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LETTERS TO THE EDITOR

SOME EXACT SOLUTIONS OF THE VIBRATION OF NON-HOMOGENEOUS MEMBRANES

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A membrane is non-homogeneous if it has density or thickness variations. Literature on the vibrations of non-homogeneous membranes are few. A composite membrane composed of joining many homogeneous strips was considered by Sato [1] and Kalotas and Lee [2], while membranes composed of two distinct pieces were studied by several authors [3–6]. Recently, Masad [7] investigated a continuously non-homogeneous rectangular membrane where the density function varies linearly with respect to an edge. Masad used numerical integration and a perturbation method to solve for the natural frequencies.

The purpose of this note is to show that the linear density variation case studied by Masad [6] has a closed form exact solution. Also, we shall present another exact solution for the vibration of a continuously non-homogeneous annular membrane. These exact solutions are important not only in their own right as specific vibration problems, but can also serve as error standards for approximate methods, whether analytic or numerical.

The equation of motion is

$$\nabla^2 w + k^2 \rho(\mathbf{x}) w = 0, \tag{1}$$

where w is the displacement, all lengths are normalized by dimension L, $\rho(\mathbf{x})$ is a density function with a mean of unity, and k is the constant normalized frequency

$$k = (\text{frequency}) L[(\text{mean density per area})/(\text{tension per length})]^{1/2}. \tag{2}$$

The boundary condition is that $w = 0$ on the perimeter of domain σ. Then $\rho(\mathbf{x})$ satisfies

$$\frac{1}{\sigma}\int\!\!\int \rho(\mathbf{x})\,\mathrm{d}\sigma = 1. \tag{3}$$

Note that the total mass is fixed for whatever density variation. Thus it is more relevant when compared with the homogeneous membrane where $\rho(\mathbf{x}) = 1$.

Consider the rectangular membrane with normalized dimensions $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant l$. The linear density function that satisfies equation (3) is

$$\rho(x) = 1 + b(x - 0{\cdot}5) > 0. \tag{4}$$

Here, $0 \leqslant b < 2$ is a parameter describing the inhomogeneity. Let

$$w(x, y) = \sin(\alpha y) f(x), \tag{5}$$

where $\alpha = n\pi/l$. Substitution into equation (1) gives the Stokes equation

$$f_{zz} + zf = 0, \tag{6}$$

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where

$$z \equiv \left[k^2\left(1-\frac{b}{2}\right)-\alpha^2+bk^2x\right](bk^2)^{-2/3}. \tag{7}$$

The general solution to equation (6) is in terms of Bessel functions,

$$f = c_1\, z^{1/2}\mathrm{J}_{-1/3}\left(\tfrac{2}{3}\, z^{3/2}\right)+c_2\, z^{1/2}\mathrm{J}_{1/3}\left(\tfrac{2}{3}\, z^{3/2}\right). \tag{8}$$

The condition for non-trivial solutions satisfying $f(0)=f(1)=0$ is

$$(z_0\, z_1)^{1/2}\left[\mathrm{J}_{-1/3}\left(\tfrac{2}{3}\, z_0^{3/2}\right)\mathrm{J}_{1/3}\left(\tfrac{2}{3}\, z_1^{3/2}\right)-\mathrm{J}_{-1/3}\left(\tfrac{2}{3}\, z_1^{3/2}\right)\mathrm{J}_{1/3}\left(\tfrac{2}{3}\, z_0^{3/2}\right)\right]=0, \tag{9}$$

where $z_0 = z|_{x=0}$ and $z_1 = z|_{x=1}$. For a given α, b, the eigenvalue k is obtained from equation (9) by simple root search. Since linear density (or tapered thickness) has some importance, we tabulated the gravest frequency in Table 1. For given l, the frequency for $b = 0$ is that of the homogeneous membrane, i.e. $(1 + l^{-2})^{1/2}$, while the frequency for other b values are for non-homogeneous membranes *of the same total mass*.

For Table 1, we conclude the following: (1) For a given aspect ratio (or area) the gravest frequency first decreases with taper b, then increases with b: the minimum is slightly below the homogeneous frequency. (2) As b is varied, there exist plateaus of very small frequency changes (perhaps a frequency lock): the frequency may jump to a higher plateau for larger b, and these jumps occur more often at low aspect ratio l. These interesting phenomena should be investigated experimentally.

Another exact solution exists for a non-homogeneous annular membrane when the density is proportional to the inverse square of the radius. Let the annulus be described by $aL \leqslant r \leqslant L$. Let

$$\rho = \frac{c}{r^2}, \qquad c = \frac{1-a^2}{-2\ln a} > 0, \qquad a \neq 0, \tag{10}$$

TABLE 1

The gravest frequency for a rectangular membrane with linear density: *vertical lines indicate some plateau separations*

	b								
l	0·0	0·25	0·5	0·75	1·0	1·25	1·5	1·75	1·99
0·2	16·0190	16·954	18·451	20·381	24·981	27·745	33·498	45·776	223·52
0·4	8·4590	8·438 ∣	10·093	12·366	15·017	17·918	21·018	24·301	113·28
0·6	6·1062	6·098	6·075 ∣	8·222	8·248 ∣	10·963	11·032 ∣	17·158	76·538
0·8	5·0290	5·024	5·011	4·991 ∣	7·465	7·489 ∣	10·430	13·594	56·505
1·0	4·4429	4·439	4·430	4·416 ∣	7·074	7·096	7·118 ∣	10·205	46·777
1·2	4·0894	4·087	4·080	4·068	4·053 ∣	6·873	6·895 ∣	10·042	39·793
1·4	3·8607	3·858	3·852	3·843	3·830	3·813 ∣	6·757	6·778 ∣	32·120
1·6	3·7047	3·702	3·697	3·689	3·677	3·663 ∣	6·666	6·687 ∣	29·777
1·8	3·5939	3·592	3·587	3·579	3·569	3·555	3·539 ∣	6·623	26·438
2·0	3·5124	3·510	3·506	3·499	3·489	3·476	3·461 ∣	6·578	23·103
5·0	3·2038	3·202	3·199	3·193	3·186	3·176	3·165	3·151 ∣	9·746
10·0	3·1573	3·156	3·152	3·147	3·140	3·131	3·120	3·107 ∣	6·406
50·0	3·1422	3·141	3·137	3·132	3·125	3·116	3·105	3·093	3·079

TABLE 2

The gravest frequency for an annular membrane with inverse square density

a	k(non-homo)	k(homo)	a	k(non-homo)	k(homo)
0	0	2·4048	0·4	5·064	5·183
0·0001	1·464	2·587	0·5	6·162	6·246
0·001	1·690	2·654	0·6	7·770	7·828
0·01	2·071	2·801	0·7	10·417	10·455
0·1	2·943	3·314	0·8	15·676	15·698
0·2	3·574	3·816	0·9	31·401	31·412
0·3	4·245	4·412	0·99	314·16	314·16

where the value of c is from the constant mass condition (3). The governing equation is

$$w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} + \frac{k^2 c}{r^2} w = 0. \tag{11}$$

Equation (11) yields the solution

$$w = \cos(n\theta) \sin[\sqrt{k^2 c - n^2} \ln r], \quad n = 0, 1, 2, \ldots . \tag{12}$$

For non-trival solutions the boundary conditions give

$$\sqrt{k^2 c - n^2} \ln a = m\pi, \qquad m = 1, 2, 3, \ldots, \tag{13}$$

or

$$k = \sqrt{\frac{1}{c}\left[\left(\frac{m\pi}{\ln a}\right)^2 + n^2\right]}. \tag{14}$$

The gravest frequency is when $n = 0$ and $m = 1$, where

$$k = \pi\sqrt{\frac{2}{(1 - a^2)|\ln a|}}. \tag{15}$$

In comparison, the gravest frequency for a homogeneous annulus is the lowest eigenvalue of the Bessel function equation:

$$J_0(k) Y_0(ka) - J_0(ka) Y_0(k) = 0. \tag{16}$$

The results are shown in Table 2. We conclude for the inverse square density distribution that the gravest frequency for the non-homogeneous annulus membrane is lower than that of the homogeneous membrane, especially when the inner radius a is small. In fact $k \to 0$ as $a \to 0$ for the non-homogeneous case while $k \to$ first zero of J_0 for the homogeneous case.

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