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VIBROIMPACTS OF A DUFFING OSCILLATOR UNDER SINUSOIDAL FORCE

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1. INTRODUCTION

The dynamics of one-degree-of-freedom linear oscillators contacting rigid or elastic stops has been a subject of considerable interest due to its application in many branches of engineering [1–3]. A few references are included here for brevity, as many more can be found in the papers referred to here. This system can exhibit simple as well as complex periodic and chaotic motions. In most of the previous investigations, the basic system was modelled as a linear oscillator. Here, the effect of non-linearity on dynamic characteristics of a viscously damped Duffing oscillator with hardening or softening spring contacting rigid snubbers under external sinusoidal force is studied using a central difference method [1, 2]. An extremely small time step was used to reduce accumulation of errors due to the repeated nature of impacts [4, 5]. The effect of non-linearity on periodic one impact/n cycles motions was studied in detail. Results indicate that typical non-linear behaviour with a jump phenomenon present in the impactless case also occurs in this case. The displacement amplitude at resonant peaks reduces with increase in the hardening non-linearity. However, the frequency ranges of one impact/n cycles motions expand significantly with increase of the hardening non-linearity. The opposite behaviour was observed in case of a softening spring. The effects of system parameters and external excitation on periodic motions, velocities after impacts, and the average number of impacts were also investigated.

2. THEORY

A damped Duffing oscillator excited by a sinusoidal force $F \sin \Omega t$ was considered. It consists of a linear and cubic non-linear spring with stiffness coefficients K_1 and K_2 respectively, a viscous dashpot with damping constant C and mass M. Rigid stops were located at distances d_1 and d_2 on the right and left respectively. The collisions between the oscillator and stops were considered instantaneous and represented by the coefficient of restitution e. The differential equation of motion of M between impacts is

$$M\ddot{x} + C\dot{x} + K_1 x + K_2 x^3 = F \sin \Omega t, \tag{1}$$

where x is the absolute displacement of M and the dot represents differentiation with respect to time t. An exact closed form solution of displacement between impacts is not available; hence a numerical approach based on the central difference method was used. The displacement x_{i+1} at time t_{i+1} can be calculated using the previous displacements x_i and x_{i-1} at times t_i and t_{i-1} respectively, as [6]

$$\begin{aligned} x_{i+1} = [\{2M/\Delta t)^2 - K_1\}x_i + \{C/(2\Delta t) - M/(\Delta t)^2\}x_{i-1} \\ - K_2 x_i^3 + F \sin \Omega t_i]/\{M/(\Delta t)^2 + C/(2\Delta t)\}]., \end{aligned} \tag{2}$$

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where Δt is the time step. The x_{-1} required to start the simulation with assumed initial displacement x_0 and velocity $\dot{x}_0$ at $t=0$ is given as $x_{-1} = x_0 - \dot{x}_0 \Delta t + (\Delta t)^2 \ddot{x}_0)/2$ and $\ddot{x}_0 = (-C\dot{x}_0 - K_1 x_0 - K_2 x_0^3)/M$. The unknown times of collisions on stop 1 or 2 were obtained by iteratively solving $(x_{i+1} + d_2) < 10^{-12}$ for $x(t) < 0$ and $(x_{i+1} - d_1) < 10^{-12}$ for $x(t) > 0$. An efficient approach based on combining overshoot, bisection and constant time step was used. The velocities just before and after were obtained as $\dot{x}_- = (x_{i+1} - x_i)/(t_{i+1} - t_i)$ and $\dot{x}_+ = -e\dot{x}_-$ respectively. The x_{i+1} and $\dot{x}_+$ were used as the new initial conditions for the next computation cycle. A computer program based on the above equations and additional equations valid for joint motions was developed and an extremely small $\Delta t = (\Omega/2\pi)/20\,000$ was used. All computations were performed using double precision arithmetic on a DEC-station 2100 digital computer.

3. RESULTS AND DISCUSSION

The results of an impact oscillator with linear spring ($K_2 = 0$) were compared with previous theoretical results and they agree at least upto five decimal places. As an example, a case with $M = K_1 = F = 1{\cdot}0$, $C = K_2 = 0{\cdot}0$, $e = 0{\cdot}8$, $d_1 = 0$ and $\Omega = 6$ was considered, and the system exhibits stable periodic one-impact/three-cycles motion [3]. The numerical values of the non-dimensional parameter $M\Omega\dot{x}_+/F$ obtained using 50, 500 and 20 000 steps per cycle were 8·375134, 8·230265 and 8·228571, respectively, and the agreement with the corresponding theoretical value of 8·228571 [3] is obviously best for the smallest time step. This and other results clearly indicated that an extremely small step produced the most reliable results at the cost of computation time. Accurate results are required for calculating the Feigenbaum number and the frequency at which chaos starts. Results of a Duffing oscillator contacting a single stop are presented first.

The effects of the non-linearity parameter K_2 on the non-dimensional displacement, x_{max}/A, the velocity immediately after impact, $\dot{x}_{ia}/A\Omega$ and the time durations between impacts, $(t_{i+1} - t_i)\Omega/2\pi$, are presented in Figure 1. The x_{max} and A are the maximum displacements of the Duffing oscillator with impacts and that of a simple linear oscillator without impacts, respectively. A comparison of Figure 1(e) with Figure 1(b) indicates that pattern of periodic and aperiodic motions occurring for a hardening Duffing oscillator ($K_2 = 1$) looks similar to that for the impact oscillator ($K_2 = 0$). The basic pattern in the direction of increasing frequency is multi-impact/cycle, two unequispaced impacts/cycle, one equispaced impact/cycle, period doubling motions and chaotic motion. A similar pattern repeats as frequency increases, however the period increases from one to two to three cycles. However, as K_2 increases the whole pattern moves towards the direction of the higher frequency. The widely studied periodic one equispaced impact per n cycles motion corresponds to the horizontal lines at $(t_{i+1} - t_i)\Omega/2\pi = 1, 2$, etc., as seen in the second row of Figure 1. It is important to note that the frequency range of this widely studied motion increases with increase in K_2. As an example, the frequency ranges of the (1, 1) motions for $K_2 = 0$, 1, 10 and 100 are $(1{\cdot}4 - 2{\cdot}6 = 1{\cdot}2)$, $(1{\cdot}6 - 3{\cdot}0 = 1{\cdot}4)$, $(2{\cdot}3 - 4{\cdot}2 = 1{\cdot}9)$ and $(3{\cdot}6 - 6{\cdot}2 = 2{\cdot}6)$ respectively. The x_{max}/A peaks within the frequency range of one equispaced impact per n cycles motion and the peak moves right with increase in K_2. As an example, for $n = 1$, the peak for $K_2 = 0$ occurs at $\Omega = 2$ and increases to $\Omega = 2{\cdot}9$, 4·3 and 6·3 for $K_2 = 1$, 10 and 100, respectively. Additionally, the response curve is symmetric about the peak for $K_2 = 0$ and is asymmetric for $K_2 > 0$, and this behaviour is similar to the response of a hardening system with a jump. $x_{max}/A > 1$ for most of the frequency range, and this indicates that due to impacts and non-linear stiffness the maximum displacement is larger than that of a linear impactless system and may reach a high value. The variation of $\dot{x}_{ia}/A\Omega$ with Ω is shown in column 3 of Figure 1 and looks

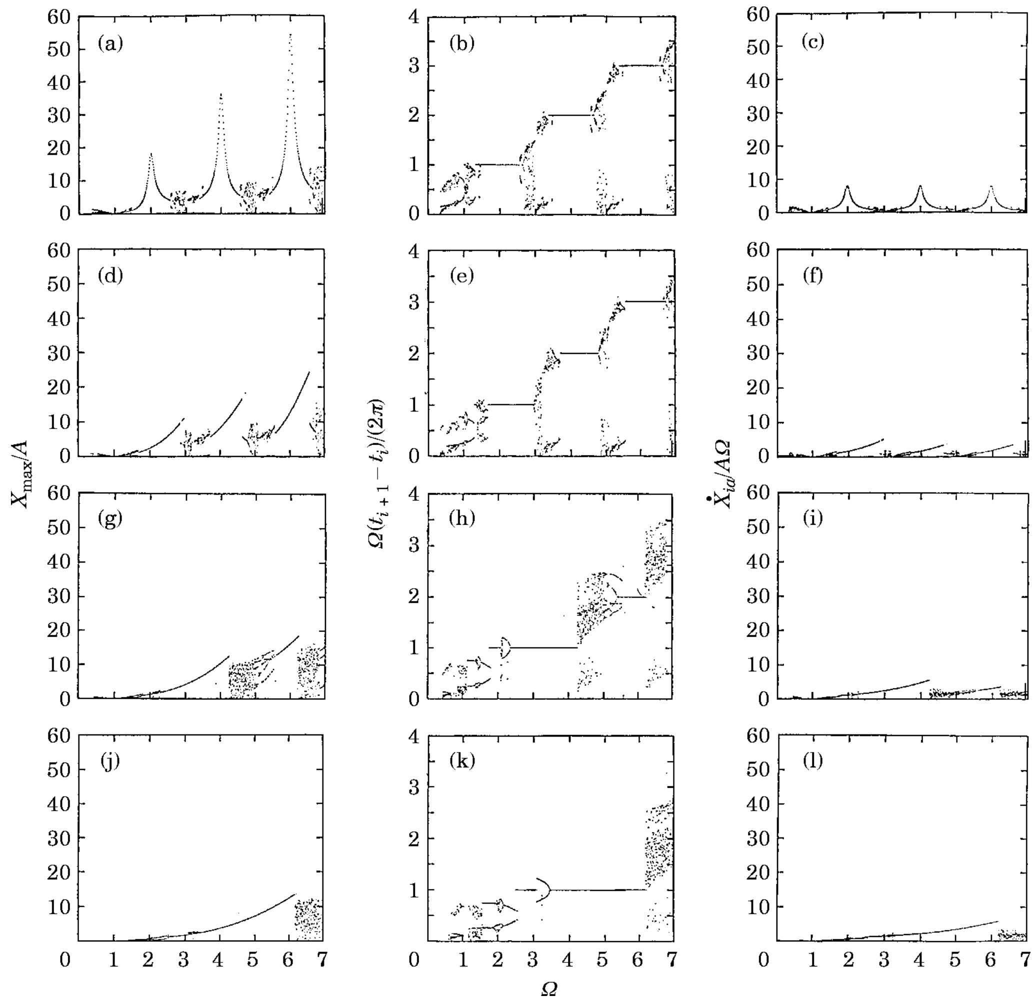

Figure 1. The effect of K_2 on X_{max}/A, $\dot{X}_{ia}/A\Omega$ and $\Omega(t_{i+1}-t_i)/(2\pi)$ of a Duffing oscillator contacting a single stop at zero gap. $M=K_1=F=1{\cdot}0$, $C=0{\cdot}0$, $e=0{\cdot}8$ and $d_1=0{\cdot}0$. (a–c) $K_2=0$; (d–f) $K_2=1$; (g–i) $K_2=10$; (j–l) $K_2=100$.

similar to that of x_{max}/A. However, the maximum values are smaller, and are less than 10 with peaks of nearly indentical heights.

The behaviour of a stiff system at $K_2=100$ in the higher excitation frequency of Ω up to 35 is shown in Figure 2, which indicates that, for all practical purposes, the high frequency response above $\Omega=15$ is aperiodic. x_{max}/A reached a maximum of 160; however, $\dot{x}_{ia}/A\Omega$ was generally below 5. In Figure 2(c) it is indicated that $(t_{i+1}-t_i)\Omega/2\pi$ has two bands, one at low values and the other at high values, and this second band expands with frequency. This band is within two lines joining points (35, 16) and (35, 12) to the origin. The variation of the average time between impacts, shown in Figure 2(d), indicates that it increases with Ω in a linear fashion, and values are clustered around a line joining a point (35, 14) to the origin.

The effect of damping on response was also investigated and it indicated the expected behaviour, where large peaks in x_{max}/A seen in Figure 1 for $\xi=0{\cdot}0$ were significantly reduced for $\xi>0{\cdot}0$ [1, 2]. The damping affects the response for small values of K_2, while the response for the same damping at a large $K_2=100$ is less, because in this case stiffness

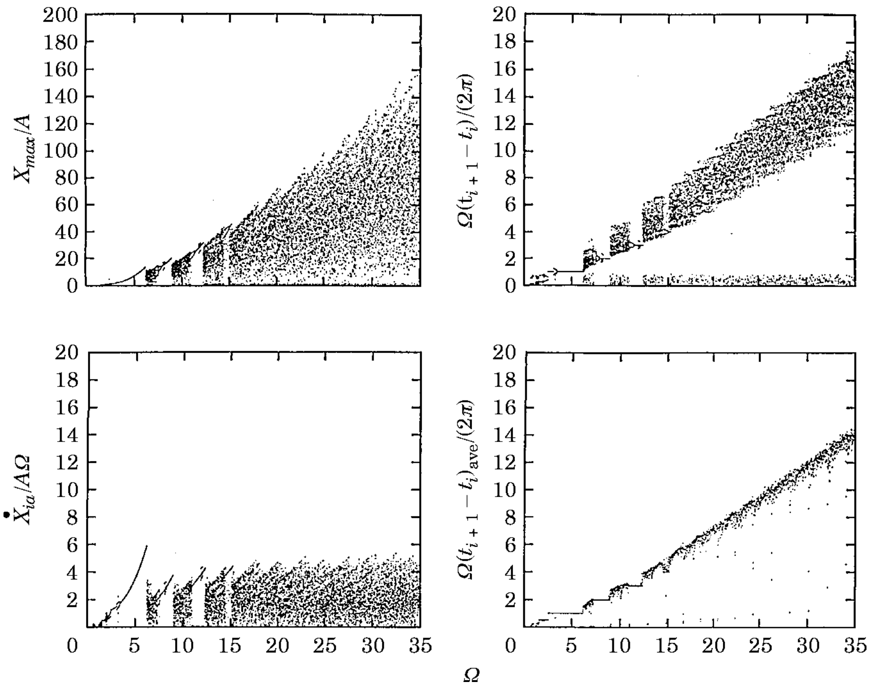

Figure 2. The effect of large K_2 on $X_{\max}/A$, $\dot{X}_{ia}/A\Omega$, $\Omega(t_{i+1} - t_i)/(2\pi)$ and $\Omega(t_{i+1} - t_i)_{\text{ave}}/(2\pi)$ in the high frequency range. The other parameters are the same as in Figure 1.

is predominant. Thus, the effect of damping is more of a quantitative nature. The effects of changing K_2 and F according to $K_2F^2 =$ constant and for zero gap indicate that the behaviour was nearly identical when presented using the non-dimensional parameters used above. Additionally, it was observed that the response when the gap was zero was not very sensitive to an increase in force, as an increasing force form 1 to 10 to 100 produced results identical to those shown in Figure 1(a). Additionally, the effect of a weakly softening spring ($K_2 = -0{\cdot}1$) indicated that the response looked similar to that shown in Figure 1(a) when the same parameters were used. However, the frequency ranges of $(1, n)$ motions move towards the left, curves bend towards the left and a jump also occurs on the left side. The results of an oscillator contacting two stops located with identical small gaps, $d_1 = d_2$ and $1 \gg d_1/A$, indicated that results for $K_2 = 0$ and 1 do not differ significantly and look similar to each other, and this suggests that the effect of non-linearity K_2 on the response in this case is not significant, because the K_2x^3 term remains small. Hence, the results for $K_2 = 0$ can be helpful in assessing the behaviour for non-zero K_2 when gaps are small compared to A.

4. CONCLUSIONS

The behaviour of a Duffing oscillator excited by a sinusoidal force contacting a stop located on one or both sides was investigated. The results were obtained using a central difference method with a very small time step. The dynamic responses of a Duffing oscillator and a linear oscillator contacting a single stop can differ significantly. The frequency ranges of $(1, n)$ motions significantly expand and move towards the direction of increasing frequency for the hardening spring, and the reverse behaviour is found for the softening spring.

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