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FREE VIBRATION OF THICK CIRCULAR CYLINDRICAL SHELLS SUBJECTED TO AXIAL STRESSES

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A two-dimensional higher-order shell theory is applied to the free vibration problems of a simply supported cylindrical shell subjected to axial stresses. The effects of higher-order deformations such as shear deformations with thickness changes and rotatory inertia on natural frequencies of a thick elastic circular cylindrical shells are studied. Based on the power series expansion of displacement components, a set of fundamental dynamic equations of a two-dimensional higher-order shell theory is derived through Hamilton's principle. Several sets of truncated approximate theories which can take into account the complete effects of higher-order deformations are applied to solve the vibration problems of a simply supported thick circular cylindrical shell. In order to assure the accuracy of the present theory, the convergence of the natural frequencies is examined and the results are compared with those obtained in existing theories.

1. INTRODUCTION

A great many significant contributions can be found on vibrations of circular cylindrical shells in the literature based upon two-dimensional shell theory. Most of them have been developed for thin circular cylinders and very little for thick cylinders. Although thin shells have been the thrust of the primary applications to aerospace and marine structures, much attention is now being paid to thick shell structures. In architectural thick concrete shell structures such as domes and cylinders, the tensile stress induced by external loads is counteracted by the prestressing stress introduced in the in-plane circumferential or axial direction of the shell. It is easy to control the distribution of the prestressing stress in the shell section through the prestressing strands. Usually, two approaches have been used to analyze thick shell structures, i.e., one is based on the three-dimensional elasticity theory and the other, approximate two-dimensional *shell* theory.

It is very complicated to obtain effective solutions of the three-dimensional vibration problems of thick elastic shells and therefore few papers dealing with the free vibrations of thick circular cylindrical shells have appeared. Based on the three-dimensional theory of elasticity, Armenàkas *et al*. [1] presented a volume containing tables of natural frequencies and graphs of representative mode shapes of harmonic elastic waves propagating in an infinitely long isotropic hollow cylinder. The tables may be used directly in obtaining the frequency of standing waves propagating in simply supported shells of finite length. A finite element method was presented by Bradford and Dong [2] for the vibration and stability analyses of initially stressed orthotropic cylindrical shells. The formulation is capable of treating a three-dimensional initial stress state which is radially

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symmetric. Using the ordinary Ritz's method based on the three-dimensional theory of elasticity, Singal and Williams [3] analyzed the free vibration problem of thick circular cylindrical shells and rings. For the various experimental models with free–free boundary conditions, calculated and measured resonant frequencies were compared to assess the accuracy of the analysis. In the free vibration problem of a homogeneous isotropic thick cylindrical shell or panel with simply supported boundaries, a three-dimensional solution method has been presented by Soldatos and Hadjigeorgiou [4]. The governing equations of three-dimensional linear elasticity were solved by using an iterative mathematical approach to obtain the natural frequencies of a thick cylindrical shell which is composed of fictious layers. Recently, So and Leissa [5] developed a three-dimensional method of analysis for the free vibration frequencies of elastic hollow circular cylinders having all surfaces free. The Ritz method based upon the local co-ordinates was used to achieve accurate frequencies for free hollow circular cylinders of finite length and comparisons were made with other three-dimensional results.

In order to take into account the influence of transverse shear deformation and rotatory inertia, a number of authors derived modified shell theories in the past. Mirsky and Herrmann [6] developed a Timoshenko–Mindlin-type shear deformation theory by introducing the shear correction coefficient κ^2, as was done in Timoshenko beams and Mindlin plates. The dynamic shear coefficients were determined by considering the thickness-shear motions in axial and circumferential directions, respectively. By expanding the shell displacement components in power series of the thickness co-ordinate, there exist approximate two-dimensional shell theories. Upon using certain truncations of the power series, a higher-order shell theory which can take into account the first order effects of transverse shear deformations has been applied to cylindrical shells by Bhimaraddi [7]. Based upon a realistic parabolic variation for shear strains with zero values at the external surfaces, the shear correction factors are not required in the theory. Transverse normal strain is assumed to be zero and transverse normal stress in the direction of the shell thickness is excluded. However, two-dimensional higher-order theories of circular cylindrical shells which take into account the complete effects of shear deformations with thickness changes and rotatory inertia have not been investigated. Recently, it has been pointed out that neglecting higher-order deformations such as shear deformations and thickness changes will lead to an overprediction of the natural frequency for shallow circular arches [8] and thick circular rings [9].

This paper presents a two-dimensional higher-order theory of thick circular cylindrical shells which can take into account the complete effects of both shear deformations with thickness changes and rotatory inertia. Several sets of the governing equations of truncated approximate theories are applied to the analysis of the free vibration problem of a simply supported circular cylindrical shell subjected to axial stresses. On the basis of the power series expansions of displacement components, a fundamental set of dynamic equations of a two-dimensional higher-order theory for the vibration problem of thick circular cylindrical shells is derived through Hamilton's principle. The equations of motion of a shell subjected to axial stresses are expressed in terms of the displacement components. Following the Navier solution procedure, the displacement components are expanded into Fourier series that satisfy the simply supported boundary conditions. Natural frequencies of a circular cylindrical shell subjected to axial stresses are obtained by solving the eigenvalue problem numerically. The convergence properties of the present numerical solutions are shown to be accurate for the natural frequencies with respect to the order of approximate theories. A comparison of the obtained natural frequencies is also made with those of existing theories. Natural frequency and buckling stress for a simply supported circular cylindrical shell subjected to axial stress can be expressed analytically with reference to the corresponding natural frequency for the shell without axial stress.

The present results obtained by various sets of approximate theories are considered to be accurate enough for the natural frequencies of thick circular cylindrical shells by taking into account the effects of shear deformations and rotatory inertia. It may be noticed that the two-dimensional higher-order shell theory in the present paper is useful for the vibration problem of extremely thick circular cylindrical shells.

2. FUNDAMENTAL EQUATIONS OF CIRCULAR CYLINDRICAL SHELLS

Consider a circular cylindrical shell of mean radius of curvature R, thickness H and length L. As shown in Figure 1, a curvilinear co-ordinate system (x, y, z) is defined on the middle surface of the circular cylindrical shell, where the x-axis is taken along the middle surface in the circumferential direction with the y-axis in the axial direction and the z-axis in the direction normal to the tangent to the middle surface. The dynamic displacement components in a shell are expressed as

$$u \equiv u(x, y, z; t), \qquad v \equiv v(x, y, z; t), \qquad w \equiv w(x, y, z; t), \tag{1}$$

where t denotes time. The displacement components may be expanded into power series of the thickness co-ordinate z as follows:

$$u = \sum_{n=0}^{\infty} \overset{(n)}{u} z^n, \qquad v = \sum_{n=0}^{\infty} \overset{(n)}{v} z^n, \qquad w = \sum_{n=0}^{\infty} \overset{(n)}{w} z^n, \tag{2}$$

where $n = 0, 1, 2, \ldots, \infty$.

2.1. STRAIN-DISPLACEMENT RELATIONS

Strain components may be expanded as follows:

$$\gamma_{\alpha\beta} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}_{\alpha\beta} z^n, \qquad \gamma_{\alpha z} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}_{\alpha z} z^n, \qquad \gamma_{zz} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}_{zz} z^n, \tag{3}$$

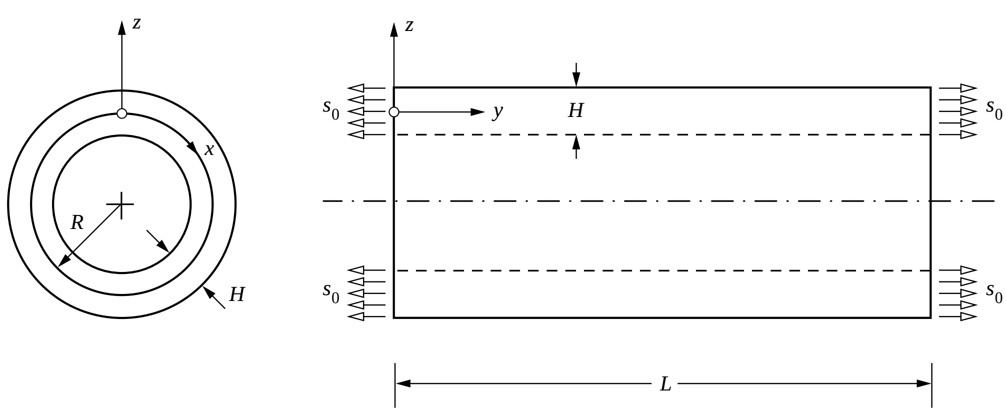

Figure 1. Co-ordinate and geometry of circular cylindrical shell.

where Greek lower case subscripts indicate the co-ordinate x or y. Strain-displacement relations can be written as (Yokoo and Matsunaga [10])

$$\overset{(n)}{\gamma}_{xx} = \overset{(n)}{u}_{,x} - \frac{1}{R}\overset{(n)}{w} - \frac{1}{R}\left(\overset{(n-1)}{u}_{,x} - \frac{1}{R}\overset{(n-1)}{w}\right), \qquad \overset{(n)}{\gamma}_{yy} = \overset{(n)}{v}_{,y}, \qquad \overset{(n)}{\gamma}_{zz} = (n+1)\overset{(n+1)}{w},$$

$$\overset{(n)}{\gamma}_{xy} = \overset{(n)}{\gamma}_{yx} = \frac{1}{2}\left(\overset{(n)}{u}_{,y} + \overset{(n)}{v}_{,x} - \frac{1}{R}\overset{(n-1)}{u}_{,y}\right), \tag{4}$$

$$\overset{(n)}{\gamma}_{xz} = \frac{1}{2}\left\{(n+1)\overset{(n+1)}{u} - \frac{n-1}{R}\overset{(n)}{u} + \overset{(n)}{w}_{,x}\right\}, \qquad \overset{(n)}{\gamma}_{yz} = \frac{1}{2}\left\{(n+1)\overset{(n+1)}{v} + \overset{(n)}{w}_{,y}\right\},$$

where a comma indicates partial differentiation with respect to the co-ordinate subscripts that follow. No restrictive assumptions are made concerning the order of H/R.

2.2. EQUATIONS OF MOTION

Consider a true cylindrical shell subjected to a uniformly distributed initial axial stress, s_0, which is assumed to be constant in the axial direction. Since it is assumed that the initial deformation due to the axial stress is axisymmetric and is uniformly distributed in the axial direction, there is no influence of the initial deformation in the present problem. Introducing stress components $s^{\alpha\beta}$, $s^{\alpha z}$ and s^{zz}, Hamilton's principle is applied to derive the equations of dynamic equilibrium and natural boundary conditions of a shell. An additional work due to the initial axial stress which is assumed to remain unchanged during vibration is taken into consideration. Both the outer and inner surfaces of a shell are assumed to be traction free. The principle for the present problem may be expressed for an arbitrary time interval t_1 to t_2 as follows:

$$\int_{t_1}^{t_2}\int_V [s^{xx}\delta\gamma_{xx} + s^{yy}\delta\gamma_{yy} + 2(s^{xy}\delta\gamma_{xy} + s^{xz}\delta\gamma_{xz} + s^{yz}\delta\gamma_{yz}) + s^{zz}\delta\gamma_{zz}$$
$$- \rho(\dot{u}\delta\dot{u}_{,x} + \dot{v}\delta\dot{v}_{,y} + \dot{w}\delta\dot{w}_{,z}) + s_0\,(u_{,y}\,\delta u_{,y} + v_{,y}\,\delta v_{,y} + w_{,y}\,\delta w_{,y})]\,\mathrm{d}V\,\mathrm{d}t = 0, \tag{5}$$

where the over-dot indicates partial differentiation with respect to time, ρ denotes the mass density, and $\mathrm{d}V$, the volume element. The volume element is given in terms of normal curvilinear co-ordinates defined for the middle surface S by

$$\mathrm{d}V = \mu\,\mathrm{d}z\,\mathrm{d}S, \qquad \mu = 1 - \frac{z}{R}. \tag{6}$$

The initial axial stress is assumed to be expressed as the following power series:

$$s_0 = \sum_{l=0}^{\infty} \overset{(l)}{s}_0\, z^l, \tag{7}$$

where $l = 0, 1, 2, \ldots, \infty$.

By performing the variation as indicated in equation (5), the equations of motion are obtained as follows:

$$\delta \overset{(n)}{u}: \quad \left(\overset{(n)}{N}{}^{xx} - \frac{1}{R}\overset{(n+1)}{N}{}^{xx}\right)_{,x} + \left(\overset{(n)}{N}{}^{xy} - \frac{1}{R}\overset{(n+1)}{N}{}^{xy}\right)_{,y} - n\overset{(n-1)}{Q}{}^{x} + (n-1)\frac{1}{R}\overset{(n)}{Q}{}^{x}$$

$$+ \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \overset{(l)}{s_0}\,\overset{(m)}{u}_{,yy} f(n+m+l+1) = \rho \sum_{m=0}^{\infty} f(n+m+1)\overset{(m)}{\ddot{u}}, \tag{8}$$

$$\delta \overset{(n)}{v}: \quad \overset{(n)}{N}{}^{xy}_{,x} + \overset{(n)}{N}{}^{yy}_{,y} - n\overset{(n-1)}{Q}{}^{y}$$

$$+ \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \overset{(l)}{s_0}\,\overset{(m)}{v}_{,yy} f(n+m+l+1) = \rho \sum_{m=0}^{\infty} f(n+m+1)\overset{(m)}{\ddot{v}}, \tag{9}$$

$$\delta \overset{(n)}{w}: \quad \frac{1}{R}\left(\overset{(n)}{N}{}^{xx} - \frac{1}{R}\overset{(n+1)}{N}{}^{xx}\right) + \overset{(n)}{Q}{}^{x}_{,x} + \overset{(n)}{Q}{}^{y}_{,y} - n\overset{(n-1)}{T}$$

$$+ \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \overset{(l)}{s_0}\,\overset{(m)}{w}_{,yy} f(n+m+l+1) = \rho \sum_{m=0}^{\infty} f(n+m+1)\overset{(m)}{\ddot{w}}, \tag{10}$$

where $n, m = 0, 1, 2, \ldots, \infty$.

The stress resultants are defined as follows:

$$\overset{(n)}{N}{}^{\alpha\beta} = \int_{-H/2}^{+H/2} \mu s^{\alpha\beta} z^n \, \mathrm{d}z, \qquad \overset{(n)}{Q}{}^{\alpha} = \int_{-H/2}^{+H/2} \mu s^{\alpha z} z^n \, \mathrm{d}z, \qquad \overset{(n)}{T} = \int_{-H/2}^{+H/2} \mu s^{zz} z^n \, \mathrm{d}z. \tag{11}$$

The following functions are defined as

$$f(k) \equiv g(k) - \frac{1}{R} g(k+1), \tag{12}$$

where k is an integer and

$$g(k) \equiv \int_{-H/2}^{+H/2} z^{k-1} \, \mathrm{d}z = \frac{1}{k}\left(\frac{H}{2}\right)^k [1 - (-1)^k] = \begin{cases} 0 & (k\text{: even}) \\ \dfrac{2}{k}\left(\dfrac{H}{2}\right)^k & (k\text{: odd}). \end{cases} \tag{13}$$

2.3. CONSTITUTIVE RELATIONS

For elastic and isotropic materials, the constitutive relations can be written as

$$s^{xx} = (D_{00} + E_1)\gamma_{xx} + E_1(\gamma_{yy} + \gamma_{zz}), \qquad s^{yy} = (D_{00} + E_1)\gamma_{yy} + E_1(\gamma_{xx} + \gamma_{zz}),$$
$$s^{xy} = s^{yx} = D_{00}\gamma_{xy}, \qquad s^{xz} = D_{00}\gamma_{xz}, \qquad s^{yz} = D_{00}\gamma_{yz}, \tag{14}$$
$$s^{zz} = (D_{00} + E_1)\gamma_{zz} + E_1(\gamma_{xx} + \gamma_{yy}),$$

where $\delta^{\alpha\beta}$ is Kronecker's delta and Lamé's constants D_{00} and E_1 are defined by using Young's modulus E and Poisson's ratio v as follows:

$$D_{00} \equiv \frac{E}{1+v}, \qquad E_1 \equiv \frac{vE}{(1+v)(1-2v)}. \tag{15}$$

2.4. BOUNDARY CONDITIONS

For the equations of boundary conditions along the boundaries on the middle surface:

$$\overset{(n)}{u} \quad \text{or} \quad v_x \overset{(n)}{N^{xx}} + v_y \overset{(n)}{N^{yx}} + \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} v_y \overset{(l)}{s_0} \overset{(m)}{u}_{,y} f(n+m+l+1),$$

$$\overset{(n)}{v} \quad \text{or} \quad v_y \overset{(n)}{N^{yy}} + v_x \overset{(n)}{N^{xy}} + \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} v_y \overset{(l)}{s_0} \overset{(m)}{v}_{,y} f(n+m+l+1), \tag{16}$$

$$\overset{(n)}{w} \quad \text{or} \quad v_x \overset{(n)}{Q^{x}} + v_y \overset{(n)}{Q^{y}} + \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} v_y \overset{(l)}{s_0} \overset{(m)}{w}_{,y} f(n+m+l+1),$$

are to be prescribed.

2.5. STRESS RESULTANTS IN TERMS OF THE EXPANDED DISPLACEMENT COMPONENTS

Stress resultants can be expressed in terms of the expanded displacement components.

$$\overset{(n)}{N^{xx}} = \sum_{m=0}^{\infty} \left\{ (D_{00} + E_1) \left[\overset{(m)}{u}_{,x} - \frac{1}{R} \overset{(m)}{w} - \frac{1}{R} \left(\overset{(m-1)}{u}_{,x} - \frac{1}{R} \overset{(m-1)}{w} \right) \right] \right.$$

$$\left. + E_1 [\overset{(m)}{v}_{,y} + (m+1) \overset{(m+1)}{w}] \right\} f(n+m+1), \tag{17}$$

$$\overset{(n)}{N^{yy}} = \sum_{m=0}^{\infty} \left\{ (D_{00} + E_1) \overset{(m)}{v}_{,y} + E_1 \left[\overset{(m)}{u}_{,x} + (m+1) \overset{(m+1)}{w} \right. \right.$$

$$\left. \left. - \frac{1}{R} \overset{(m)}{w} - \frac{1}{R} \left(\overset{(m-1)}{u}_{,x} - \frac{1}{R} \overset{(m-1)}{w} \right) \right] \right\} f(n+m+1), \tag{18}$$

$$\overset{(n)}{N^{xy}} = \overset{(n)}{N^{yx}} = \frac{D_{00}}{2} \sum_{m=0}^{\infty} \left(\overset{(m)}{u}_{,y} + \overset{(m)}{v}_{,x} - \frac{1}{R} \overset{(m-1)}{u}_{,y} \right) f(n+m+1), \tag{19}$$

$$\overset{(n)}{Q^{x}} = \frac{D_{00}}{2} \sum_{m=0}^{\infty} \left[(m+1) \overset{(m+1)}{u} - \frac{m-1}{R} \overset{(m)}{u} + \overset{(m)}{w}_{,x} \right] f(n+m+1), \tag{20}$$

$$\overset{(n)}{Q^{y}} = \frac{D_{00}}{2} \sum_{m=0}^{\infty} [(m+1) \overset{(m+1)}{v} + \overset{(m)}{w}_{,y}] f(n+m+1), \tag{21}$$

$$\overset{(n)}{T} = \sum_{m=0}^{\infty} \left\{ (D_{00} + E_1)(m+1) \overset{(m+1)}{w} + E_1 \left[\overset{(m)}{u}_{,x} + \overset{(m)}{v}_{,y} \right. \right.$$

$$\left. \left. - \frac{1}{R} \overset{(m)}{w} - \frac{1}{R} \left(\overset{(m-1)}{u}_{,x} - \frac{1}{R} \overset{(m-1)}{w} \right) \right] \right\} f(n+m+1), \tag{22}$$

where $n, m = 0, 1, 2, \ldots, \infty$.

2.6. EQUATIONS OF MOTION IN TERMS OF THE EXPANDED DISPLACEMENT COMPONENTS

The equations of motion can be expressed in terms of the expanded displacement components as

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\Bigg[\Bigg\{(D_{00}+E_1)\Bigg[\overset{(m)}{u}_{,x}-\frac{1}{R}\overset{(m)}{w}-\frac{1}{R}\Bigg(\overset{(m-1)}{u}_{,x}-\frac{1}{R}\overset{(m-1)}{w}\Bigg)\Bigg]_{,x}+E_1\,[\overset{(m)}{v}_{,y}+(m+1)\overset{(m+1)}{w}]_{,x}\\
&\quad+\frac{D_{00}}{2}\Bigg(\overset{(m)}{u}_{,y}+\overset{(m)}{v}_{,x}-\frac{1}{R}\overset{(m-1)}{u}_{,y}\Bigg)_{,y}\Bigg\}\Bigg[f(n+m+1)-\frac{1}{R}f(n+m+2)\Bigg]\\
&\quad+\Bigg\{\frac{n-1}{R}\frac{D_{00}}{2}\Bigg[(m+1)\overset{(m+1)}{u}-\frac{m-1}{R}\overset{(m)}{u}+\overset{(m)}{w}_{,x}\Bigg]-\rho\ddot{\overset{(m)}{u}}\Bigg\}f(n+m+1)\\
&\quad-\frac{nD_{00}}{2}\Bigg[(m+1)\overset{(m+1)}{u}-\frac{m-1}{R}\overset{(m)}{u}+\overset{(m)}{w}_{,x}\Bigg]f(n+m)\\
&\quad+\sum_{l=0}^{\infty}\overset{(l)}{s_0}\,\overset{(m)}{u}_{,yy}f(n+m+l+1)\Bigg]=0, \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\Bigg[\Bigg\{\frac{D_{00}}{2}\Bigg(\overset{(m)}{u}_{,y}+\overset{(m)}{v}_{,x}-\frac{1}{R}\overset{(m-1)}{u}_{,y}\Bigg)_{,x}+\Bigg\{(D_{00}+E_1)\overset{(m)}{v}_{,y}+E_1\Bigg[\overset{(m)}{u}_{,x}+(m+1)\overset{(m+1)}{w}\\
&\quad-\frac{1}{R}\overset{(m)}{w}-\frac{1}{R}\Bigg(\overset{(m-1)}{u}_{,x}-\frac{1}{R}\overset{(m-1)}{w}\Bigg)\Bigg]\Bigg\}_{,y}-\rho\ddot{\overset{(m)}{v}}\Bigg\}f(n+m+1)-\frac{nD_{00}}{2}\Bigg[(m+1)\overset{(m+1)}{v}\\
&\quad+\overset{(m)}{w}_{,y}\Bigg]f(n+m)+\sum_{l=0}^{\infty}\overset{(l)}{s_0}\,\overset{(m)}{v}_{,yy}f(n+m+l+1)\Bigg]=0, \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\Bigg[\frac{1}{R}\Bigg\{(D_{00}+E_1)\Bigg[\overset{(m)}{u}_{,x}-\frac{1}{R}\overset{(m)}{w}-\frac{1}{R}\Bigg(\overset{(m-1)}{u}_{,x}-\frac{1}{R}\overset{(m-1)}{w}\Bigg)\Bigg]\\
&\quad+E_1\,[\overset{(m)}{v}_{,y}+(m+1)\overset{(m+1)}{w}]\Bigg\}\Bigg[f(n+m+1)-\frac{1}{R}f(n+m+2)\Bigg]\\
&\quad+\Bigg\{\frac{D_{00}}{2}\Bigg(\Bigg[(m+1)\overset{(m+1)}{u}-\frac{m-1}{R}\overset{(m)}{u}+\overset{(m)}{w}_{,x}\Bigg]_{,x}+[(m+1)\overset{(m+1)}{v}+\overset{(m)}{w}_{,y}]_{,y}\Bigg)\\
&\quad-\rho\ddot{\overset{(m)}{w}}\Bigg\}f(n+m+1)-n\Bigg\{(D_{00}+E_1)\,(m+1)\overset{(m+1)}{w}+E_1\Bigg[\overset{(m)}{u}_{,x}+\overset{(m)}{v}_{,y}-\frac{1}{R}\overset{(m)}{w}
\end{aligned}
$$

$$-\frac{1}{R}\left(\overset{(m-1)}{u}_{,x}-\frac{1}{R}\overset{(m-1)}{w}\right)\Bigg]\Bigg\}f(n+m)+\sum_{l=0}^{\infty}\overset{(l)}{s}_0\,\overset{(m)}{w}_{,yy}f(n+m+l+1)\Bigg]=0. \tag{25}$$

2.7. Mth order approximate theory

Since the fundamental equations mentioned above are complex, approximate theories of various orders may be considered for the present problem. A set of the following combination of displacement components for Mth ($M \geqslant 1$) order approximate equations is proposed:

$$u=\sum_{m=0}^{2M-1}\overset{(m)}{u}z^m,\qquad v=\sum_{m=0}^{2M-1}\overset{(m)}{v}z^m,\qquad w=\sum_{m=0}^{2M-2}\overset{(m)}{w}z^m, \tag{26}$$

where $m = 0, 1, 2, 3, \ldots, M$.

The total number of the unknown displacement components is $(6M - 1)$. In the above cases of $M = 1$, an assumption of plane strains is inherently imposed. Another set of the governing equations of the lowest order approximate theory ($M = 1$) is derived with the use of an assumption that the normal stress s^{zz} is zero.

Under the assumption of plane state of stresses, the shear strains γ_{xz} and γ_{yz} must vanish through the thickness of a shell and the lowest order approximate theory reduces to the classical shell theory.

3. FOURIER SERIES SOLUTION FOR CIRCULAR CYLINDRICAL SHELL

A simply supported circular cylindrical shell subjected to initial axial stress is analyzed for natural frequencies and vibration modes.

Following the Navier solution procedure, displacement components are assumed for the circumferential wave number $r \geqslant 1$ as

$$\overset{(n)}{u}=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\overset{(n)}{u}_{rs}\cos\frac{rx}{R}\sin\frac{s\pi y}{L}\cdot e^{\mathbf{i}\omega t},\qquad \overset{(n)}{v}=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\overset{(n)}{v}_{rs}\sin\frac{rx}{R}\cos\frac{s\pi y}{L}\cdot e^{\mathbf{i}\omega t},$$

$$\overset{(n)}{w}=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\overset{(n)}{w}_{rs}\sin\frac{rx}{R}\sin\frac{s\pi y}{L}\cdot e^{\mathbf{i}\omega t},\qquad (0\leqslant x\leqslant 2\pi R,\ 0\leqslant y\leqslant L), \tag{27}$$

where the displacement mode number $r = 1, 2, 3, \ldots, \infty$ and $s = 1, 2, 3, \ldots, \infty$, ω denotes the circular frequency and $\mathbf{i}$, the imaginary unit. When the circumferential wave number $r = 0$, the following two types of displacement mode may be assumed:

$$\overset{(n)}{u}=\sum_{s=1}^{\infty}\overset{(n)}{u}_{0s}\sin\frac{s\pi y}{L}\cdot e^{\mathbf{i}\omega t},\qquad \overset{(n)}{v}=0,\qquad \overset{(n)}{w}=0, \tag{28}$$

$$\overset{(n)}{u}=0,\qquad \overset{(n)}{v}=\sum_{s=1}^{\infty}\overset{(n)}{v}_{0s}\cos\frac{s\pi y}{L}\cdot e^{\mathbf{i}\omega t},\qquad \overset{(n)}{w}=\sum_{s=1}^{\infty}\overset{(n)}{w}_{0s}\sin\frac{s\pi y}{L}\cdot e^{\mathbf{i}\omega t}. \tag{29}$$

Equations (28) and (29) correspond to torsional and axisymmetric vibration modes, respectively.

The equations of motion are rewritten in terms of the generalized displacement components $\overset{(n)}{u}_{rs}$, and $\overset{(n)}{v}_{rs}$ and $\overset{(n)}{w}_{rs}$. The present theory yields $(6M - 1)$-frequencies for each

TABLE 1

Convergence of natural frequencies and comparison with previously published results ($L/R = 2,\ \nu = 0{\cdot}3,\ r = s = 1,\ M = 1$–$5$)

H/R	Ω	CST	FST	$M = 1$	$M = 1\dagger$	$M = 2$	$M = 3$	$M = 4$	$M = 5$
0·05	Ω_1	0·04848	0·04848	0·05014	0·04848	←	←	←	←
	Ω_2	0·1055	0·0155	0·1056	←	←	←	←	←
0·10	Ω_1	0·09739	0·09740	0·1008	0·09741	0·09736	←	←	←
	Ω_2	0·2113	0·2116	0·2117	0·2116	←	←	←	←
0·20	Ω_1	0·1981	0·1981	0·2054	0·1982	0.1978	←	←	←
	Ω_2	0·4247	0·4268	0·4279	0·4269	0·4272	←	←	←
0·40	Ω_1	0·4207	0·4178	0·4354	0·4189	0·4163	←	←	←
	Ω_2	0·8662	0·8782	0·8858	0·8791	0·8815	←	←	←
0·50	Ω_1	0·5460	0·5381	0·5622	0·5405	0·5361	0·5360	←	←
	Ω_2	1·0984	1·1165	1.1298	1.1181	1.1224	1.1225	←	←
0·80	Ω_1	0.9387	0·9837	0·9871	0·9492	0·9365	0·9360	←	←
	Ω_2	1·8613	1·8743	1·9128	1·8802	1·8931	1·8936	1·8938	←
1·00	Ω_1	1·3207	1·2354	1·3045	1·2552	1·2337	1·2327	←	←
	Ω_2	2·4330	2·4034	2·4629	2·4129	2·4332	2·4339	2·4345	2·4347

CST: classical shell theory (The Donnell theory [12]); FST: first order shear deformation shell theory (shear coefficient $\kappa^2 = 5/6$). $M = 1$: plane strain in thickness direction; $M = 1\dagger$: plane stress in thickness direction (FST, $\kappa^2 = 1$).

combination of the displacement mode numbers r and s. In the following analysis, the axial stress is assumed to be distributed uniformly in the thickness direction. Only the first term of the expanded axial stress (7) is considered, i.e., $s_0 = \overset{(0)}{s_0}$.

The dimensionless natural frequency and the initial axial stress in the y direction for vibration problems are defined as follows:

$$\Omega = \omega H\sqrt{\rho/G}, \qquad \Lambda = 2\pi R H s_0 / P_c, \tag{30}$$

where G is the shear modulus and P_c is the minimum classical buckling load for the bending problem of a simply supported straight beam of length L with circular cross-section of radius R defined by

$$G = E/2(1+\nu), \qquad P_c = \pi^2 EI/L^2, \qquad I = \pi R^4/4. \tag{31}$$

4. EIGENVALUE PROBLEM OF A THICK CIRCULAR CYLINDRICAL SHELL

Equations (23)–(25) can be rewritten by collecting the coefficients for the generalized displacements of any fixed values r and s. The generalized displacement vector $\{\mathbf{U}\}$ for the Mth order approximate theory is expressed as

$$\{\mathbf{U}\}^T = \{\overset{(0)}{u_{rs}}, \ldots, \overset{(2M-1)}{u_{rs}}; \overset{(0)}{v_{rs}}, \ldots, \overset{(2M-1)}{v_{rs}}; \overset{(0)}{w_{rs}}, \ldots, \overset{(2M-2)}{w_{rs}}\}. \tag{32}$$

Eigenvalue problems to determine the natural frequency are generalized as follows:

$$([\mathbf{K}] - \Omega^2[\mathbf{M}])\{\mathbf{U}\} = 0, \tag{33}$$

where matrix $[\mathbf{K}]$ denotes the stiffness matrix which contains the effects of axial stress and matrix $[\mathbf{M}]$, the mass matrix.

In order to analyze the eigenvalue problems, equation (33) may be rewritten as follows:

$$\left([\mathbf{K}]^{-1}[\mathbf{M}] - \frac{1}{\Omega^2}[\mathbf{I}]\right)\{\mathbf{U}\} = 0 \rightarrow \det\left([\mathbf{K}]^{-1}[\mathbf{M}] - \frac{1}{\Omega^2}[\mathbf{I}]\right) = 0, \tag{34}$$

where matrix [**I**] denotes the unit matrix. The matrix $[\mathbf{K}]^{-1}[\mathbf{M}]$ is called the dynamic matrix in the vibration problem. The power method [11] is used to obtain the numerical solution of the eigenvalue problems. Although all the eigenvalues and eigenvectors can be computed by this method for each deformation mode of r and s, the dominant eigenvalues which correspond to the lower natural frequencies are of most concern.

5. NUMERICAL EXAMPLES

5.1. NUMERICAL EXAMPLES

In architectural thick concrete shell structures, an arbitrary distribution of the initial axial stress can be introduced by the well-controlled prestressing stress to counteract the tensile stress in the concrete. It is easy to control the distribution of the prestressing stress in the shell section through the prestressing strands. In the numerical examples, the initial axial stress is assumed to be distributed uniformly in the thickness direction. Natural frequencies of a thick elastic circular cylindrical shell with simply supported boundaries

TABLE 2

Comparison of the lowest natural frequencies $\bar{\Omega}$ of circular cylindrical shells ($H/R = 0{\cdot}18,\ L/R = 2,\ \nu = 0{\cdot}3$)

s		$r = 1$	$r = 2$	$r = 3$	$r = 4$
	P	0·05645	0·03871	0·04868	0·07627
	E	0·05652	0·03929	0·04996	0·07821
	B	0·05653	0·03944	0·05009	0·07833
1	S1	0·05639	0·03890	0·04897	0·07644
	S2	0·05639	0·03891	0·04899	0·07651
	F	0·05668	0·03999	0·05259	0·08521
	CST	0·05652	0·03982	0·05237	0·08484
	P	0·09473	0·08520	0·08961	0·10973
	E	0·09402	0·08545	0·09093	0·11205
	B	0·09409	0·08562	0·09109	0·11202
2	S1	0·09484	0·08542	0·08989	0·10988
	S2	0·09485	0·08545	0·08944	0·10999
	F	0·09624	0·08903	0·09793	0·12532
	CST	0·09662	0·08899	0·09758	0·12470
	P	0·19082	0·19554	0·20570	0·22265
	E	0·18894	0·19467	0·20616	0·22450
	B	0·18832	0·19403	0·20544	022361
4	S1	0·19004	0·19472	0·20476	0·22148
	S2	0·19027	0·19498	0·20507	0·22186
	F	0·21994	0·22970	0·24841	0·27771
	CST	0·22145	0·23050	0·24832	0·27669
	P	0·50529	0·51073	0·51985	0·53269
	E	0·50338	0·50937	0·51934	0·53325
	B	0·49818	0·50418	0·51416	0·52808
8	S1	0·49690	0·50223	0·51117	0·52374
	S2	0·49850	0·50386	0·51285	0·52548
	F	0·72479	0·73058	0·74057	0·75492
	CST	0·72343	0·72829	0·73719	0·75058

P: present solution ($M = 5$); E: elasticity [1]; B: higher order theory [7]; S: first order shear deformation theory (S1, $\kappa^2 = \pi^2/12$; S2; $\kappa^2 = 5/6$). F: Flügge's theory [13]; CST: Timoshenko shell theory [12].

TABLE 3

Comparison of the first three natural frequencies $\bar{\omega}$ with previously published results ($L/R = 1$, $\nu = 0{\cdot}3$, $s = 1$)

		Present solution			Soldatos and Hadjigeorgiou [4]			Armenàkas *et al.* [1]		
H/R	r	I	II	III	I	II	III	I	II	III
0·1	1	1·06590	2·38349	3·96351	1·06238	2·37453	3·96340	1·06226	2·37443	3·96340
	2	0·88288	2·72809	4·49133	0·88260	2·71595	4·48757	0·88233	2·71586	4·48741
	3	0·80615	3·16683	5·24651	0·80963	3·15331	5·23675	0·80925	3·15325	5·23646
	4	0·89180	3·67573	6·13965	0·89905	3·66217	6·12255	0·89877	3·66194	6·12235
	5	1·11188	4·23777	7·11638	1·12216	4·22491	7·09133	—	—	—
0·2	1	1·19889	2·41052	3·95339	1·18908	2·37580	3·95284	1·18889	2·37566	3·95272
	2	1·09606	2·76467	4·48168	1·10121	2·71841	4·46607	1·10092	2·71819	4·46586
	3	1·17627	3·20783	5·23559	1·19793	3·15695	5·19520	1·19755	3·15658	5·19492
	4	1·45296	3·71741	6·12174	1·48975	3·66670	6·05090	1·48933	3·66639	6·05026
	5	1·86426	4·27798	7·08158	1·91389	4·22968	6·97752	—	—	—
0·3	1	1·35117	2·45200	3·93519	1·33761	2·37781	3·93343	1·33727	2·37754	3·93340
	2	1·30364	2·81830	4·46208	1·32371	2·72196	4·42468	1·32335	2·72149	4·42440
	3	1·47300	3·26556	5·20782	1·52805	3·16159	5·11234	1·52764	3·16095	5·11162
	4	1·84116	3·77428	6·06879	1·92695	3·67122	5·90307	1·92660	3·67046	5·90169
	5	2·33555	4·33145	6·97350	2·44628	4·23196	6·73463	—	—	—

are analyzed in the following numerical examples. Since no restrictive assumptions are made concerning the order of thickness–curvature ratio, the limit of this parameter is taken to be $H/R = 1{\cdot}0$. The length parameter L/R is varied from 1 to 20 for short to long circular cylindrical shells. Poisson's ratio is fixed to be $\nu = 0{\cdot}3$. Only the first term of the expanded axial stress in equation (7) is considered. All the numerical results are shown in the dimensionless quantities.

Although the present sets of approximate theories of any order can easily be applied to a moderately thick shell, higher orders of the expanded two-dimensional theories may be necessary to obtain reasonably accurate solutions for an extremely thick shell. It is noticed that the proper order of present approximate theories may be estimated according to the level of thickness parameters H/R and H/L of the shell.

5.2. CONVERGENCE OF THE FIRST TWO NATURAL FREQUENCIES AND COMPARISON WITH THOSE OF EXISTING THEORIES

In order to verify the accuracy of the present solutions, the convergence properties of the first two natural frequencies Ω_1 and Ω_2 of circular cylindrical shells without axial stress for the displacement mode $r = s = 1$ are shown in Table 1. It is different from the case of plates that two types of flexural and extensional displacement modes are not separated from each other in the case of shells. For each combination of r and s values, the present Mth order approximate theory yields $(6M - 1)$-frequencies in general. The dominant first two eigenvalues which correspond to the lowest two natural frequencies are of most concern. The lower natural frequency Ω_1 is predominantly flexural modes with some shear deformations, whereas the upper frequency Ω_2 is predominantly extensional modes with thickness changes. A direct comparison of the present frequencies with those from the classical shell theory (CST) in which the effects of extension and rotatory inertia are included is made. The present natural frequencies are also compared with the results of a first order shear deformation theory (FST) which corresponds to the Mindlin plate theory in which a shear correction factor κ^2 is introduced to correct the contradictory shear stress distribution over the thickness of the shell. It is noticed that the proper order of the present approximate theories may be estimated according to the level of H/R and H/L. Since the present results for $M = 1$–4 converge accurately enough within the present order of approximate theories, only the more accurate numerical results for $M = 5$ are discussed in the following.

In Table 2, a comparison of the lowest natural frequency Ω_1 of circular cylindrical shells with the geometric parameters $H/R = 0{\cdot}18$ and $L/R = 2{\cdot}0$ is made with the results in Table 2 of Bhimaraddi's paper [7]. The form of dimensionless natural frequencies $\bar{\Omega}$ in Table 2 is different from that of the first equation of equations (30), i.e., $\bar{\Omega} = \Omega/\pi$. Another comparison is also made with the results of Soldatos and Hadjigeorgiou [4] and Armenàkas

TABLE 4

First two natural frequencies $\Omega_{1,2}$ of circular cylindrical shells ($L/R = 2$, $\nu = 0{\cdot}3$, $M = 5$)

		$s_1 = 1$		$s = 2$		$s = 3$	
H/R	r	Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2
0·05	0	0·07852	0·07860	0·08108	0·1572	0·08471	0·2358
	1	0·04848	0·1056	0·07245	0·1680	0·08141	0·2420
	2	0·02822	0·1355	0·05711	0·1922	0·07391	0·2589
	3	0·02012	0·1739	0·04554	0·2231	0·06674	0·2834
	4	0·02273	0·2175	0·04151	0·2591	0·06344	0·3134
	5	0·03183	0·2636	0·04538	0·2989	0·06568	0·3476
	6	0·04441	0·3111	0·05521	0·3413	0·07336	0·3850
	7	0·05953	0·3594	0·06903	0·3857	0·08549	0·4249
	8	0·07689	0·4082	0·08574	0·4313	0·1011	0·4667
	9	0·09634	0·4572	0·1048	0·4778	0·1194	0·5100
	10	0·1178	0·5064	0·1261	0·5251	0·1401	0·5544
0·1	0	0·1574	0·1575	0·1670	0·3151	0·1912	0·4726
	1	0·09736	0·2116	0·1507	0·3371	0·1867	0·4853
	2	0·05963	0·2717	0·1249	0·3858	0·1787	0·5196
	3	0·05639	0·3485	0·1140	0·4479	0·1775	0·5690
	4	0·08133	0·4355	0·1261	0·5198	0·1900	0·6292
	5	0·1203	0·5278	0·1572	0·5993	0·2169	0·6976
	6	0·1682	0·6227	0·2010	0·6842	0·2562	0·7725
	7	0·2232	0·7192	0·2537	0·7727	0·3052	0·8522
	8	0·2844	0·8166	0·3133	0·8639	0·3618	0·9358
	9	0·3511	0·9147	0·3786	0·9569	0·4247	1·0223
	10	0·4226	1·0131	0·4489	1·0512	0·4928	1·1111
0·2	0	0·3176	0·3178	0·3668	0·6354	0·4957	0·9528
	1	0·1978	0·4272	0·3391	0·6818	0·4933	0·9799
	2	0·1397	0·5480	0·3100	0·7820	0·4983	1·0518
	3	0·1834	0·7016	0·3327	0·9073	0·5304	1·1535
	4	0·2887	0·8753	0·4110	1·0514	0·5976	1·2757
	5	0·4227	1·0593	0·5273	1·2100	0·6967	1·4132
	6	0·5750	1·2488	0·6677	1·3790	0·8205	1·5627
	7	0·7405	1·4414	0·8242	1·5553	0·9628	1·7216
	8	0·9158	1·6359	0·9922	1·7368	1·1186	1·8879
	9	1·0985	1·8317	1·1686	1·9220	1·2845	2·0600
	10	1·2867	2·0284	1·3513	2·1100	1·4581	2·2366

(*continued opposite*)

TABLE 4—*continued*

		$s=1$		$s=2$		$s=3$	
H/R	r	Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2
	0	0·6555	0·6558	0·8924	1·3082	1·3513	1·9354
	1	0·4163	0·8815	0·8431	1·4161	1·3476	2·0185
	2	0·3686	1·1249	0·8313	1·6295	1·3713	2·1805
	3	0·5724	1·4311	0·9511	1·8831	1·4650	2·3953
	4	0·8849	1·7754	1·1823	2·1689	1·6365	2·6429
0·4	5	1·2405	2·1400	1·4813	2·4809	1·8727	2·9159
	6	1·6180	2·5159	1·8191	2·8127	2·1560	3·2097
	7	2·0077	2·8984	2·1796	3·1591	2·4720	3·5206
	8	2·4040	3·2851	2·5537	3·5162	2·8103	3·8455
	9	2·8038	3·6748	2·9362	3·8811	3·1637	4·1816
	10	3·2052	4·0664	3·3238	4·2520	3·5276	4·5267
	0	0·8359	0·8373	1·2030	1·6648	1·8355	2·4704
	1	0·5360	1·1225	1·1364	1·8097	1·8239	2·5592
	2	0·5053	1·4260	1·1276	2·0814	1·8446	2·7707
	3	0·7949	1·8068	1·2898	2·3964	1·9577	3·0412
	4	1·2110	2·2346	1·5904	2·7489	2·1715	3·3476
0·5	5	1·6738	2·6878	1·9739	3·1336	2·4675	3·6827
	6	2·1582	3·1553	2·4035	3·5431	2·8229	4·0421
	7	2·6531	3·6314	2·8593	3·9709	3·2190	4·4215
	8	3·1531	4·1131	3·3303	4·4120	3·6426	4·8158
	9	3·6549	4·5985	3·8101	4·8626	4·0847	5·2182
	10	4·1567	5·0864	4·2949	5·3190	4·5393	5·6159
	0	1·8600	1·9049	2·9652	3·5116	4·3884	5·4053
	1	1·2328	2·4347	2·7386	3·8418	4·1913	4·7753
	2	1·2561	3·0007	2·6672	4·2445	4·1199	5·0535
	3	1·9257	3·7281	2·9692	4·5665	4·2711	5·3048
	4	2·8166	4·5275	3·5565	4·9474	4·6317	5·6327
1·0	5	3·7736	5·1865	4·3044	5·4385	5·1489	6·0784
	6	4·7477	5·7934	5·1274	6·0295	5·7622	6·6376
	7	5·7119	6·4495	5·9675	6·7005	6·4184	7·2902
	8	6·6250	7·1557	6·7695	7·4407	7·0721	8·0120
	9	7·3964	7·9602	7·4763	8·2526	7·6833	8·7816
	10	7·9927	8·8689	8·0638	9·1241	8·2242	9·5849

et al. [1] for the first three frequencies of a specific case for the axial mode number $s = 1$ and $r = 1–5$ in Table 3. The dimensionless frequency for this table is defined by $\bar{\omega} = (1/\sqrt{2})\,(L/H)\Omega$. It is observed that the present frequencies are in good agreement with elasticity results [1].

5.3. NATURAL FREQUENCIES WITHOUT AXIAL STRESS

The first two natural frequencies for each wave number of $r = 0–10$ and $s = 1 - 3$ are shown in Table 4 for $L/R = 2{\cdot}0$ and all the values of H/R. The results are obtained for $M = 5$ with sufficient numerical accuracy in the approximate two-dimensional theories for the natural frequencies of thick circular cylindrical shells by taking into account the effects of higher-order deformations and rotatory inertia. For $r = 0$, the first frequencies correspond to axisymmetric vibration modes and the second ones correspond to torsional modes.

TABLE 5

Lowest natural frequency Ω_0 with vibration mode numbers and corresponding critical buckling stress Λ_{cr}

H/R		L/R = 2	3	4	6	8	10	20
0·05	Ω_0	$0{\cdot}02012^{31}$	$0{\cdot}01316^{31}$	$0{\cdot}01022^{21}$	$0{\cdot}005731^{21}$	$0{\cdot}004282^{21}$	$0{\cdot}003743^{21}$	$0{\cdot}001361^{11}$
	Λ_{cr}	$0{\cdot}004092^{31}$	$0{\cdot}008862^{31}$	$0{\cdot}01689^{21}$	$0{\cdot}02689^{21}$	$0{\cdot}04745^{21}$	$0{\cdot}08851^{21}$	$0{\cdot}01872^{11}$
0·10	Ω_0	$0{\cdot}05639^{31}$	$0{\cdot}03481^{21}$	$0{\cdot}02404^{21}$	$0{\cdot}01641^{21}$	$0{\cdot}01432^{21}$	$0{\cdot}01000^{11}$	$0{\cdot}002725^{11}$
	Λ_{cr}	$0{\cdot}01607^{31}$	$0{\cdot}03100^{21}$	$0{\cdot}04673^{21}$	$0{\cdot}1102^{21}$	$0{\cdot}2653^{21}$	$0{\cdot}3159^{11}$	$0{\cdot}3753^{11}$
0·20	Ω_0	$0{\cdot}1397^{21}$	$0{\cdot}08807^{21}$	$0{\cdot}06816^{21}$	$0{\cdot}04781^{11}$	$0{\cdot}02972^{11}$	$0{\cdot}02009^{11}$	$0{\cdot}005473^{11}$
	Λ_{cr}	$0{\cdot}04932^{21}$	$0{\cdot}09923^{21}$	$0{\cdot}1878^{21}$	$0{\cdot}4679^{11}$	$0{\cdot}5714^{11}$	$0{\cdot}6375^{11}$	$0{\cdot}7569^{11}$
0·40	Ω_0	$0{\cdot}3686^{21}$	$0{\cdot}2532^{21}$	$0{\cdot}1782^{11}$	$0{\cdot}09720^{11}$	$0{\cdot}06038^{11}$	$0{\cdot}04082^{11}$	$0{\cdot}01112^{11}$
	Λ_{cr}	$0{\cdot}1717^{21}$	$0{\cdot}4101^{21}$	$0{\cdot}6420^{11}$	$0{\cdot}9669^{11}$	$1{\cdot}1792^{11}$	$1{\cdot}3158^{11}$	$1{\cdot}5624^{11}$
0·50	Ω_0	$0{\cdot}5053^{21}$	$0{\cdot}3316^{11}$	$0{\cdot}2258^{11}$	$0{\cdot}1229^{11}$	$0{\cdot}07634^{11}$	$0{\cdot}05161^{11}$	$0{\cdot}01407^{11}$
	Λ_{cr}	$0{\cdot}2581^{21}$	$0{\cdot}5627^{11}$	$0{\cdot}8246^{11}$	$1{\cdot}2367^{11}$	$1{\cdot}5080^{11}$	$1{\cdot}6827^{11}$	$2{\cdot}0010^{11}$
0·80	Ω_0	$0{\cdot}9360^{11}$	$0{\cdot}5641^{11}$	$0{\cdot}3807^{11}$	$0{\cdot}2061^{11}$	$0{\cdot}1279^{11}$	$0{\cdot}08642^{11}$	$0{\cdot}02357^{11}$
	Λ_{cr}	$0{\cdot}5535^{11}$	$1{\cdot}0177^{11}$	$1{\cdot}4650^{11}$	$2{\cdot}1736^{11}$	$2{\cdot}6456^{11}$	$2{\cdot}9489^{11}$	$3{\cdot}5097^{11}$
1·00	Ω_0	$1{\cdot}2328^{11}$	$0{\cdot}7375^{11}$	$0{\cdot}4957^{11}$	$0{\cdot}2677^{11}$	$0{\cdot}1660^{11}$	$0{\cdot}1122^{11}$	$0{\cdot}03061^{11}$
	Λ_{cr}	$0{\cdot}7681^{11}$	$1{\cdot}3916^{11}$	$1{\cdot}9870^{11}$	$2{\cdot}9337^{11}$	$3{\cdot}5653^{11}$	$3{\cdot}9765^{11}$	$4{\cdot}7355^{11}$

The lowest natural frequencies for all the parameters considered are shown in Table 5, with classification numbers of two figures on the right shoulder of natural frequencies defining the vibration mode. The first and second figures denote the wave numbers of r and s, respectively. Although the wave mode number $r = s = 1$ appears remarkably for the lowest frequencies, higher wave mode numbers of r can be observed in the case of thinner and shorter shells.

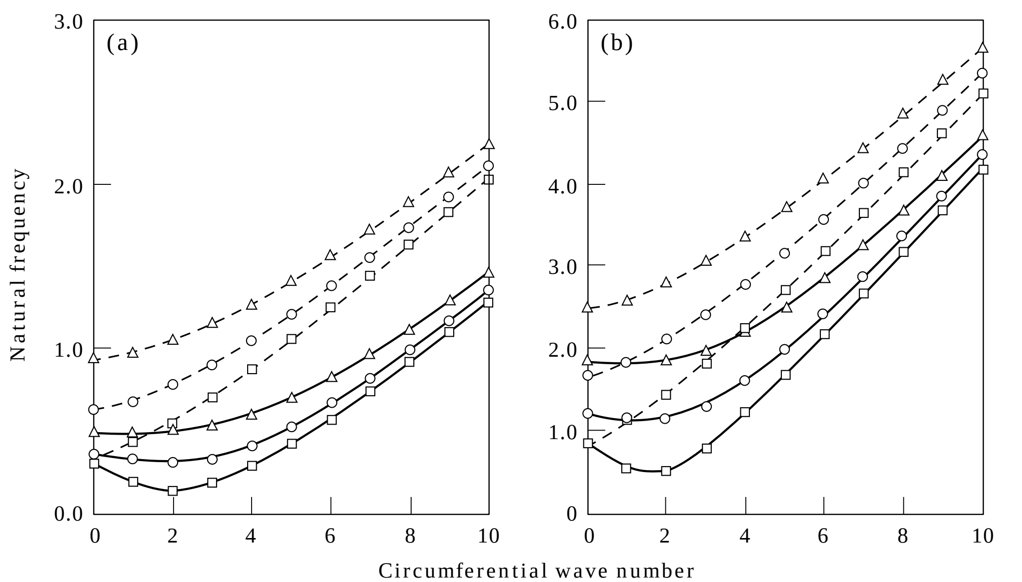

Figure 2. Natural frequency versus circumferential mode number curves ($L/R = 2$, $r = 0$–10). Ω_1: –□–, $s=1$; –○–, $s=2$; –△–, $s=3$. Ω_2 = —□—, $s=1$; —○—, $s=2$; —△—, $s=3$. (a) $H/R = 0{\cdot}2$; (b) $H/R = 0{\cdot}5$.

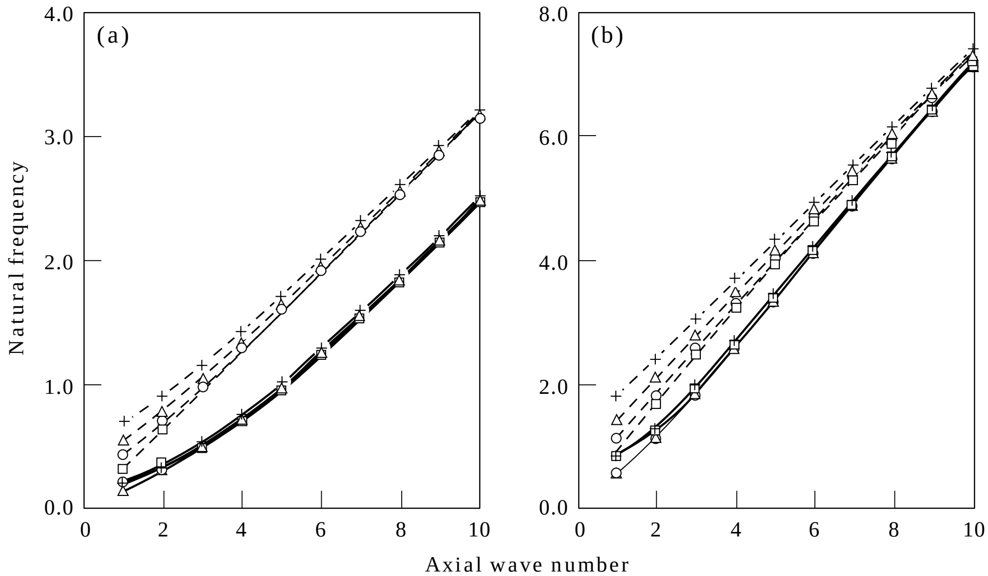

Figure 3. Natural frequency versus axial mode number curves ($L/R = 2$, $s = 1$–10). Ω_1: –□–, $r=0$; –○–, $r=1$; –△–, $r=2$; –+–, $r = 3$. Ω_2 = —□—, $r=0$; —○—, $r=1$; —△—, $r=2$; —+—, $r=3$. (a) $H/R = 0{\cdot}2$; (b) $H/R = 0{\cdot}5$.

5.4. NATURAL FREQUENCIES VERSUS WAVE MODE NUMBER CURVES

The first two natural frequencies without axial stress are plotted in Figures 2 and 3. Figure 2 shows the variations of the first two natural frequencies for $s = 1$–3 with respect to $r = 0$–10. Although, in general, the natural frequencies increase as the circumferential wave number r grows, the lowest frequencies occur at specific higher modes in the case of the first natural frequencies. The first and second natural frequencies for $r = 0$ and $s = 1$ are very close to each other. Figure 3 shows the variations of the first two natural frequencies for $r = 0$–3 with respect to $s = 1$–10. Both of the first and second natural frequencies increase monotonically as the axial wave number s grows, the lowest frequencies occurring at the first modes $s = 1$.

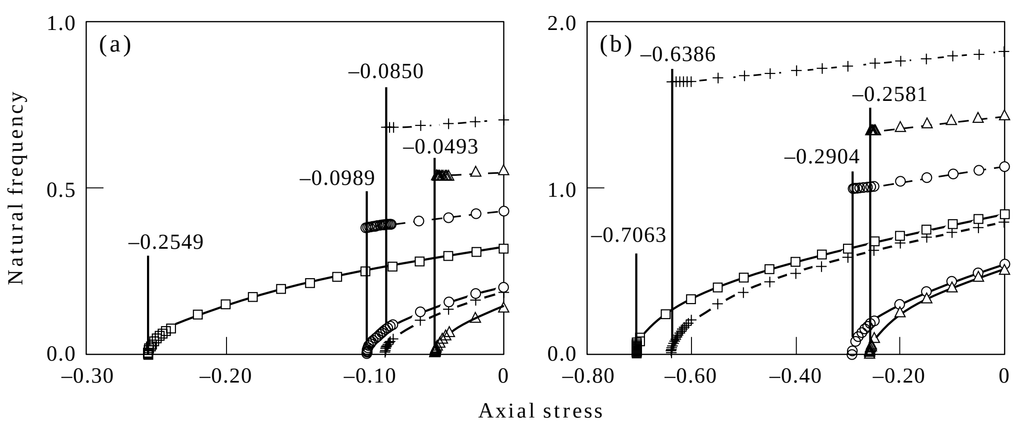

Figure 4. Natural frequency versus axial stress curves ($L/R = 2$, $s = 1$). Ω_1: –□–, $r=0$; –○–, $r=1$; –△–, $r=2$; –+–, $r = 3$. Ω_2 = —□—, $r=0$; —○—, $r=1$; —△—, $r=2$; —+—, $r=3$. (a) $H/R = 0{\cdot}2$; (b) $H/R = 0{\cdot}5$.

5.5. NATURAL FREQUENCIES VERSUS AXIAL STRESS CURVES

The natural frequency of a circular cylindrical shell subjected to initial axial stress can be obtained by solving numerically the eigenvalue problem (34) and plotted for $r = 0$–3 and $s = 1$ in Figure 4. In the case of $r = 0$, the lowest two natural frequencies which correspond to axisymmetric and torsional vibration modes, respectively, are very close to each other and cannot be distinguished in the figure. When the first natural frequency goes to zero, the critical buckling stresses are also shown in the figure for each vibration mode.

However, in the case of a simply supported circular cylindrical shell subjected to initial axial stress Λ, the natural frequency Ω_a can be expressed explicitly with reference to the natural frequency Ω_0 of a shell without axial stress. The relation between Ω_a and Ω_0 can be otbained from a comparison of the equations of motion as follows

$$\Omega_a^2 = \Omega_0^2 + \frac{(1+\nu)s^2\pi^4}{4}\left(\frac{R}{H}\right)^3\left(\frac{H}{L}\right)^4 \Lambda. \tag{35}$$

When the natural frequency Ω_a goes to zero under the axial stress Λ, elastic buckling occurs and the critical buckling stress Λ_{cr} relates with the natural frequency Ω_0 as

$$\Lambda_{cr} = -\frac{4}{(1+\nu)s^2\pi^4}\left(\frac{H}{R}\right)^3\left(\frac{L}{H}\right)^4 \Omega_0^2. \tag{36}$$

The critical buckling stress of simply supported circular cylindrical shells subjected to initial axial stress can be predicted from the natural frequency of the shell without axial stress. The calculated critical buckling stresses corresponding to the lowest natural frequencies and vibration mode numbers are also shown in Table 5. These buckling stresses do not necessarily coincide with the lowest critical buckling stresses of the shells which may occur at different displacement mode numbers from the case of the lowest natural frequencies.

6. CONCLUSIONS

Natural frequencies of thick circular cylindrical shells calculated by using the previously published thin shell theories are usually overpredicted. In order to analyze the complete effects of higher-order deformations on the natural frequencies of thick circular cylindrical shells, various orders of the expanded approximate shell theories have been presented. It has been shown that shear deformations and thickness changes have an important effect on the natural frequencies of thick circular cylindrical shells with or without axial stress, and the following conclusions may be drawn from the present analysis.

The natural frequencies of thick circular cylindrical shells calculated by using the classical thin shell theory are usually overpredicted. It has been pointed out that shear deformations and rotatory inertia have an important effect on the natural frequencies of thick circular cylindrical shells. It is very important to take into account the complete effects of higher-order deformations such as shear deformations with thickness changes and rotatory inertia for the analysis of vibration problems of thick circular cylindrical shells.

In order to verify the accuracy of the present results, the convergence properties of the numerical solutions according to the order of approximate theories have ben examined. Without the assumption of $H/R \ll 1$, the present results obtained for $M = 5$ are considered to be accurate enough for thick circular cylindrical shells. It may be noted that the two-dimensional higher-order shell theory in the present paper can predict the natural frequencies of a thick circular cylindrical shell.

In the case of a simply supported circular cylindrical shell subjected to axial stress, the natural frequency can be expressed explicitly with reference to the natural frequency of a shell without axial stress. When the natural frequency goes to zero under axial compressions, elastic buckling occurs. The critical buckling stress can also be expressed in terms of the natural frequency of a shell without axial stress.

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