Journal of Sound and Vibration (1998) **210**(5), 673–682

THE FORCED VIBRATION OF A BEAM WITH VISCOELASTIC BOUNDARY SUPPORTS

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(*Received* 8 *January* 1997, *and in final form* 27 *September* 1997)

A method of analysis for the forced vibration of a beam with viscoelastic boundary supports is proposed based on complex normal mode analysis. The viscoelastic support regions are first described in terms of equivalent complex stiffness coefficients, and then using the complex modes of the beam system with complex stiffness at the boundary points, the equations of motion are completely uncoupled. The modal equation has precisely the same form as the equation of a structurally damped single-degree-of-freedom system. Effects of the viscoelastic supports on the forced vibration response are discussed as an example.

1. INTRODUCTION

Surface damping treatments with viscoelastic materials have been successfully used for many years to reduce vibration and noise of structures especially for the beam and plate-like structures. Although such surface damping treatments are effective in vibration and noise control in general, it is not always possible or desirable to implement them in real situations depending on the given constraints. In such cases the damping treatments at the boundary supports can be an alternative solution [1].

Recently, Kang proposed a systematic method to estimate the modal properties of beams and plates with viscoelastic boundary supports [2]. For beam vibrations, the viscoelastic support regions are first described analytically in terms of equivalent complex stiffness parameters, and then a characteristic equation for the beam structure supported at its ends by springs with the complex stiffness is derived. Subsequently, natural frequencies and modal loss factors of the assembled beam are obtained by solving the transcendental characteristic equations numerically. A similar approach can be applied to plates with viscoelastic boundary supports.

The current study involves forced vibrations of a beam with viscoelastic boundary supports. In the proposed approach, the equation of the motion is uncoupled by using the damped complex modes of the beam. The modal equation has precisely the same form as the equation of a structurally damped single-degree-of-freedom system.

2. EQUIVALENT SYSTEM FOR A BEAM WITH VISCOELASTIC BOUNDARY SUPPORTS

Figure 1(a) shows a beam of width b, thickness h and length l with viscoelastic boundary supports whose thickness and length are given by H and L, respectively. The Young's

0022–460X/98/100673 + 10 $25.00/0/sv971353

modulus of the viscoelastic material takes a complex number as expressed below

$$E^* = E_r\,(1 + j\eta), \tag{1}$$

where E_r is the real part of the complex modulus, called the storage modulus, and η is the loss factor of the material given by the ratio of the imaginary part of the complex modulus, i.e., loss modulus to the storage modulus. The stiffness parameters for the viscoelastic boundary support region of a beam system will also be complex quantities represented in terms of the geometric characteristics of the support layers and their material properties.

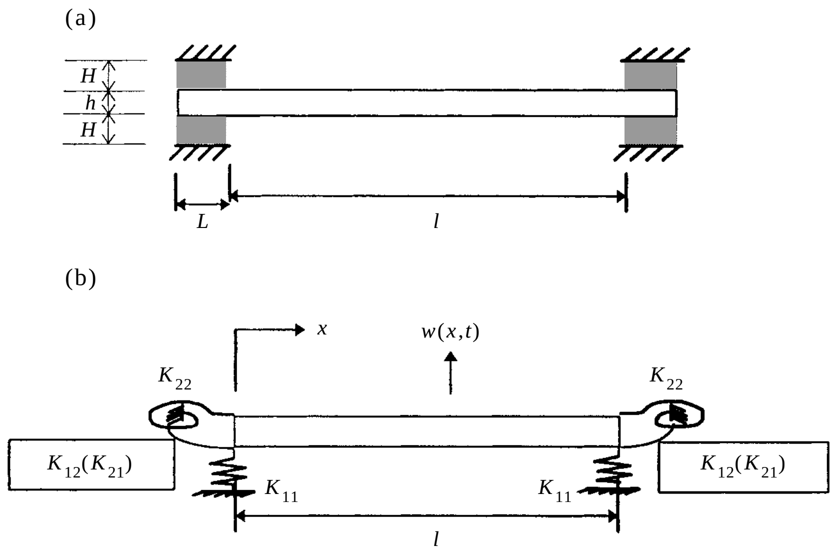

Figure 1. (a) Beam system with viscoelastic supports; (b) equivalent stiffness representation of the support region.

It can be shown easily that when the material properties and geometry of the top and the bottom viscoelastic layers are the same at either boundary point, the transverse motion of the support region at the boundary point of the beam system is uncoupled from the longitudinal motion. Hence, the stiffness parameters can be represented as shown in Figure 1(b) and the beam system becomes self-adjoint.

The equivalent stiffness parameters K_{11}, $K_{12}\,(=K_{21})$ and K_{22} shown in Figure 1(b) are frequency dependent complex quantities as is the modulus of the viscoelastic insertion material. The parameter $K_{12}\,(=K_{21})$ represents the coupling between the translational motion in the transverse direction and the rotational motion. Detailed derivations of the expressions for these parameters can be found in Kang and Kim [2, 3].

3. EIGENVALUE PROBLEM OF BEAM WITH COMPLEX STIFFNESS AT BOUNDARIES

The equation for transverse vibration of a uniform Bernoulli–Euler beam of length l is given by

$$EI\,\frac{\partial^4 w(x,\,t)}{\partial x^4} + \rho A\,\frac{\partial^2 w(x,\,t)}{\partial t^2} = 0, \tag{2}$$

where $w(x,\,t)$ is the transverse deflection at a distance x along the beam and at time t, EI is the flexural rigidity, ρ is the mass density, and A represents the cross-sectional area of the beam. Boundary conditions of a beam system as shown in Figure 1(b) can be described as follows:

$$\begin{Bmatrix} -EI\dfrac{\partial^3 w(0,t)}{\partial x^3} \\ EI\dfrac{\partial^2 w(0,t)}{\partial x^2} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} w(0,t) \\ \dfrac{\partial w(0,t)}{\partial x} \end{Bmatrix} \quad \text{at } x = 0, \tag{3a}$$

$$\begin{Bmatrix} EI\dfrac{\partial^3 w(l,t)}{\partial x^3} \\ EI\dfrac{\partial^2 w(\mathrm{l},t)}{\partial x^2} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} w(l,t) \\ -\dfrac{\partial w(l,t)}{\partial x} \end{Bmatrix} \quad \text{at } x = l. \tag{3b}$$

Let the transverse displacement, $w(x, t)$, be represented by

$$w(x,t) = W(x)\,\mathrm{e}^{\,j\bar{\Omega}t}, \tag{4}$$

where $\bar{\Omega}$ is the complex frequency of free vibration and $W(x)$ the shape of vibration. Substituting equation (4) into equation (2) gives $W(x)$ as follows:

$$W(x) = A\sin\beta\frac{x}{l} + B\cos\beta\frac{x}{l} + C\sinh\beta\frac{x}{l} + D\cosh\beta\frac{x}{l}, \tag{5a}$$

where β is a frequency parameter defined by:

$$\beta^4 = \frac{\bar{\Omega}^2\rho A l^4}{EI}. \tag{5b}$$

Substitution of the solution form given by equations (4) and (5) into the boundary conditions given by equation (3) results in a set of four homogeneous algebraic equations with five unknown constants, A, B, C, D and β in matrix form as follows:

$$[M(\beta)]\begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = 0, \tag{6}$$

where $[M(\beta)]$ is a 4×4 square matrix consisting of transcendental functions of the frequency parameters β. In order for the above equation to have a non-trivial solution, the determinant of the matrix must be zero, i.e.,

$$\det\,[M(\beta)] = 0. \tag{7}$$

Transcendental equation (7) can be numerically solved for the dimensionless natural frequency parameter β_n (n = mode number) [2, 4]. Once the eigenvalues are obtained, the corresponding mode shapes can be determined by equations (5a) and (6). Since the stiffness parameters are complex quantities and, hence, the characteristic equation is complex, roots of equation (7) should be also complex. Consequently, the natural frequency given by $\bar{\Omega}_n = \beta_n^2\sqrt{EI/\rho A l^4}$ and the corresponding mode shape function $W_n(x)$ will also be complex.

4. FORCED VIBRATION RESPONSES FOR HARMONIC EXCITATIONS

Let $W_m(x)$ and $W_n(x)$ be the complex mode shape functions corresponding to the natural frequencies ω_m and ω_n, respectively. Since the beam system with the same viscoelastic layers at the top and bottom of the beam at either boundary point is self-adjoint as mentioned above, the orthogonal relation of the complex modes are given as follows (see the Appendix for derivations):

$$\int_0^l \rho A W_m W_n \, \mathrm{d}x = \delta_{mn} M_{mn}, \tag{8}$$

where δ_{mn} is Kronecker delta function.

Now, consider the vibration response of the above beam system subject to a harmonic loading given by $q(x, t) = Q(x)\,\mathrm{e}^{j\Omega t}$. That is, equation of the beam system is given by

$$EI \frac{\partial^4 w(x, t)}{\partial x^4} + \rho A \frac{\partial^2 w(x, t)}{\partial t^2} = Q(x)\,\mathrm{e}^{j\Omega t}. \tag{9}$$

Since the vibration response should also be harmonic, the solution to equation (9) can be assumed as follows:

$$w(x, t) = \sum_{n=1}^{\infty} G_n W_n(x)\,\mathrm{e}^{j\Omega t}. \tag{10}$$

Substituting the expression (10) into equation (9) leads to

$$\sum_{n=1}^{\infty} G_n \left[EI \frac{\mathrm{d}^4 W_n(x)}{\mathrm{d}x^4} - \rho A \Omega^2 W_n(x) \right] = Q(x). \tag{11}$$

Recalling that each vibration mode satisfies the homogeneous equation given by

$$EI \frac{\mathrm{d}^4 W_n(x)}{\mathrm{d}x^4} = \rho A \bar{\Omega}_n^2 W_n(x), \tag{12}$$

and substituting this relation into equation (11) yields:

$$\sum_{n=1}^{\infty} G_n (\bar{\Omega}_n^2 - \Omega^2) \rho A W_n(x) = Q(x). \tag{13}$$

Utilizing the orthogonality of eigenfunctions given by equation (8), equation (13) can be uncoupled and the equation of motion for each mode takes the following form:

$$\bar{\Omega}_n^2 G_n - \Omega^2 G_n = \frac{Q_n}{M_n} \qquad (n = 1, 2, 3, \ldots), \tag{14}$$

where

$$M_n = \int_0^l \rho A W_n^2(x) \, \mathrm{d}x, \tag{15a}$$

$$Q_n = \int_0^l Q(x) W_n(x) \, \mathrm{d}x. \tag{15b}$$

The relationship between the complex natural frequency $\bar{\Omega}_n$ and the associated real natural frequency Ω_n and modal loss factor η_n can be formulated as in a single-degree-of-freedom system with structural damping as follows:

$$\bar{\Omega}_n = \Omega_n \, (1 + j\eta_n)^{1/2}. \tag{16}$$

From equations (14) and (16), the coefficient G_n is given by

$$G_n = \frac{Q_n}{M_n \, [\Omega_n^2(1 + j\eta_n) - \Omega^2]}, \tag{17}$$

which is precisely the same form as the forced response of a structurally damped single-degree-of-freedom system [5]. Substituting equation (17) back into equation (10), the forced vibration at a position x along the beam is given by

$$w(x, t) = \sum_{n=1}^{\infty} \frac{W_n \, (x) Q_n \, \mathrm{e}^{\,j\Omega t}}{M_n \, [\Omega_n^2(1 + j\eta_n) - \Omega^2]} \tag{18}$$

or

$$w(x, t) = \sum_{n=1}^{\infty} \frac{W_n \, (x) \int_0^l Q(x) W_n \, (x) \, \mathrm{d}x \, \mathrm{e}^{\,j\Omega t}}{\int_0^l \rho A W_n^2(x) \, \mathrm{d}x [\Omega_n^2(1 + j\eta_n) - \Omega^2]}. \tag{19}$$

It is to be noted here that the above formulations are a little different from those by Mead [6] for a constrained viscoelastic damped sandwich beam given where the system is not self-adjoint and, hence, its mode shape functions are not orthogonal to each other but orthogonal to the mode shape functions of the adjoint system. That is, a non-self-adjoint system, biorthogonality defined as follows:

$$\int_0^l \rho A W_m \, \bar{W}_n \, \mathrm{d}x = 0 \qquad (m \neq n), \tag{20}$$

where the over-bar which signifies the mode shape function of the adjoint system [7, 8], is used to decouple the equations of motion.

5. CROSS CHECKING OF THE PROPOSED TECHNIQUE BY COMPARISON WITH THE MODAL STRAIN ENERGY METHOD

In order to check the validity of the proposed technique in the prediction of the forced response of the beam system, the Modal Strain Energy methods (MSE) [9] was employed for a comparison of the prediction results. For numerical comparison, a case in which a unit harmonic concentrated load was placed at the mid-point of the beam system is adapted. For the MSE, $W_n \, (x)$ were taken from mode functions of the associated undamped beam system and the modal loss factor of each mode was estimated by the corresponding modal strain energy. Since the accuracy which the MSE can achieve becomes lower with the increase in system damping, the modal loss factors of the system for the comparison study were kept to around 0·05. The responses by the two methods are given in Figure 2, where one can see that the discrepancies between the two methods are negligibly small. Hence, the accuracy of the proposed technique can be confirmed.

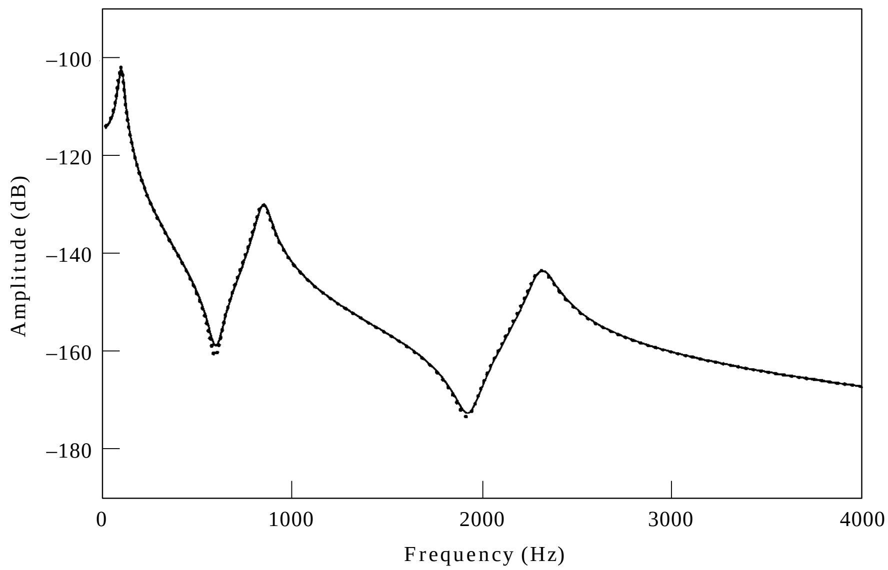

Figure 2. Comparison of the FRF amplitude at the beam center between the complex method (——) and the MSE method (· · · ·).

6. NUMERICAL RESULTS FOR VARIOUS SUPPORT LENGTHS

As an example, the forced vibration of a beam with various viscoelastic boundary support lengths driven by a harmonic load concentrated at the center of the beam ($x = l/2$) is computed numerically here.

The geometrical and physical parameters of the beam are taken as $l = 0{\cdot}2$ m, $A = 0{\cdot}0016\ \mathrm{m}^2$, $EI = 71{\cdot}68\ \mathrm{N \cdot m^2}$, $\rho = 7850\ \mathrm{kg/m^3}$, and structural damping $\eta_s = 0{\cdot}01$. The thickness of the viscoelastic layer is taken as $H = 0{\cdot}254$ mm and the material

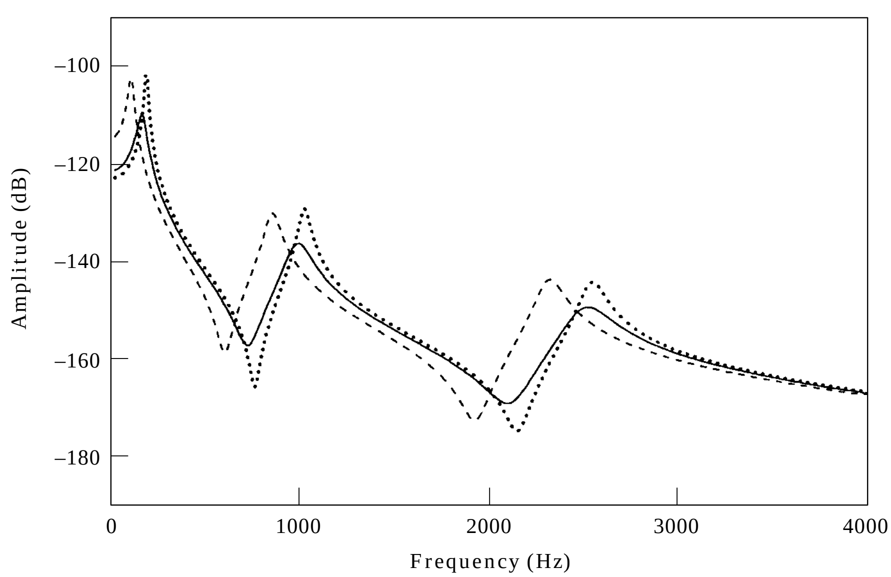

Figure 3. Forced displacement responses at the beam center for three values of support length. – – –, $L = 4$ mm; ——, $L = 8$ mm; · · · ·, $L = 16$ mm.

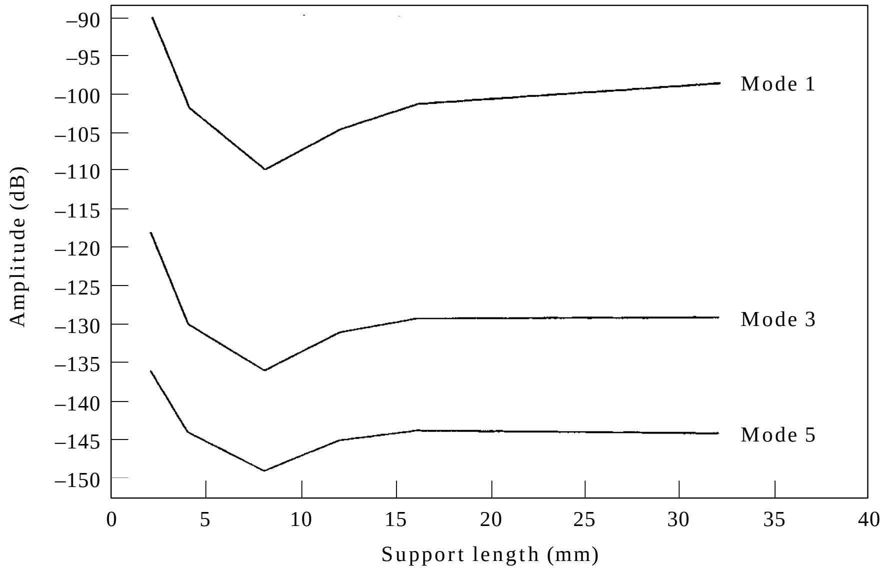

Figure 4. Variation of peak amplitude with support length.

is assumed to be 3M-SJ2015 damping tape, properties of which are taken from Kang [2].

The responses were obtained for three values of the viscoelastic support length by sweeping the excitation frequency over a given range. For a given excitation frequency Ω, the displacement amplitude and its phase angle relative to the excitation at every point along the beam were computed using equation (19). The displacement responses at the beam center under the load are shown in Figure 3 for the three support lengths $L = 4$, 8 and 16 mm.

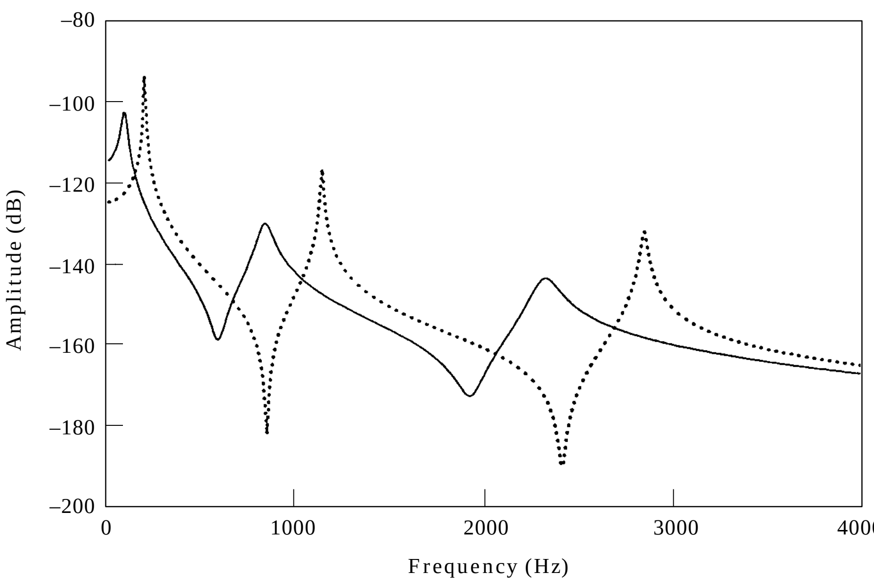

Figure 5. Displacement amplitude versus forcing frequency response. Comparison between the viscoelastic boundary supported beam (——, $L = 8$ mm) and the boundary clamped beam (· · · ·).

Figure 4 shows variations of the amplitude of each mode with the support length, where it can be seen that there exists an optimum support length around 0·008 m at which the resonance amplitudes are minimized. Figure 5 shows comparatively the response of the boundary clamped beam together with the one supported on the optimum viscoelastic boundary supports, which shows that the displacement amplitude can be reduced roughly by up to 10 dB by appropriate damping treatment at the boundary point.

7. CONCLUSIONS

A method for the forced response analysis of a beam with viscoelastic boundary supports is presented in this paper. Under the assumption that the viscoelastic layers at the top and the bottom of the beam boundary are the same, the equation is uncoupled by the mode shape functions of the original system, that is, without introducing the adjoint system. The method is basically not much different from the conventional modal analysis except that eigenvalues and eigenfunctions are all complex quantities and can be obtained only by numerical computations. An example has been presented to illustrate the effects of the viscoelastic support length on the forced vibration response. It is shown that selection of an optimum support length can minimize the transverse amplitude of the beam for the harmonic excitation.

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APPENDIX

A.1. ORTHOGONALITY OF THE COMPLEX MODES FOR A SELF-ADJOINT SYSTEM

The complex mode shape function $W(x)$ of the beam with complex stiffness at the boundary must satisfy the equation of motion (2) and the boundary conditions given by equation (3) as follows:

$$EIW''''(x) = \bar{\Omega}^2 \rho A W(x), \tag{A1}$$

$$\begin{Bmatrix} -EIW'''(0) \\ EIW''(0) \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} W(0) \\ W'(0) \end{Bmatrix} \qquad \text{at } x = 0, \tag{A2}$$

$$\begin{Bmatrix} EIW'''(l) \\ EIW''(l) \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} W(l) \\ -W'(l) \end{Bmatrix} \qquad \text{at } x = l, \tag{A3}$$

where the prime denote differentiation with respect to the spatial coordinates.

Let $W_m(x)$ and $W_n(x)$ be the complex mode shape functions corresponding to the natural frequencies $\bar{\Omega}_m$ and $\bar{\Omega}_n$. Then they satisfy the following relationships:

$$EIW_m''''(x) - \bar{\Omega}_m^2 \rho A W_m(x) = 0, \tag{A4}$$

$$EIW_n''''(x) - \bar{\Omega}_n^2 \rho A W_n(x) = 0. \tag{A5}$$

Multiplying equations (A4) and (A5) respectively by $W_n(x)$ and $W_m(x)$, subtracting the resulting equation from the other, and integrating from 0 to l gives

$$\int_0^l [EIW_m'''' W_n - \bar{\Omega}_m^2 \rho A W_m W_n] \, \mathrm{d}x - \int_0^l [EIW_n'''' W_m - \bar{\Omega}_n^2 \rho A W_n W_m] \, \mathrm{d}x = 0 \tag{A6a}$$

or

$$\int_0^l \rho A W_m W_n \, \mathrm{d}x = \frac{1}{(\bar{\Omega}_m^2 - \bar{\Omega}_n^2)} \int_0^l (EIW_m'''' W_n - EIW_m W_n'''') \, \mathrm{d}x. \tag{A6b}$$

Each term of the right-hand side of equation (A6b) can be rewritten using integration by parts, considering the boundary conditions (A2) and (A3) as follows:

$$\begin{aligned}
&\int_0^l EIW_m'''' W_n \, \mathrm{d}x \\
&\quad = EI\left[W_n W_m''' |_0^l - W_n' W_m'' |_0^l + \int_0^l W_n'' W_m'' \, \mathrm{d}x \right] \\
&\quad = EI\Big[W_m'''(l) W_n(l) - W_m'''(0) W_n(0) - W_m''(l) W_n'(l) \\
&\quad + W_m''(0) W_n'(0) + \int_0^l W_n'' W_m'' \, \mathrm{d}x \Big] \\
&\quad = K_{11}[W_m(l) W_n(l) + W_m(0) W_n(0)] + K_{22}[W_n'(l) W_m'(l) + W_n'(0) W_m'(0)] \\
&\quad - K_{12}[W_m'(l) W_n(l) - W_m'(0) W_n(0)] - K_{21}[W_m(l) W_n'(l) - W_m(0) W_n'(0)] \\
&\quad + EI \int_0^l W_n'' W_m'' \, \mathrm{d}x
\end{aligned} \tag{A7}$$

and

$$\int_0^l EIW_m\, W_n'''' \,\mathrm{d}x$$

$$= EI\left[W_m\, W_n''' \,|_0^l - W_m'\, W_n'' \,|_0^l + \int_0^l W_m''\, W_n'' \,\mathrm{d}x \right]$$

$$= EI\left[W_n''' (l) W_m (l) - W_n''' (0) W_m (0) - W_n'' (l) W_m' (l) \right.$$

$$\left. + W_n'' (0) W_m' (0) + \int_0^l W_m''\, W_n'' \,\mathrm{d}x \right]$$

$$= K_{11} [W_n (l) W_m (l) + W_n (0) W_m (0)] + K_{22} [W_m' (l) W_n' (l) + W_m' (0) W_n' (0)]$$
$$- K_{12} [W_n' (l) W_m (l) - W_n' (0) W_m (0)] - K_{21} [W_n (l) W_m' (l) - W_n (0) W_m' (0)]$$
$$+ EI \int_0^l W_m''\, W_n'' \,\mathrm{d}x. \tag{A8}$$

From the above equations it is obvious that

$$\int_0^l EIW_m''''\, W_n \,\mathrm{d}x = \int_0^l EIW_m\, W_n'''' \,\mathrm{d}x. \tag{A9}$$

Consequently equation (A6b) reduces to

$$\int_0^l \rho A W_m\, W_n \,\mathrm{d}x = 0, \tag{A10}$$

which proves the orthogonality of the damped complex modes.