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TRANSVERSE VIBRATIONS OF A THIN RECTANGULAR PLATE SUBJECTED TO A NON-UNIFORM STRESS DISTRIBUTION FIELD

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1. INTRODUCTION

The problem of transverse vibrations of a thin rectangular plate subjected to a non-uniform, in-plane loading (see Figure 1) must be tackled in two steps:

First, the solution of the corresponding plane stress problem is required which is governed by Airy's biharmonic equation

$$\nabla^4 U = 0, \tag{1}$$

whose solution, satisfying appropriate boundary conditions, allows the determination of the components of the stress tensor [1]

$$\sigma_{\bar{x}} = \frac{\partial^2 U}{\partial \bar{y}^2}, \qquad \sigma_{\bar{y}} = \frac{\partial^2 U}{\partial \bar{x}^2}, \qquad \tau_{\bar{x}\bar{y}} = -\frac{\partial^2 U}{\partial \bar{x}\,\partial \bar{y}}. \tag{2}$$

Second, once expressions (2) are determined, multiplying them by the plate thickness, h, one obtains the stress resultants

$$N_{\bar{x}} = N_{\bar{x}}\,(\bar{x}, \bar{y}), \qquad N_{\bar{y}} = N_{\bar{y}}\,(\bar{x}, \bar{y}), \qquad N_{\bar{x}\bar{y}} = N_{\bar{x}\bar{y}}\,(\bar{x}, \bar{y}), \tag{3}$$

and substitutes them in the vibrating plate equation which, for normal modes, is conveniently expressed in the form

$$D\left(\frac{\partial^4 W}{\partial \bar{x}^4} + 2\frac{\partial^4 W}{\partial \bar{x}^2\,\partial \bar{y}^2} + \frac{\partial^4 W}{\partial \bar{y}^4}\right) - \left(N_{\bar{x}}\frac{\partial^2 W}{\partial \bar{x}^2} + 2N_{\bar{x}\bar{y}}\frac{\partial^2 W}{\partial \bar{x}\,\partial \bar{y}} + N_{\bar{y}}\frac{\partial^2 W}{\partial \bar{y}^2}\right) - \rho h \omega^2 W = 0. \tag{4}$$

In some instances researchers have mistakingly omitted that first step in the analytical solution assuming that the stress resultants field is given directly by [2]

$$N_{\bar{x}} = Sf(\bar{y}), \qquad N_{\bar{y}} = N_{\bar{x}\bar{y}} = 0, \tag{5}$$

where $Sf(\bar{y})$ is the applied stress resultant at $\bar{x} = \pm a$; see Figure 1.

It can be shown that this procedure is only valid if $f(\bar{y})$ is either constant or a linear function of $\bar{y}$ [3].

The present study deals with the determination of the fundamental frequency coefficient of transverse vibration of the structural system, shown in Figure 1, for several combinations of boundary conditions. As a first order approximation for Airy's stress

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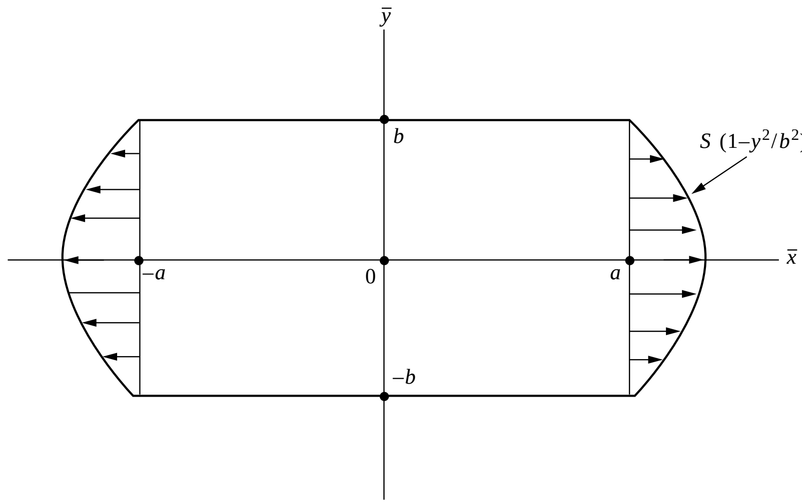

Figure 1. Vibrating structural system under study.

function $U(\bar{x}, \bar{y})$, use is made of the expression $U_a(\bar{x}, \bar{y})$ presented in reference [1]† which has been obtained using the Rayleigh–Ritz method:

$$U(\bar{x}, \bar{y}) \cong U_a(\bar{x}, \bar{y}) = \frac{S}{2}\bar{y}^2\left(1 - \frac{1}{6}\frac{\bar{y}^2}{b^2}\right) + \alpha_1(\bar{x}^2 - a^2)^2(\bar{y}^2 - b^2)^2, \tag{6}$$

where

$$\alpha_1 = \frac{S}{a^4b^2}\frac{1}{(64/7) + (256/49)(b^2/a^2) + (64/7)(b^4/a^4)}.$$

Substituting equation (6) in equations (2) and (3) and introducing $\bar{x} = 2ax$, $\bar{y} = 2by$ and $\lambda = a/b$, one obtains

$$N_x = Sg_1(x, y), \qquad N_y = Sg_2(x, y), \qquad N_{xy} = Sg_{12}(x, y), \tag{7}$$

where

$$g_1(x, y) = 1 - 4y^2 + \frac{4\beta}{\lambda^2}(4x^2 - 1)^2(12y^2 - 1),$$

$$g_{12}(x, y) = -\frac{64\beta}{\lambda^3}(4x^3 - x)(4y^3 - y), \qquad g_2(x, y) = \frac{4\beta}{\lambda^4}(12x^2 - 1)(4y^2 - 1)^2,$$

$$\beta = \frac{\lambda^6}{(64/7)\lambda^4 + (256/49)^2 + (64/7)}.$$

†Since in the present study one is interested in stress resultants, the thickness of the plate, h, is assumed to be included in the parameter S.

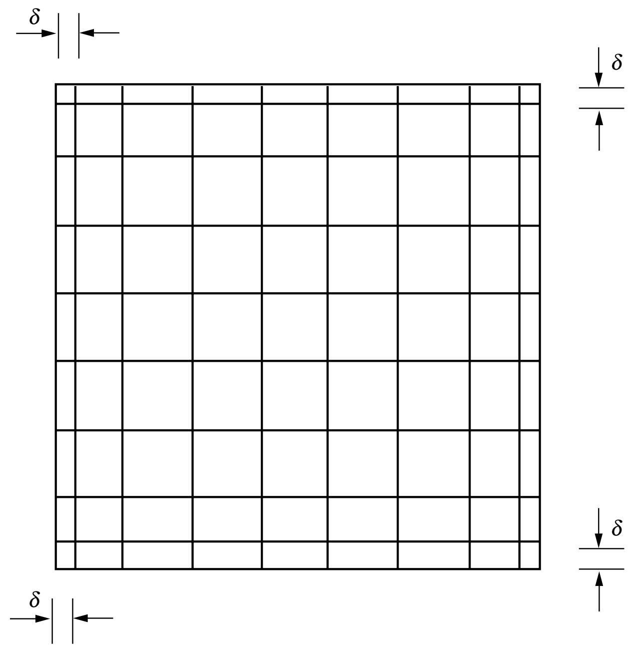

Figure 2. Partition of the plate domain when using the differential quadrature method ($\delta = 10^{-4}$).

Accordingly the governing partial differential equation results, after substitution of equation (7) and the dimensionless variables x and y into equation (4):

$$\frac{\partial^4 W}{\partial x^4} + 2\lambda^2 \frac{\partial^4 W}{\partial x^2\,\partial y^2} + \lambda^4 \frac{\partial^4 W}{\partial y^4} - S_1 \left(g_1 \frac{\partial^2 W}{\partial x^2} + 2\lambda g_{12} \frac{\partial^2 W}{\partial x\,\partial y} + \lambda^2 g_2 \frac{\partial^2 W}{\partial y^2} \right) - \Omega^2 W = 0, \quad (8)$$

where $\Omega = \sqrt{\rho h/D}\,(2a)^2\omega$, $S_1 = 4a^2 S/D$.

The solution of equation (8) subject to the corresponding boundary conditons at the edges $\bar{x} = -a$, $\bar{y} = -b$; $\bar{x} = a$, and $\bar{y} = b$ (Figure 1) will be approached in the present study using the differential quadrature method [4–6].

2. SOLUTION BY MEANS OF THE DIFFERENTIAL QUADRATURE METHOD

Due to the efforts of Bert and associates, the method of differential quadrature is already well established in the technical literature [4–6].

Following references [4–6] the plate domain is partitioned as shown in Figure 2. For all the situations considered, the number of nodal points in each direction was $N = 9$.

Using the notation introduced by Bert and co-workers [4–6] one obtains, in the case of a simply supported rectangular plate,

$$\sum_{k_1=2}^{N-1} D_{ik_1} W_{k_1 j} + 2\lambda^2 \sum_{k_1=2}^{N-1} \sum_{k_2=2}^{N-1} B_{ik_1} B_{jk_2} W_{k_1 k_2} + \lambda^4 \sum_{k_2=2}^{N-1} D_{jk_2} W_{ik_2}$$

$$- S_1 \left[g_{1ij} \sum_{k_1=2}^{N-1} B_{ik_1} W_{k_1 j} + 2\lambda g_{12ij} \sum_{k_1=2}^{N-1} \sum_{k_2=2}^{N-1} A_{ik_1} A_{jk_2} W_{k_1 k_2} \right.$$

TABLE 1

Values of Ω_1 in the case of a simply supported rectangular plate, boundary conditions: SS ($x = -a$), SS ($y = -b$), SS ($x = a$), SS ($y = b$)

λ	$S_1 = 0$	1	10	20	50
1[a]	19·732	19·928	21·544	23·208	27·604
Reference [7]	19·74	—	—	—	—
1·5[a]	32·078	32·180	33·086	34·061	36·824
Reference [7]	32·08	—	—	—	—
2[a]	49·351	49·414	49·972	50·583	52·364

[a]Present results.

TABLE 2

Values of Ω_1 in the case of a SS–C–SS–SS rectangular plate

λ	$S_1 = 0$	1	10	20	50
1[a]	23·636	23·801	25·153	26·574	30·441
Reference [7]	23·65	—	—	—	—
1·5[a]	42·532	42·606	43·269	43·991	46·082
Reference [7]	42·53	—	—	—	—
2[a]	69·337	69·380	69·758	70·175	71·402

[a]Present results.

TABLE 3

Values of Ω_1 in the case of a SS–C–SS–C rectangular plate

λ	$S_1 = 0$	1	10	20	50
1[a]	28·952	29·090	30·207	31·402	34·735
Reference [7]	28·95	—	—	—	—
1·5[a]	56·366	56·420	56·902	57·431	58·979
Reference [7]	56·35	—	—	—	—
2[a]	95·288	95·317	95·575	95·859	96·699

[a]Present results.

TABLE 4

Values of Ω_1 in the case of a C–SS–SS–SS rectangular plate, obtained by means of the differential quadrature method

λ	$S_1 = 0$	1	10	20	50
1	23·636	23·823	25·357	26·948	31·195
1·5	35·048	35·151	36·062	37·043	39·818
2	51·675	51·739	52·314	52·943	54·775

TABLE 5

Values of Ω_1 in the case of a C–C–SS–SS rectangular plate

λ	$S_1 = 0$	1	10	20	50
1[a]	27·043	27·205	28·541	29·944	33·767
Reference [7]	27·06	—	—	—	—
1·5[a]	44·892	44·969	45·657	46·405	48·563
Reference [7]	44·89	—	—	—	—
2[a]	71·090	71·135	71·532	71·968	73·255

[a] Present results.

TABLE 6

Values of Ω_1 in the case of a C–C–SS–C rectangular plate, obtained by means of the differential quadrature method

λ	$S_1 = 0$	1	10	20	50
1	31·830	31·971	33·114	34·330	37·707
1·5	58·203	58·260	58·771	59·329	60·957
2	95·603	96·633	96·906	97·207	98·096

TABLE 7

Values of Ω_1 in the case of a C–SS–C–SS rectangular plate

λ	$S_1 = 0$	1	10	20	50
1[a]	28·952	29·116	30·454	31·870	35·766
Reference [7]	28·95	—	—	—	—
1·5[a]	39·116	39·212	40·066	40·993	43·655
2[a]	54·800	54·863	55·428	56·049	57·873
Reference [7]	54·75	—	—	—	—

[a]Present results.

TABLE 8

Values of Ω_1 in the case of a C–C–C–SS rectangular plate

λ	$S_1 = 0$	1	10	20	50
1[a]	31·830	31·978	33·183	34·470	38·051
Reference [7]	31·83	—	—	—	—
1·5[a]	48·195	48·270	48·937	49·668	51·799
Reference [7]	48·17	—	—	—	—
2[a]	73·467	73·512	73·911	74·352	75·661

[a]Present results.

$$+\lambda^2 g_{2ij} \sum_{k_2=2}^{N-1} B_{jk_2} W_{ik_2}\Bigg] - \Omega^2 W_{ij} = 0, \qquad (i, j = 3, \ldots, N-2),$$

$$\sum_{k_1=2}^{N-1} B_{2k_1} W_{k_1 j} = 0, \qquad (j = 3, \ldots, N-1),$$

$$\sum_{k_2=2}^{N-1} B_{2k_2} W_{ik_2} = 0, \qquad (i = 2, \ldots, N-2),$$

$$\sum_{k_1=2}^{N-1} B_{(N-1)k_1} W_{k_1 j} = 0, \qquad (j = 2, \ldots, N-2),$$

$$\sum_{k_2=2}^{N-1} B_{(N-1)k_2} W_{ik_2} = 0, \qquad (i = 3, \ldots, N-1), \tag{9}$$

where the D's, B's and A's are the coefficients of the linear combinations of fourth, second and first order derivatives of the displacement amplitude $W(x, y)$, respectively.

Similar systems of equations are obtained for other combinations of boundary conditions.

Admittedly the present analysis has been facilitated by the fact that the non-uniform stress field is known in advance. It seems quite advantageous, in a general situation, to obtain the solution of the stress problem and then the vibrational response, in a unified manner, by means of the differential quadrature technique.

3. NUMERICAL RESULTS

All calculations have been performed for $\lambda = a/b = 1$, 1·5 and 2, while values taken for the stress resultant parameter, $S_1 = 4a^2S/D$, were 0, 1, 10, 20 and 50.

The fundamental frequency parameters Ω_1 are listed in Tables 1–9. The eigenvalues have been compared with values available in the literature ($S_1 = 0$, [7]). It was observed that good agreement was achieved.

As expected, the effect of S_1 becomes less important as a/b increases.

TABLE 9

Values of Ω_1 in the case of a clamped rectangular plate

λ	$S_1 = 0$	1	10	20	50
1[a]	36·008	36·139	37·206	38·353	41·592
Reference [7]	35·99	—	—	—	—
1·5[a]	60·816	60·872	61·376	61·932	63·572
Reference [7]	60·77	—	—	—	—
2[a]	98·396	98·427	98·702	99·008	99·921
Reference [7]	98·33	—	—	—	—

[a]Present results.

4. CONCLUSIONS

As shown, the differential quadrature method allows for the straightforward solution of a complex elastodynamics problem. If the solution of the plane stress problem is not known in advance, the differential quadrature technique may be used to one's advantage by solving first the plane stress problem and then to tackle the transverse vibrations problem.

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