICET Fellowships in the Physics Department of Universidad Nacional del Sur.

The present study has been sponsored by Secretaría General de Ciencia y Technología of Universidad Nacional del Sur and the Comisión de Investigaciones Científicas (Buenos Aires Province).

## REFERENCES

1. A. W. LEISSA and Y. NARITA 1980 *Journal of Sound and Vibration* **70**, 221–229. Natural frequencies of simply supported circular plates.
2. P. A. A. LAURA, G. C. PARDOEN, L. E. LUISONI and D. R. AVALOS 1981 *Fibre Science and Technology* **15**, 65–77. Transverse vibrations of axisymmetric polar orthotropic circular plates elastically restrained against rotation along the edges.
3. D. J. GUNARATNAM and A. P. BHATTACHARYA 1989 *Journal of Sound and Vibration* **137**, 383–392. Transverse vibrations and stability of polar orthotropic circular plates: high-level relationships.
4. H. A. LARRONDO, D. R. AVALOS and P. A. A. LAURA 1993 *Journal of Sound and Vibration* **168**, 31–37. Axisymmetric vibrations and buckling of non-uniform, cylindrically aelotropic circular plates carrying a central concentrated mass.
5. M. D. SANCHEZ, D. A. VEGA, S. A. VERA and P. A. A. LAURA 1997 *Institute of Applied Mechanics*, *Publication IMA* No. 97-32. Vibrations of circular plates of rectangular orthotropy.
6. S. G. LEKHNITSKII 1968 *Anisotropic Plates*. New York: Gordon and Breach.
7. P. A. A. LAURA 1995 *Ocean Engineering* **22**, 235–250. Optimization of variational methods.
8. A. W. LEISSA 1969 *Vibration of Plates* (NASA SP 160). Washington, D.C.: U.S. Government Printing Service.