



## COMPUTATION OF NORMAL FORMS VIA A PERTURBATION TECHNIQUE

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Considering qualitative behavior of a non-linear dynamical system often leads to first simplifying the differential equations or finding their normal forms. A perturbation technique for computing normal forms is presented. This technique, associated with the method of multiple scales, can be used to systematically find a unique form for a given set of differential equations. The technique is discussed in detail through the analysis of Hopf bifurcation. It is shown that for Hopf bifurcation, the method only requires solving two dimensional matrix systems for any higher order normal forms of a general  $n$ -dimensional system. With the aid of the symbolic language Maple, this approach is straightforward, and is computationally efficient and fast. Furthermore, a simple verification scheme is given for verifying the normal forms and associated non-linear transformations obtained using any methodology. Examples are presented to demonstrate the applicability of the perturbation technique and the computation efficiency of the Maple program.

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### 1. INTRODUCTION

Non-linear vibrations have a relatively long history and continue to form a basis for the study of more complex behavior patterns such as *bifurcation* and *instability* associated with dynamical systems. One of the basic tools in the study of qualitative behavior of a non-linear system near a bifurcation point is the theory of *normal forms* (e.g., see references [1–4]). The basic idea of the method of normal forms is employing successive co-ordinate transformations to systematically construct the simplest possible form of the original differential equations. The simple form can exhibit all possible dynamical properties of the original system in the neighborhood of the bifurcation point. Normal forms are generally not uniquely defined and finding a normal form for a given system of differential equations is not a simple task. In particular, finding the explicit formulas of normal forms in terms of the coefficients of the original non-linear system is not easy. Several researchers have, based on the normal form theory, developed symbolic computation programs for some bifurcation cases (e.g. references [2–4]). The normal form theory is usually applied together with center manifold theory [5]. The idea of the center manifold theory is also successive non-linear transformations. It reduces the original system to a center manifold associated with the part of the original system characterized by the eigenvalues with zero real parts at a bifurcation point. The center manifold may have smaller dimensions than that of the original system and therefore, given a non-linear system, the center manifold theory is often applied to simplify the system before applying the normal form theory.

A perturbation technique is presented which has been developed for computing normal forms via multiple scales [6]. The technique is systematic and can be directly applied to the original non-linear system, without the application of center manifold theory. In fact,

this approach has combined the two steps involved in using center manifold theory and normal form theory into one step to simultaneously obtain the normal forms and non-linear transformations. The main objective is not to study bifurcations, but to illustrate the procedure of the technique and to show the computational efficiency of the method. Thus, Hopf bifurcation will be chosen as an example for this purpose. It will be shown that the perturbation technique systematically leads to a *unique normal form* for a given set of differential equations up to any order. This approach is straightforward and can be easily implemented in a symbolic computation system. With the aid of a symbolic language such as Maple [7], one can use this approach to find the explicit expressions of normal forms in terms of the coefficients of the original system. It will also be shown that for Hopf bifurcation the method only requires solving two-dimensional matrix algebraic equations for any higher order normal forms of a general  $n$ -dimensional system. In order to verify the normal forms and the associated non-linear transformations obtained using any method, a simple approach called *system verification* is given. A number of examples are presented to show the applicability of the perturbation technique and the computation efficiency of the Maple program. All the results presented have been verified by the system verification scheme.

The perturbation technique is described in the next section. Section 3 outlines the symbolic computation. An example is given in section 4 to show the applicability of the technique, and then in section 5, the “system verification” scheme is described and used to verify the results obtained in section 4. A mechanical oscillator is presented in section 6 to illustrate the practical usefulness of the technique. In section 7, Hilbert’s 16th problem is considered to further demonstrate the computation efficiency of the technique. Finally, concluding remarks are drawn in the last section.

## 2. PERTURBATION TECHNIQUE

The method of *multiple scales* or *multiple time scales* is frequently used for analyzing second-order non-linear ordinary differential equations (ODE’s) [8], usually given in the form

$$\ddot{x} + x = \varepsilon f(x, \dot{x}), \quad (1)$$

where the dot indicates the differentiation with respect to time  $t$ , and  $\varepsilon$  is a small parameter ( $0 < \varepsilon \ll 1$ ).  $f$  is a non-linear function, assumed to be analytic and thus can be expressed as a Taylor series. Nayfeh has extended this technique to consider a more general two-dimensional system consisting of first-order ODE’s [9]:

$$\dot{x}_1 = x_2 + \varepsilon f_1(x_1, x_2), \quad \dot{x}_2 = -x_1 + \varepsilon f_2(x_1, x_2),$$

where  $f_1$  and  $f_2$  are non-linear functions. Nayfeh obtained the normal forms and periodic solutions up to second order when  $f_1$  and  $f_2$  are cubic non-linear terms [9].

A perturbation technique associated with multiple scales is developed to consider a general  $n$ -dimensional system,

$$\dot{\mathbf{x}} = J\mathbf{x} + \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in R^n, \quad \mathbf{f}: R^n \rightarrow R^n, \quad (2)$$

where  $J\mathbf{x}$  represents the linear terms, the non-linear function  $\mathbf{f}$  is assumed to be analytic, and  $\mathbf{x} = \mathbf{0}$  is an equilibrium of the system, i.e.,  $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ . Further, assume that the linearized system (2) evaluated at the equilibrium  $\mathbf{0}$  has a pair of purely imaginary eigenvalues  $\pm i\omega_c$ . Without loss of generality, we may assume that  $\omega_c = 1$  (otherwise one may use an

additional transformation  $t' = \omega_c t$  to change frequency  $\omega_c$  to 1) and the Jacobian matrix of equation (2) evaluated at  $\mathbf{x} = \mathbf{0}$  is in the Jordan canonical form

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & A \end{bmatrix}, \quad A \in \mathbb{R}^{(n-2) \times (n-2)}, \quad (3)$$

where  $A$  is hyperbolic, i.e., all of its eigenvalues have non-zero real parts, either real or complex conjugate. For the most physically interesting cases, it is assumed that the unstable manifold is empty, i.e., all the eigenvalues of  $A$  have negative real parts. Equation (3) may be written in the component form

$$\begin{aligned} \dot{x}_1 &= x_2 + f_1(x_1, x_2, \dots, x_n), & \dot{x}_2 &= -x_1 + f_2(x_1, x_2, \dots, x_n), & (4, 5) \\ \dot{x}_p &= -\alpha_p x_p + f_p(x_1, x_2, \dots, x_n), & (p &= 3, 4, \dots, m_1 + 2), & (6) \\ \dot{x}_q &= -\alpha_q x_q + \omega_q x_{q+1} + f_q(x_1, x_2, \dots, x_n), \\ \dot{x}_{q+1} &= -\omega_q x_q - \alpha_q x_{q+1} + f_{q+1}(x_1, x_2, \dots, x_n), & (q &= m_1 + 3, m_1 + 5, \dots, n - 1), & (7) \end{aligned}$$

where  $\alpha_p > 0$ ,  $\alpha_q > 0$ ,  $\omega_q > 0$ , and  $m_1 + 2m_2 + 2 = n$ . The above equations show that the matrix  $A$  has  $m_1$  real eigenvalues,  $-\alpha_p$ ,  $p = 1, 2, \dots, m_1$  and  $m_2$  pairs of complex conjugate eigenvalues,  $-\alpha_q \pm i\omega_q$ ,  $q = 1, 2, \dots, m_2$ . The functions  $f_i(x_1, x_2, \dots, x_n)$  satisfy  $f_i(0, 0, \dots, 0) = 0$  and  $\partial f_i(0, 0, \dots, 0)/\partial x_j = 0$ ,  $i, j = 1, 2, \dots, n$ . That is, the Taylor expansion of  $f_i$  about  $\mathbf{x} = \mathbf{0}$  starts from the second order terms.

The underlying idea of the method of multiple scales is to consider the expansion representing the response to be a function of multiple independent variables, or scales, instead of a single time variable. Thus, one begins by introducing new independent variables according to

$$T_k = \varepsilon^k t, \quad k = 0, 1, 2, \dots \quad (8)$$

It follows that the derivatives with respect to  $t$  now become expansions in terms of the partial derivatives with respect to  $T_k$ , given by

$$\frac{d}{dt} = \frac{dT_0}{dt} \frac{\partial}{\partial T_0} + \frac{dT_1}{dt} \frac{\partial}{\partial T_1} + \frac{dT_2}{dt} \frac{\partial}{\partial T_2} + \dots = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots, \quad (9)$$

where the differentiation operator  $D_k = \partial/\partial T_k$ .

Next, suppose that the solution of equation (2) (or equivalently, equations (4)–(7)) in the neighborhood of  $\mathbf{x} = \mathbf{0}$  is represented by an expansion of the form

$$x_i(t; \varepsilon) = \varepsilon x_{i1}(T_0, T_1, \dots) + \varepsilon^2 x_{i2}(T_0, T_1, \dots) + \dots \quad (i = 1, 2, \dots, n). \quad (10)$$

It is noted that the number of independent time scales needed depends on the order to which the expansion is carried out. For example, if the expansion is carried out to  $O(\varepsilon^2)$ , then  $T_0$ ,  $T_1$  and  $T_2$  are needed. In general, if one wants to derive a normal form up to order  $n$ , then  $T_0, T_1, \dots, T_n$  should be included in the solution (10). However, one can employ the idea of the *intrinsic harmonic balancing* technique [10] and let  $n$  be open so that the process of the method will automatically take account of all necessary terms up to a certain order as the following procedure shows.

Substituting solution (10) into equations (4)–(7) with the aid of equation (9) and balancing the like powers of  $\varepsilon$  results in the ordered perturbation equations

$$\varepsilon^1: \quad D_0 x_{11} = x_{21}, \quad D_0 x_{21} = -x_{11}, \quad (11, 12)$$

$$D_0 x_{p1} = -\alpha_p x_{p1}, \quad (p = 3, 4, \dots, m_1 + 2), \quad (13)$$

$$D_0 x_{q1} = -\alpha_q x_{q1} + \omega_q x_{(q+1)},$$

$$D_0 x_{(q+1)1} = -\omega_q x_{q1} - \alpha_q x_{(q+1)1}, \quad (q = m_1 + 3, m_1 + 5, \dots, n - 1), \quad (14)$$

$$\varepsilon^2: \quad D_0 x_{12} = x_{22} - D_1 x_{11} + f_{12}(x_{11}, x_{21}, \dots, x_{n1}), \quad (15)$$

$$D_0 x_{22} = -x_{12} - D_1 x_{21} + f_{22}(x_{11}, x_{21}, \dots, x_{n1}), \quad (16)$$

$$D_0 x_{p2} = -\alpha_p x_{p2} + f_{p2}(x_{11}, x_{21}, \dots, x_{n2}), \quad (p = 3, 4, \dots, m_1 + 2), \quad (17)$$

$$D_0 x_{q2} = -\alpha_q x_{q2} + \omega_q x_{(q+1)2} + f_{q2}(x_{11}, x_{21}, \dots, x_{n1}),$$

$$D_0 x_{(q+1)2} = -\omega_q x_{q2} - \alpha_q x_{(q+1)2} + f_{(q+1)2}(x_{11}, x_{21}, \dots, x_{n1}), \quad (18)$$

$$(q = m_1 + 3, m_1 + 5, \dots, n - 1),$$

etc., where  $f_{i2} = (d^2/d\varepsilon^2)[f_i(x_1, x_2, \dots, x_n)]_{\varepsilon=0}$ . Note that the  $f_{i2}$  are functions of  $x_{i1}$  ( $i = 1, 2, \dots, n$ ) which have been solved from the  $\varepsilon^1$  order perturbation equations (11)–(14). In general, functions  $f_{ik}$  only involve variables which have been solved from the previous  $(k - 1)$  steps perturbation equations.

To find the solutions of the  $\varepsilon^1$  order equations (11)–(14), first note that these equations can be divided into two groups, one of which consists of the first two equations (11) and (12), and the other one includes the remaining equations. Second, since we are interested in the asymptotic behavior of the system, the solutions of the second group are contributed from the first two variables  $x_1$  and  $x_2$  only.

Now consider equations (11) and (12). Differentiating equation (11) and then substituting equation (12) into the resulting equation yields a simple second order ordinary differential equation

$$D_0^2 x_{11} + x_{11} = 0. \quad (19)$$

The solution of equation (19) can be written in a general form

$$x_{11} = r(T_1, T_2, \dots) \cos [T_0 + \phi(T_1, T_2, \dots)] \equiv r \cos (T_0 + \phi) \equiv r \cos \theta, \quad (20)$$

where  $r$  and  $\phi$  represent, respectively, the amplitude and phase of motion and  $\theta = \omega_c T_0 + \phi = T_0 + \phi$ . Once  $x_{11}$  is solved,  $x_{21}$  can be directly determined from equation (11). Note that solution (20) implies that

$$D_0 r = 0 \quad \text{and} \quad D_0 \phi = 0. \quad (21)$$

The asymptotic  $\varepsilon^1$  order solutions of the second group are given by

$$x_{i1} = 0, \quad i = 3, 4, \dots, n, \quad (22)$$

which actually represent the first order steady state solutions of the second group equations.

Next, to solve the  $\varepsilon^2$  order perturbation equations (15)–(18), the procedure described above can be applied again. First, differentiating equation (15) and then substituting equation (16) into the resulting equation results in

$$D_0^2 x_{12} + x_{12} = -D_1 D_0 x_{11} - D_1 x_{21} + D_0 f_{12} + f_{22}, \quad (23)$$

which is a non-linear homogeneous ordinary differential equation. Substituting the solutions  $x_{11}$  and  $x_{21}$  into the right-hand side of equation (23) gives an expression in terms of trigonometric functions  $\cos k(T_0 + \phi)$  and  $\sin k(T_0 + \phi)$ ,  $k = 0, 1, 2$ . To eliminate possible *secular* terms which may appear in  $x_{12}$ , it is required that the coefficients of the

two terms  $\cos(T_0 + \phi)$  and  $\sin(T_0 + \phi)$  equal zero, which in turn yields the explicit expressions for  $D_1 a$  and  $D_1 \phi$ . Then the solution of  $x_{12}$  is determined from the remaining terms of equation (23), and thus,  $x_{12}$  involves a particular solution only. Having found  $x_{12}$ , then  $x_{22}$  is determined from equation (15) as  $x_{22} = D_0 x_{12} + D_1 x_{11} - f_{12}(x_{11}, x_{21}, \dots, x_{m1})$ . This procedure can be carried out to any high order perturbation equations.

Finally, the normal forms, given in polar co-ordinates, can be written as

$$\frac{dr}{dt} = \frac{\partial r}{\partial T_0} \frac{\partial T_0}{\partial t} + \frac{\partial r}{\partial T_1} \frac{\partial T_1}{\partial t} + \frac{\partial r}{\partial T_2} \frac{\partial T_2}{\partial t} + \dots = D_0 r + \varepsilon D_1 r + \varepsilon^2 D_2 r + \dots, \quad (24)$$

$$\frac{d\theta}{dt} = \omega_c + \frac{\partial \phi}{\partial T_0} \frac{\partial T_0}{\partial t} + \frac{\partial \phi}{\partial T_1} \frac{\partial T_1}{\partial t} + \frac{\partial \phi}{\partial T_2} \frac{\partial T_2}{\partial t} + \dots = 1 + D_0 \phi + \varepsilon D_1 \phi + \varepsilon^2 D_2 \phi + \dots. \quad (25)$$

It should be pointed out that the particular solutions of equation (23), etc. can be found by using the method of *harmonic balancing* so that the solution is uniquely determined. Thus,  $D_i r$  and  $D_i \phi$  are also uniquely defined, which implies that the normal forms given in equations (24) and (25) are uniquely determined. It is also noted that the derivatives  $D_i r$  and  $D_i \phi$  are given in the form of a monomial as  $D_i r = (\text{constant})r^{i+1}$  and  $D_i \phi = (\text{constant})r^i$ , where the constants are given explicitly in terms of the coefficients of the original differential equations.

### 3. SYMBOLIC COMPUTATION

The procedure described in the previous section can be very easily implemented in a computer system using a symbolic language such as Maple. Software package coded in Maple language for calculating the normal forms and periodic solutions of system (2) (or equivalently (4)–(7)) has been developed. The program is slightly different from the procedure described above. The perturbation solution in the program starts from  $\varepsilon^0$  order rather than  $\varepsilon^1$  order as described in the above procedure. However, the above procedure will be identical to that given in the program if a scaling  $x_i \rightarrow \varepsilon x_i$  is used first in the above procedure. In order to obtain the normal forms for the original equation (2), one needs to use a back scaling  $\varepsilon x_i \rightarrow x_i$ , or simply setting  $\varepsilon = 1$  in equations (24) and (25).

The input file includes the data of the input functions  $f_i$ , the numbers of real eigenvalues and complex eigenvalues of the linearized systems, as well as the order of the normal forms. Table 1 is an example of an input file, where M1 ( $m_1$ ) = 1 is the number of the non-zero real eigenvalues and M2 ( $m_2$ ) = 1 is the number of the pairs of complex conjugate eigenvalues. Therefore, for this example, the system has dimension  $2 + 1 + 2 \times 1 = 5$ ,

TABLE 1

*Input file to Maple source code*

---

```

M1 := 1;
M2 := 1;
N := 2 + M1 + M2*2;
Order := 4;
func[1] := eps*(x[1]^2 - x[1]*x[3]);
func[2] := eps*(x[2]^2 + x[1]*x[4]) + eps^2*x[2]^3;
func[3] := - x[3] + eps*x[1]^2;
func[4] := - x[4] + x[5] + eps*x[1]^2;
func[5] := - x[4] - x[5] + eps*x[2]^2;

```

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Note: the Maple source code (file name *program 1*) and the above input file (name *input1*) are available from the author's web site: <http://pyu1.apmaths.uwo.ca/~pyu/pub/software>.

indicating that in addition to a pair of purely imaginary eigenvalues, it has one real eigenvalue and a pair of complex conjugate eigenvalues. Order = 4 indicates that the program will be executed up to fifth order (Order + 1) approximations. It should be noted in the first two functions `func[1]` and `func[2]` that there are no linear terms which have been included in the main program since it is assumed that  $\omega_c = 1$ . Also note that a scaling  $x_i \rightarrow \varepsilon x_i$  has been used in the input functions.

#### 4. AN EXAMPLE OF HOPF BIFURCATION

In this section, the input file given in Table 1 is used as an example to show the procedure of the method, and then this example will be used in the next section to discuss a verification scheme. This example is actually described by the original equations:

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1^2 - x_1 x_3, \\ \dot{x}_2 &= -x_1 + x_2^2 + x_1 x_4 + x_2^3, \\ \dot{x}_3 &= -x_3 + x_1^2, \\ \dot{x}_4 &= -x_4 + x_5 + x_1^2, \\ \dot{x}_5 &= -x_4 - x_5 + x_2^2.\end{aligned}\tag{26}$$

Note that applying a scaling  $x_i \rightarrow \varepsilon x_i$  to the above equations yields the equations given in the Maple input file (see Table 1). It is seen that the origin  $\mathbf{x} = \mathbf{0}$  is an equilibrium, and the linearized system of equation (26) has eigenvalues  $\pm i$ ,  $-1$  and  $-1 \pm i$  at the origin. Now executing the Maple source code *program1* yields the following output, which includes  $D_i r$ ,  $D_i \phi$  and the solutions  $x_i$ :

$$D_1 r = D_3 r = 0, \quad D_2 r = \frac{3}{40} r^3, \quad D_4 r = -\frac{14867}{68000} r^5,\tag{27}$$

$$D_1 \phi = D_3 \phi = 0, \quad D_2 \phi = -\frac{7}{12} r^2, \quad D_4 \phi = \frac{8093503}{14688000} r^4,\tag{28}$$

and

$$\begin{aligned}x_1 &= r \cos \theta + \frac{1}{6} r^2 (3 + \cos 2\theta + 2 \sin 2\theta) - \frac{1}{480} r^3 (28 \cos 3\theta - 13 \sin 3\theta) + \cdots, \\ x_2 &= -r \cos \theta - \frac{1}{6} r^2 (3 - \cos 2\theta + 2 \sin 2\theta) - \frac{1}{480} r^3 (260 \cos \theta - 168 \sin \theta \\ &\quad + 17 \cos 3\theta + 28 \sin 3\theta) + \cdots, \\ x_3 &= \frac{1}{10} r^2 (5 + \cos 2\theta + 2 \sin 2\theta) - \frac{1}{12} r^3 (5 \cos \theta + 3 \sin \theta - \cos 3\theta + \sin 3\theta) + \cdots, \\ x_4 &= \frac{1}{10} r^2 (5 + 2 \cos 2\theta + \sin 2\theta) + \frac{1}{510} r^3 (119 \cos \theta + 408 \sin \theta \\ &\quad - 23 \cos 3\theta + 44 \sin 3\theta) + \cdots, \\ x_5 &= -\frac{1}{10} r^2 (\cos 2\theta + 3 \sin 2\theta) - \frac{1}{510} r^3 (68 \cos \theta - 119 \sin \theta - 24 \cos 3\theta \\ &\quad + 57 \sin 3\theta) + \cdots,\end{aligned}\tag{29}$$

where the fourth and fifth order terms are omitted for simplicity. The normal form can then be considered from equations (27) and (28) as

$$\begin{aligned}\frac{dr}{dt} &= D_0 r + \varepsilon D_1 r + \varepsilon^2 D_2 r + \varepsilon^3 D_3 r + \varepsilon^4 D_4 r \\ &= \frac{1}{40} r^3 \left( 3 - \frac{14867}{1700} r^2 \right),\end{aligned}\quad (30)$$

$$\begin{aligned}\frac{d\theta}{dt} &= \omega_c \frac{dT_0}{dt} + D_0 \phi + \varepsilon D_1 \phi + \varepsilon^2 D_2 \phi + \varepsilon^3 D_3 \phi + \varepsilon^4 D_4 \phi \\ &= 1 - \frac{1}{12} r^2 \left( 7 - \frac{8093503}{1224000} r^2 \right).\end{aligned}\quad (31)$$

It should be noted that the back scaling  $\varepsilon x_i \rightarrow x_i$  (and so  $\varepsilon r \rightarrow r$ ) has been used in the above equations (29)–(31), or equivalently  $\varepsilon$  has been set to 1 in these equations.

Now, based on the normal form (30), one can easily find steady state solutions and their stability. The steady state solutions can be obtained by setting  $dr/dt = 0$ , which yields a trivial solution  $\bar{r} = 0$ , corresponding to the initial equilibrium  $\mathbf{x} = \mathbf{0}$ , and a non-trivial solution

$$\bar{r} = \sqrt{5100/14867},\quad (32)$$

which is a periodic motion and its asymptotic solution is given by equation (29). The stability of the periodic solution is determined by differentiating equation (30) with respect to  $r$  (which means perturbing  $r$  around  $\bar{r}$  or simply linearization with respect to  $\bar{r}$ ) to obtain

$$\left. \frac{d}{dr} \left( \frac{da}{dt} \right) \right|_{r=\bar{r}} = -\frac{1530}{29384} < 0.\quad (33)$$

Therefore, the periodic solution is stable. The frequency of the periodic motion is given by equation (31) as

$$\omega = \left. \frac{d\theta}{dt} \right|_{r=\bar{r}} = 1 - \frac{1}{12} \left( \frac{5100}{14867} \right) \left( 7 - \frac{8093503}{1224000} \frac{5100}{14867} \right) = \frac{9174269227}{10609329072}.\quad (34)$$

## 5. VERIFICATION SCHEME

When one has obtained a normal form from a set of differential equations by using certain methods, the question usually arising is how to verify them. Here, we want to ask: can we find an approach to verify the normal forms and periodic solutions found by using the perturbation technique described in the previous sections? These are different verification approaches. A common one is substituting the obtained periodic solutions, assumed up to  $n$ th order, back to the original differential equations; if the residue is  $(n + 1)$ th order, then the periodic solutions are correct. Let's call this verification scheme "solution verification" which is frequently used to check a solution. The verification scheme given below is, however, different from the "solution verification" because it simultaneously verifies both the solutions and, more importantly, the normal forms. It may be called the "system verification" scheme. The procedure of the approach is as follows: to obtain a general non-linear transformation from the vibration solution, to substitute the

non-linear transformation back to the original differential equations, and then simplify the resulting equations to obtain a new set of differential equations. If the new system is identical to the normal form obtained from the perturbation technique, then the results are correct. One may think that “solution verification” should suffice to verify the normal forms too. However, this is not true since the “solution verification” scheme *only* verifies solutions, *not* necessary the normal forms. In fact, cases have been found in which although solutions are verified to be correct (i.e., the “solution verification” gives correct results), the new system does not agree with the normal forms, implying that the normal form obtained from the perturbation technique is wrong (so the “system verification” fails).

Details of the “system verification” scheme are given below. First note that the periodic solution of equation (2) (e.g., see equation (29)) can be written in a general form:

$$\begin{aligned} x_1 &= r \cos \theta + h_1(r \cos \theta, r \sin \theta), \\ x_2 &= -r \sin \theta + h_2(r \cos \theta, r \sin \theta), \\ x_i &= h_i(r \cos \theta, r \sin \theta), \quad i = 3, 4, \dots, n. \end{aligned} \quad (35)$$

Next, if the transformation

$$y_1 = r \cos \theta, \quad y_2 = -r \sin \theta, \quad (36)$$

is introduced into equation (35), one can rewrite equation (31) in a Cartesian system:

$$x_1 = y_1 + h_1(y_1, y_2), \quad x_2 = y_2 + h_2(y_1, y_2), \quad x_i = h_i(y_1, y_2), \quad i = 3, 4, \dots, n. \quad (37)$$

The first two equations may be considered as a non-linear transformation between the co-ordinates  $(x_1, x_2)$  and  $(y_1, y_2)$ , while the remaining equations actually represent the projection of the original system to the two-dimensional center manifold [5]. Differentiating the first two equations of (37) yields

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 1 + \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & 1 + \frac{\partial h_2}{\partial y_2} \end{bmatrix} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix}, \quad (38)$$

from which we can solve for  $\dot{y}_1$  and  $\dot{y}_2$  as

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{bmatrix} 1 + \frac{\partial h_1(y_1, y_2)}{\partial y_1} & \frac{\partial h_1(y_1, y_2)}{\partial y_2} \\ \frac{\partial h_2(y_1, y_2)}{\partial y_1} & 1 + \frac{\partial h_2(y_1, y_2)}{\partial y_2} \end{bmatrix}^{-1} \begin{pmatrix} x_2 + f_1(x_1, x_2, \dots, x_n) \\ -x_1 + f_2(x_1, x_2, \dots, x_n) \end{pmatrix}, \quad (39)$$

where equations (4) and (5) have been used. Then substitute equation (37) into equation (39) so that the resulting equations involve variables  $y_1$  and  $y_2$  only. Next, expand the resulting equations in Taylor series, truncate the series to the same order of the perturbations performed through the multiple scales, then put the transformed system in a polar form. If the results given in the polar form are identical to the normal forms (24) and (25), then the normal forms as well as the asymptotic solutions (i.e., non-linear transformations) are verified to be correct! Otherwise they must be wrong! All the results presented in this paper have been verified using the “system verification” scheme implemented on a computer.



One may think that finding the inverse matrix involved in equation (39) is not easy when the order of the approximations become high. (This is true!) So, to avoid this difficulty, instead of equation (39) one may want to verify the following equation:

$$\begin{bmatrix} 1 + \frac{\partial h_1(y_1, y_2)}{\partial y_1} & \frac{\partial h_1(y_1, y_2)}{\partial y_2} \\ \frac{\partial h_2(y_1, y_2)}{\partial y_1} & 1 + \frac{\partial h_2(y_1, y_2)}{\partial y_2} \end{bmatrix} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} x_2 + f_1(x_1, x_2, \dots, x_n) \\ -x_1 + f_2(x_1, x_2, \dots, x_n) \end{pmatrix}, \quad (40)$$

by comparing the coefficients of the same order terms on both sides of the equation with the aid of equation (37). If the coefficients are equal up to the required order, one then claims that the obtained non-linear transformation and normal form are correct! This is only partially true because verifying equation (40) only verifies the transformation, not the normal form; unless one guarantees that the normal form is correct as long as the transformation is right! But there is no reason to guarantee this. So the “system verification” is necessary to be used.

It should be pointed out that the “system verification” scheme is independent of any particular procedure of methods and is particularly useful when one wants to compare two (or several) different normal forms and non-linear transformations obtained using two (several) different approaches. It is possible that both two (or several) normal forms are correct, thus how to verify the results cannot depend upon the procedure by the methodologies, but must depend on the equations, normal forms and transformations. In this case, the “system verification” scheme can be used to verify each pair of normal forms and non-linear transformations. In other words, a correct normal form must be consistent with its associated non-linear transformation. In fact, we have successfully used the “system verification” scheme to verify different normal forms of a same system, obtained using different approaches.

For the example described by equation (26), the computer output of the transformation (37) up to third-order terms is (edited by rearranging the order of terms)

$$\begin{aligned} x_1 &= y_1 + \frac{2}{3}y_1^2 - \frac{2}{3}y_1y_2 + \frac{1}{3}y_2^2 - \frac{7}{120}y_1^3 - \frac{13}{160}y_1^2y_2 + \frac{7}{40}y_1y_2^2 + \frac{13}{480}y_2^3 + \dots, \\ x_2 &= y_2 - \frac{1}{3}y_1^2 + \frac{2}{3}y_1y_2 - \frac{2}{3}y_2^2 - \frac{277}{480}y_1^3 - \frac{7}{40}y_1^2y_2 + \frac{209}{480}y_1y_2^2 + \frac{49}{120}y_2^3 + \dots, \\ x_3 &= \frac{3}{5}y_1^2 - \frac{2}{5}y_1y_2 + \frac{2}{5}y_2^2 + \frac{1}{3}y_1^3 - y_1^2y_2 + \frac{2}{3}y_1y_2^2 - \frac{2}{3}y_2^3 + \dots, \\ x_4 &= \frac{7}{10}y_1^2 - \frac{1}{5}y_1y_2 + \frac{3}{10}y_2^2 + \frac{16}{85}y_1^3 - \frac{18}{17}y_1^2y_2 + \frac{94}{255}y_1y_2^2 - \frac{182}{255}y_2^3 + \dots, \\ x_5 &= -\frac{1}{10}y_1^2 + \frac{3}{5}y_1y_2 + \frac{1}{10}y_2^2 - \frac{22}{255}y_1^3 + \frac{26}{255}y_1^2y_2 - \frac{14}{51}y_1y_2^2 - \frac{88}{255}y_2^3 + \dots. \end{aligned} \quad (41)$$

The transformed equation (39), from the computer output, truncated at fifth-order terms, is (edited)

$$\dot{y}_1 = y_2 + \frac{3}{40}y_1(y_1^2 + y_2^2) - \frac{7}{12}y_2(y_1^2 + y_2^2) - \frac{14867}{68000}y_1(y_1^2 + y_2^2)^2 + \frac{8093503}{14688000}y_2(y_1^2 + y_2^2)^2, \quad (42)$$

$$\dot{y}_2 = -y_1 + \frac{3}{40}y_2(y_1^2 + y_2^2) + \frac{7}{12}y_1(y_1^2 + y_2^2) - \frac{14867}{68000}y_2(y_1^2 + y_2^2)^2 - \frac{8093503}{14688000}y_1(y_1^2 + y_2^2)^2. \quad (43)$$

Now in order to prove that equations (42) and (43) are identical to equations (30) and (31), equation (42) is multiplied by  $y_1$  and equation (43) by  $y_2$ , and then the resulting equations are added together to yield

$$y_1\dot{y}_1 + y_2\dot{y}_2 = \frac{3}{40}(y_1^2 + y_2^2)^2 - \frac{14867}{68000}(y_1^2 + y_2^2)^3. \quad (44)$$

Similarly, multiplying equation (42) by  $y_2$  and equation (43) by  $y_1$ , and then subtracting the resulting second equation from the first equation leads to

$$y_2\dot{y}_1 - y_1\dot{y}_2 = y_1^2 + y_2^2 - \frac{7}{12}(y_1^2 + y_2^2)^2 + \frac{8093503}{14688000}(y_1^2 + y_2^2)^3. \quad (45)$$

Next, applying the transformation (36), and using the relations

$$\dot{y}_1 = \frac{dr}{dt} \cos \theta - r \frac{d\theta}{dt} \sin \theta, \quad \dot{y}_2 = -\frac{dr}{dt} \sin \theta - r \frac{d\theta}{dt} \cos \theta$$

in equations (44) and (45), one obtains

$$r \frac{dr}{dt} = \frac{3}{40} r^4 - \frac{14867}{68000} r^6, \quad \text{i.e.,} \quad \frac{dr}{dt} = \frac{3}{40} r^3 - \frac{14867}{68000} r^5,$$

which is identical to equation (30); and

$$r^2 \frac{d\theta}{dt} = r^2 - \frac{7}{12} r^4 + \frac{8093503}{14688000} r^6, \quad \text{i.e.,} \quad \frac{d\theta}{dt} = 1 - \frac{7}{12} r^2 + \frac{8093503}{14688000} r^4,$$

which is the same as equation (31).

In order to further demonstrate the computation efficiency of the perturbation technique and the Maple program, we shall consider two more examples in the next two sections. One of them is an oscillator of a mechanical system and the other is the well known Hilbert's 16th problem.

## 6. DOUBLE PENDULUM

The double pendulum system shown in Figure 1 consists of two rigid weightless links of equal length  $l$  which carry two concentrated masses  $2m$  and  $m$ , respectively. A follower force  $P$  is applied to this system.

The system energy for the three linear springs  $h_1$ ,  $h_2$  and  $h_3$  is assumed to be given by reference [11]:

$$V = \frac{1}{2}[(h_1 + h_2 + h_3 l^2)\theta_1^2 + 2(h_3 l^2 - h_2)\theta_1\theta_2 + (h_2 + h_3 l^2)\theta_2^2] - \frac{1}{6}h_3 l^2(\theta_1 + \theta_2)(\theta_1^3 + \theta_2^3), \quad (46)$$

where  $\theta_1$  and  $\theta_2$  are generalized co-ordinates which specify the configuration of the system completely.

The kinetic energy  $T$  of the system is expressed by

$$T = \frac{ml^2}{2\Omega^2} [3\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)], \quad (47)$$

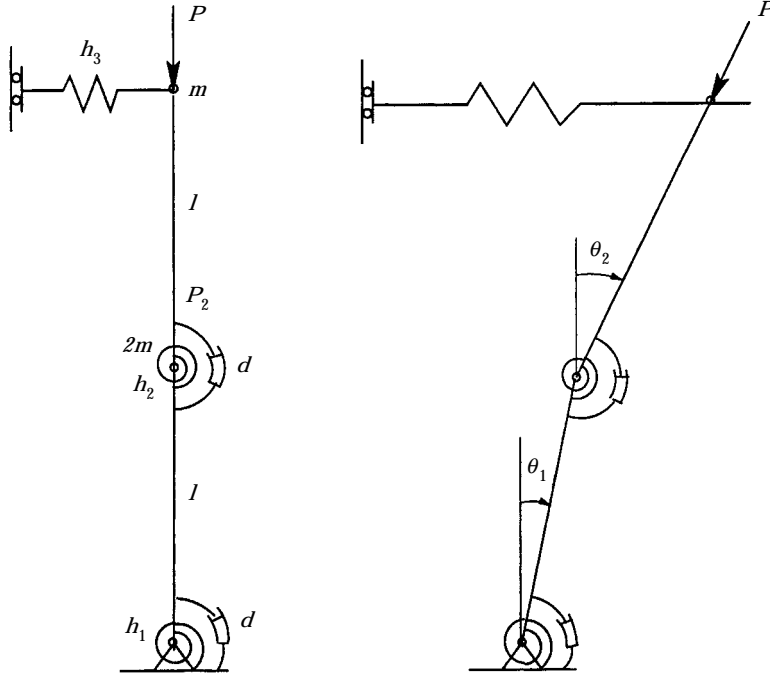


Figure 1. A double pendulum system.

where  $\Omega$  is an arbitrary value rendering the time variable non-dimensional, and the overdot denotes differentiation with respect to the non-dimensional time variable  $t'$  and  $t' = \Omega t$ .

The components of the generalized forces corresponding to the generalized co-ordinates  $\theta_1$  and  $\theta_2$  may be written as

$$Q = Pl \sin(\theta_1 - \theta_2), \quad (48)$$

and the linear damping can be expressed by

$$D = \frac{1}{2}[\dot{\theta}_1^2 + (\dot{\theta}_1 - \dot{\theta}_2)^2]d. \quad (49)$$

With the aid of the *Lagrangian equations*, one can drive a set of first order differential equations up to third order terms as follows:

$$\frac{dz_1}{dt'} = z_2,$$

$$\begin{aligned} \frac{dz_2}{dt'} = & -\frac{1}{2}(f_1 + 2f_2 - \eta)z_1 - \frac{3}{2}f_4z_2 + \frac{1}{2}(2f_2 - \eta)z_3 + f_4z_4 \\ & + \frac{1}{12}(3f_1 + 9f_2 + 2f_3 - 4\eta)z_1^3 - \frac{1}{4}(2f_1 + 9f_2 - 2f_3 - 4\eta)z_1^2z_3 \\ & + \frac{1}{4}(f_1 + 9f_2 - 4\eta)z_1z_3^2 - \frac{1}{6}(3f_2 + 3f_3 - 2\eta)z_3^3 \\ & + \frac{1}{4}f_3(3z_2 - 2z_4)(z_1 - z_3)^2 - z_2z_4(z_1 - z_3), \end{aligned}$$

$$\frac{dz_3}{dt'} = z_4,$$

$$\begin{aligned}
\frac{dz_4}{dt'} &= \frac{1}{2}(f_1 + 4f_2 - 2f_3 - \eta)z + \frac{5}{2}f_4z_2 - \frac{1}{2}(4f_2 + 2f_3 - \eta)z_3 - 2f_4z_4 \\
&\quad - \frac{1}{12}(6f_1 + 15f_2 - 2f_3 - 7\eta)z_1^3 + \frac{1}{4}(4f_1 + 15f_2 - f_3 - 7\eta)z_1^2z_3 \\
&\quad - \frac{1}{4}(2f_1 + 15f_2 - 2f_3 - 7\eta)z_1z_3^2 + \frac{1}{12}(15f_2 + 14f_3 - 7\eta)z_3^3 \\
&\quad - \frac{1}{4}f_4(5z_2 - 4z_4)(z_1 - z_3)^2,
\end{aligned} \tag{50}$$

where the state variables  $z_i$  are defined as

$$z_1 = \theta_1, \quad z_2 = \dot{\theta}_1, \quad z_3 = \theta_2 \quad \text{and} \quad z_4 = \dot{\theta}_2, \tag{51}$$

and dimensionless coefficients  $f_i$  are given by

$$f_1 = \frac{h_1\Omega^2}{mI^2}, \quad f_2 = \frac{h_2\Omega^2}{mI^2}, \quad f_3 = \frac{h_3\Omega^2}{m}, \quad f_4 = \frac{d\Omega^2}{mI^2} \quad \text{and} \quad \eta = \frac{P\Omega^2}{mI}, \tag{52}$$

in which  $f_i > 0$  ( $i = 1, 2, 3, 4$ ) due to physical conditions, and  $\eta$  (or  $P$ ) is used to indicate the system parameter.

The Jacobian matrix of equation (50) evaluated at an arbitrary point on the initial equilibrium solution  $z_i = 0$  takes the form

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{2}(f_1 + 2f_2 - \eta) & -\frac{3}{2}f_4 & \frac{1}{2}(2f_2 - \eta) & f_4 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2}(f_1 + 4f_2 - 2f_3 - \eta) & \frac{5}{2}f_4 & -\frac{1}{2}(4f_2 + 2f_3 - \eta) & -2f_4 \end{bmatrix}, \tag{53}$$

from which one may obtain the characteristic polynomial

$$P(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4, \tag{54}$$

where

$$\begin{aligned}
a_1 &= \frac{7}{2}f_4, & a_2 &= \frac{1}{2}(f_4^2 + f_1 + 6f_2 + 2f_3 - 2\eta), \\
a_3 &= \frac{1}{2}(f_1 + f_2 + 5f_3)f_4, & a_4 &= \frac{1}{2}f_1(f_2 + f_3) + f_3(2f_2 - \eta).
\end{aligned} \tag{55}$$

By using the *Hurwitz criterion* one may show that when

$$a_1 > 0, \quad a_2 > 0, \quad a_4 > 0 \quad \text{and} \quad a_3(a_1a_2 - a_3) - a_4a_1^2 > 0, \tag{56}$$

the initial equilibrium solution  $z_i = 0$  is stable.

It is easy to verify that at the critical point, defined by

$$f_1 = \frac{1}{2}, \quad f_2 = \frac{5}{8}, \quad f_3 = \frac{1}{8}, \quad f_4 = \frac{1}{2}, \quad \text{and} \quad \eta_c = \frac{3}{2}, \tag{57}$$

the polynomial  $P(\lambda)$  has a pair of purely imaginary and two distinct negative eigenvalues:

$$\lambda_{1,2} = \pm \frac{1}{2}i, \quad \lambda_3 = -\frac{1}{2} \quad \text{and} \quad \lambda_4 = -\frac{5}{4}. \tag{58}$$

Choosing  $\eta$  (initially,  $P$ ) as the parameter, and then using the parameter transformation

$$\eta = \eta_c + \mu = \frac{3}{2} + \mu, \tag{59}$$

and the state variable transformation

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{bmatrix} -\frac{1}{20} & \frac{7}{20} & -2 & -1 \\ -\frac{7}{40} & -\frac{1}{40} & 1 & \frac{5}{4} \\ 0 & \frac{1}{2} & 0 & 1 \\ -\frac{1}{4} & 0 & 0 & -\frac{5}{4} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad (60)$$

one may transform equation (50) into the system

$$\begin{aligned} \frac{dx_1}{dt'} &= X_1(x_1, x_2, x_3, x_4; \mu) = \frac{1}{2}x_2 + \frac{8}{145}\mu x_1 + \frac{24}{145}\mu x_2 + g_1(x_1, x_2, x_3, x_4), \\ \frac{dx_2}{dt'} &= X_2(x_1, x_2, x_3, x_4; \mu) = -\frac{1}{2}x_1 + \frac{9}{145}\mu x_1 + \frac{27}{145}\mu x_2 + g_2(x_1, x_2, x_3, x_4), \\ \frac{dx_3}{dt'} &= X_3(x_1, x_2, x_3, x_4; \mu) = -\frac{1}{2}x_3 + g_3(x_1, x_2, x_3, x_4), \\ \frac{dx_4}{dt'} &= X_4(x_1, x_2, x_3, x_4; \mu) = -\frac{5}{4}x_4 + g_4(x_1, x_2, x_3, x_4), \end{aligned} \quad (61)$$

where functions  $g_i(x_1, x_2, x_3, x_4)$ , listed in the Appendix, are polynomials starting from second order terms. The Jacobian matrix of equation (61), evaluated on the initial equilibrium solution  $x_i = 0$  at the critical point  $\mu_c = 0$ , is now in the Jacobian canonical form

$$J_c = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{5}{4} \end{bmatrix}. \quad (62)$$

Now, the Maple program can be applied to equations (61) to obtain the normal forms. Since here  $\omega_c = 1/2$ , we need another time scale  $t' = (1/2)t''$ , or we simply first multiply the right hand side of equation (61) by 2 and then execute the Maple program. When the normal form is obtained from the output, it should be multiplied by a back scaling  $1/2$  (i.e., the  $\omega_c$ ). Having done this, one may finally find the following forms up to third order:

$$\frac{dr}{dt'} = \frac{7}{58}\mu r - \frac{3019}{296960}r^3, \quad (63)$$

$$\frac{d\theta}{dt'} = \frac{1}{2} + \frac{3}{58}\mu - \frac{19597}{296960}r^2. \quad (64)$$

Note that the  $\mu$  terms in equations (63) and (64) are given by the formulas [10]:

$$\begin{aligned} \left. \frac{d\alpha(\mu)}{d\mu} \right|_{\mu=0} &= \frac{1}{2} \left( \frac{\partial^2 X_1}{\partial x_1 \partial \mu} + \frac{\partial^2 X_2}{\partial x_2 \partial \mu} \right)_{x_i=0, \mu=0} = \frac{1}{2} \left( \frac{8}{145} + \frac{27}{145} \right) = \frac{7}{58}, \\ \left. \frac{d\omega(\mu)}{d\mu} \right|_{\mu=0} &= \frac{1}{2} \left( \frac{\partial^2 X_1}{\partial x_2 \partial \mu} - \frac{\partial^2 X_2}{\partial x_1 \partial \mu} \right)_{x_i=0, \mu=0} = \frac{1}{2} \left( \frac{24}{145} - \frac{9}{145} \right) = \frac{3}{58}, \end{aligned} \quad (65)$$

where  $\alpha(\mu)$  and  $\omega(\mu)$  are, respectively, the real and imaginary parts of the complex conjugate eigenvalues of the Jacobian matrix of equation (61) evaluated on the equilibrium  $x_i = 0$  in the vicinity of the critical point  $\mu_c = 0$ , satisfying  $\alpha(0) = 0$  and  $\omega(0) = \omega_c = 1/2$ .

The steady state solutions are determined by setting  $dr/dt' = 0$ , which yields  $\bar{r}_1 = 0$  and

$$\bar{r}_2^2 = \frac{1025360}{87551} \mu. \quad (66)$$

The solution  $\bar{r}_1 = 0$  is actually the initial equilibrium  $x_i = 0$  while equation (66) represents periodic motion of a family of limit cycles. Equation (66) describes the *pitchfork bifurcation path* (or *parameter-amplitude* relationship), shown in Figure 2(a). The stability of the steady state solutions are determined from

$$\frac{d}{dr} \left( \frac{dr}{dt'} \right)_{r=\bar{r}} = \frac{7}{58} \mu - \frac{3019}{146480} \bar{r}^2 \begin{cases} < 0 & \Rightarrow \text{stable,} \\ > 0 & \Rightarrow \text{unstable.} \end{cases} \quad (67)$$

Evaluating equation (67) on the initial equilibrium  $\bar{r}_1 = 0$  results in  $(d/dr)(dr/dt') = (7/58)\mu$ . Thus, the initial equilibrium is stable when  $\mu < 0$  and unstable when  $\mu > 0$ . Similarly, evaluating equation (67) on the periodic solution (66) yields  $(d/dr)(dr/dt') = -(7/29)\mu$ , which indicates the periodic solution is stable if  $\mu > 0$  and unstable if  $\mu < 0$ . This shows the classical theory of stability exchange between the initial equilibrium and the periodic solutions at the critical point  $\mu = 0$ . The frequency of the periodic motion is given by equation (64) as

$$\omega = \left. \frac{d\theta}{dt'} \right|_{r=\bar{r}_2} = \frac{1}{2} + \frac{3}{58} \mu - \left( \frac{19597}{296960} \right) \left( \frac{7}{58} \right) \left( \frac{296960}{3019} \right) \mu = \frac{1}{2} - \frac{2209}{3019} \mu. \quad (68)$$

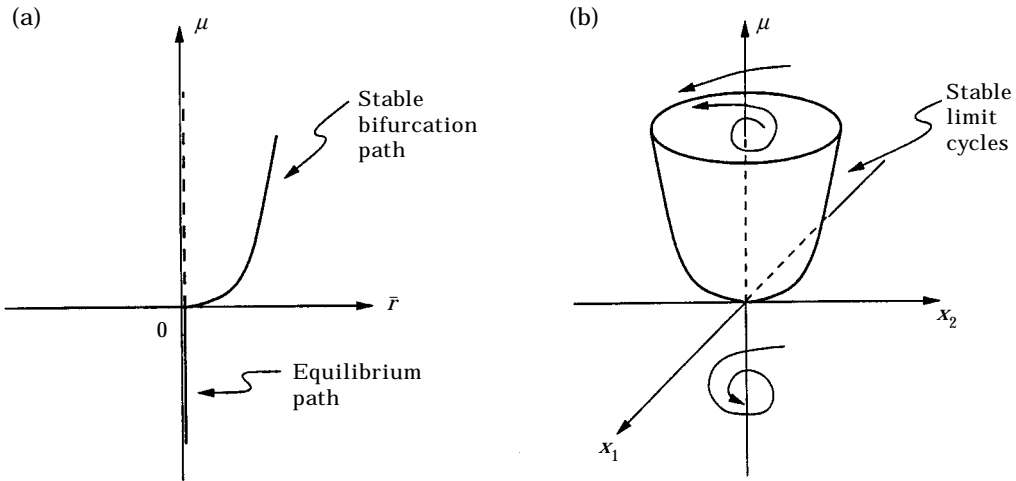


Figure 2. Periodic solution for the double pendulum: (a) pitchfork bifurcation path; and (b) Hopf bifurcation solutions.

The periodic asymptotic solutions up to third-order terms are also obtained from executing the Maple program as given in the following equations:

$$\begin{aligned}
x_1 &= \bar{r} \cos \omega t' + \bar{r}^3 \left( \frac{4519}{11712000} \cos 3\omega t' - \frac{24281}{113216000} \sin 3\omega t' \right), \\
x_2 &= -\bar{r} \sin \omega t' + \bar{r}^3 \left( \frac{179}{1484800} \cos \omega t' - \frac{105997}{1484800} \sin \omega t' - \frac{5681}{3072000} \cos 3\omega t' \right. \\
&\quad \left. - \frac{354443}{89088000} \sin 3\omega t' \right), \\
x_3 &= \bar{r}^3 \left( \frac{17473}{10764800} \cos \omega t' - \frac{14639}{10764800} \sin \omega t' + \frac{4519}{11712000} \cos 3\omega t' - \frac{24281}{113216000} \sin 3\omega t' \right), \\
x_4 &= \bar{r}^3 \left( \frac{17473}{10764800} \cos \omega t' - \frac{14639}{10764800} \sin \omega t' + \frac{4519}{11712000} \cos 3\omega t' \right. \\
&\quad \left. - \frac{24281}{113216000} \sin 3\omega t' \right). \tag{69}
\end{aligned}$$

The periodic solutions are depicted in Figure 2(b).

It is seen from the Appendix that equation (61) is quite involved, compared to the simple example given in the previous section. Therefore, a symbolic computation program is necessary to be used for saving computation time and avoiding mistakes. It is known that though a recursive program can, in principle, be executed to any a required high order, the computation time increases very fast as the order of the normal forms and solutions increases. All the results including the normal forms and periodic solutions (non-linear transformations) for the case study of Hopf bifurcation of the double pendulum system are obtained by executing the Maple program, and execution on an IBM RS6000 workstation is virtually instant. In order to show the efficiency of the program, we have run the program for this example up to 11th order on the same IBM RS6000 workstation. It only took about half a minute of CPU time to get all the results (normal form and periodic solution), and the output of the 11th order normal forms (asymptotic solutions are omitted here) is given as follows (edited by rearranging the terms):

$$\begin{aligned}
\frac{dr}{dt} &= \frac{7}{58} \mu r - \frac{3019}{292960} r^3 - \frac{5426431103033}{9359981543424000} r^5 + \frac{1100905917258337546093}{19668025883000611799040000} r^7 \\
&\quad + \frac{26620573881148541396526584893146067}{464942313595613669023895519232000000000} r^9 \\
&\quad + \frac{2817957571023925669033252817555329038029913727}{91591701618311335044844277883343994880000000000000} r^{11},
\end{aligned}$$

$$\begin{aligned}
\frac{d\theta}{dt} = & \frac{1}{2} + \frac{3}{58}\mu - \frac{19597}{296960}r^2 - \frac{104608971027463}{9359981543424000}r^4 - \frac{466952276797470137509781}{147510194122504588492800000}r^6 \\
& - \frac{502431434305919305691349692533338169}{464942313595613669023895519232000000000}r^8 \\
& + \frac{16613721812708589604284432256148783297417621609}{40707422941471704464375234614819553280000000000000}r^{10}. \tag{70}
\end{aligned}$$

We know that the memory needed for executing a symbolic program increases extraordinarily fast as the order of the normal form increases. For a test, the Maple program has been executed on the same IBM RS6000 workstation to get the normal form and asymptotic solutions for the double pendulum example up to 21st order, and the machine did not crash. The CPU time taken for this case is about 1 h. In practice, of course, we usually do not need such high order approximations to study a physical system like the double pendulum. However, in degenerate bifurcation cases, we do need to derive high order normal forms. In order to further show the efficiency of our Maple program, we now consider another example.

## 7. HILBERT'S 16TH PROBLEM

In this section, the method and Maple program is used to consider the well known Hilbert's 16th problem.

### 7.1. QUADRATIC FORM

A simplified quadratic form of the Hilbert's 16th problem is given by [12]

$$\begin{aligned}
\dot{x} &= -y + \lambda_1 x - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\
\dot{y} &= x + \lambda_1 y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2. \tag{71}
\end{aligned}$$

The fundamental question about this quadratic differential equation is: what is the maximal number of limit cycles the system can have? In general, this is a very difficult question. If it is restricted to the neighborhood of the isolated fixed point  $(x, y) = (0, 0)$ , then the question is related to degenerate Hopf bifurcations [13]. Suppose equation (71) is transformed into the polar co-ordinate normal form:

$$\dot{r} = v_1 r + v_3 r^3 + v_5 r^5 + v_7 r^7 + \dots, \quad \dot{\theta} = 1 + O(r^2), \tag{72, 73}$$

where  $v_i$ ,  $i = 1, 3, 5, \dots$ , called *focus number*, are functions of the parameters  $\lambda_i$ ,  $i = 1, 2, \dots, 6$ . Now the maximal number of limit cycles which may exist in the vicinity of the origin depends upon the  $v_i$ 's. For example, if we choose the  $\lambda_i$ 's such that  $v_1 = v_3 = 0$  but  $v_5 \neq 0$ , then equation (72) can be truncated as  $\dot{r} = v_5 r^5$  which may be perturbed to obtain equation  $\dot{r} = \mu_1 r + \mu_2 r^3 + v_5 r^5$ , where  $\mu_1$  and  $\mu_2$  are small perturbation parameters. Thus, the maximal number of limit cycles which may exist in the vicinity of the fixed point  $(0, 0)$  is 2.



Bautin [12] used the method of *successive functions* to obtain the coefficients of the normal form as follows:

$$\begin{aligned}
v_1 &= 2\pi\lambda_1, \\
v_3 &= -\frac{\pi}{4}\lambda_5(\lambda_3 - \lambda_6), \quad (\lambda_1 = 0), \\
v_5 &= \frac{\pi}{24}\lambda_2\lambda_4(\lambda_3 - \lambda_6)(5\lambda_3 + \lambda_4 - 5\lambda_6), \quad (\lambda_5 = 0), \\
v_7 &= -\frac{125\pi}{32}\lambda_2(\lambda_3 - \lambda_6)^3(\lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2), \quad (\lambda_4 = -5(\lambda_3 - \lambda_6)), \quad (74)
\end{aligned}$$

where the equations in the brackets give the conditions when the previous  $v_i$ 's are equal to zero. For example, if  $\lambda_1 = 0$ , then  $v_1 = 0$  but  $v_3 \neq 0$ ; if both  $\lambda_1$  and  $\lambda_5$  equal zero, then  $v_1 = v_3 = 0$ , but  $v_5 \neq 0$ , etc.

Executing our Maple program yields

$$\begin{aligned}
D_0r &= \lambda_1r, \\
D_2r &= -\frac{1}{8}\lambda_5(\lambda_3 - \lambda_6)r^3, \quad (\lambda_1 = 0), \\
D_4r &= \frac{1}{48}\lambda_2\lambda_4(\lambda_3 - \lambda_6)(5\lambda_3 + \lambda_4 - 5\lambda_6)r^5, \quad (\lambda_5 = 0), \\
D_6r &= \frac{25}{64}\lambda_2(\lambda_3 - \lambda_6)^3(\lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2)r^7, \quad (\lambda_4 = -5(\lambda_3 - \lambda_6)). \quad (75)
\end{aligned}$$

Note the difference between the notations  $v_i$  and  $D_i r$ :  $D_{i+1}r$  corresponds to  $v_i$ . It can be seen that with a scaling  $2\pi$ , the coefficient of  $r$  in  $D_0r$ , the coefficient of  $r^3$  in  $D_2r$  and the coefficient of  $r^5$  in  $D_4r$  agree with Bautin's results  $v_1$ ,  $v_3$  and  $v_5$ , respectively. However, the coefficient of  $r^7$  in  $D_6r$  is different from  $v_7$ , not only in scaling but also in the sign. This difference is important as it can critically change the behavior of the system. So which set of results is correct? Recently, several researchers have shown that the result of  $v_7$  in Bautin's solution is wrong. The same results of equation (75) have been obtained, for example, by Shi [14], and Qin and Liu [15] using symbolic computations, and by Farr *et al.* [13] using the Liapunov Schmidt method.

Further, if we let the factor  $\lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2$  in  $D_6r$  (or  $v_7$ ) equal zero, then our Maple program has been executed to show that  $D_0r = D_2r = D_4r = \dots = D_{20}r = 0$ . This seems to imply that all high order focus numbers are zero. That is, the origin is a *center*. Bautin [12] has proved that the origin is indeed a center, i.e.,  $v_{2k-1} = 0 \forall k > 0$  if the following conditions are satisfied:

$$\lambda_1 = \lambda_5 = 0, \quad \lambda_4 = -5(\lambda_3 - \lambda_6), \quad \lambda_2^2 = \lambda_6(\lambda_3 - 2\lambda_6). \quad (76)$$

## 7.2. CUBIC FORM

Next, consider the cubic system of Hilbert's 16th problem (e.g., see references [16, 17]):

$$\begin{aligned}
\dot{x} &= y + \lambda x, \\
\dot{y} &= -x + \lambda y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3. \quad (77)
\end{aligned}$$

Jin and Wang [16] considered the case under the conditions:

$$\lambda = a_{11} = a_{12} = 0, \quad a_{21} = -3a_{03}, \quad a_{20} = -\frac{1}{2}a_{02} \text{ and } a_{30} = -\frac{1}{12}a_{02}^2, \quad (78)$$

with  $\lambda = a_{02}a_{03} \neq 0$ . They used symbolic computer programs to find the following results:

$$\begin{aligned} u_1 = v_3 = v_5 = 0, \quad v_7 &= -\frac{1}{1680} a_{03}(288a_{03}^2 - a_{02}^4), \\ v_9 = \frac{23}{6480} a_{02}^2 a_{03}(288a_{03}^2 - a_{02}^4), \quad v_{11} &= \frac{1}{997920} a_{03}(999a_{03}^2 + 790a_{02}^4)(288a_{03}^2 - a_{02}^4). \end{aligned} \quad (79)$$

Note that there is a common factor  $(288a_{03}^2 - a_{02}^4)$  in all the three equations. Executing our Maple program yields

$$\begin{aligned} D_0 r &= D_2 r = D_4 r = 0, \\ D_6 r &= -\frac{1}{6144} a_{03}(288a_{03}^2 - a_{02}^4)r^7, \\ D_8 r &= -\frac{61}{196608} a_{02}^2 a_{03}(288a_{03}^2 - a_{02}^4)r^9, \\ D_{10} r &= -\frac{1}{4529848320} a_{03}(195840a_{03}^2 + 251933a_{02}^4)(288a_{03}^2 - a_{02}^4)r^{11}. \end{aligned} \quad (80)$$

It is seen that significant discrepancy exists in these two sets of coefficients:  $D_6 r$  and  $v_7$  differ by a scaling;  $D_8 r$  and  $v_9$  have, in addition to scaling, opposite signs; the worst is in  $D_{10} r$  and  $v_{11}$  where they even have a different factor. This again shows that verification schemes are important and needed for verifying the results. The above results obtained from our Maple program have been verified by the ‘‘system verification’’ scheme. Since Jin and Wang did not report their non-linear transformation in reference [16], we cannot use the ‘‘system verification’’ scheme to verify the results given in equation (79).

Similar to the case of quadratic form, if, in addition, we let the common factor  $(288a_{03}^2 - a_{02}^4)$  equal zero, then the execution of our Maple program on another workstation has shown that  $D_0 r = D_2 r = \dots = D_{60} r = 0$ . (The workstation, with more than 30 users, took 1 week to get the results up to 60th order.) This suggests a clue that all high order focus numbers are zero. Indeed, Jin and Wang [16] raised the conjecture and later Christopher and Lloyd [17] showed that the origin is indeed a center (i.e.,  $v_{2k-1} = 0 \forall k > 0$ ) if the following conditions,

$$\lambda = a_{11} = a_{12} = 0, \quad a_{21} = -3a_{03}, \quad a_{20} = -\frac{1}{2} a_{02}, \quad a_{30} = -\frac{1}{12} a_{02}^2, \quad a_{02}^4 = 288a_{03}^2, \quad (81)$$

are satisfied.

The purpose of using the perturbation technique developed in this paper to reconsider Hilbert’s 16th problem is not to verify the known conclusions, but to test the efficiency of the method and the Maple program. It might be very difficult or impossible to use some other methodology such as the matrix approach (based on the normal form theory) to calculate these coefficients up to as high as 60th order terms!

## 8. CONCLUDING REMARKS

A perturbation technique for computing normal forms has been developed. A special bifurcation problem is studied in detail in this paper to demonstrate the computation efficiency. It has been shown that with the aid of multiple scales, the application of the

perturbation technique is straightforward and systematic. One of the advantages using this method is that it is very easy to implement the approach in a computer system via a symbolic computation language. Based on this method, Maple programs have been developed for the study of the Hopf bifurcation. Furthermore, a “system verification” scheme is given for verifying normal forms and non-linear transformations (periodic asymptotic solutions) obtained using any approach. It is expected that the perturbation technique can be straightforwardly extended to systems whose linearized system evaluated at a critical point has, at least, a pair of purely imaginary eigenvalues. The author is currently engaged in work on several other systems. For example, the linearized equations of these systems may have two pairs of purely imaginary eigenvalues, or one zero and one pair of purely imaginary eigenvalues at a critical point.

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## APPENDIX: FUNCTIONS

The functions  $g_i(x_1, x_2, x_3, x_4)$  given in equation (61) are as follows:

$$\begin{aligned}
g_1 = & -\frac{83069}{11136000}x_1^3 - \frac{931423}{11136000}x_2^3 + \frac{631}{174}x_3^3 + \frac{1318}{87}x_4^3 - \frac{88487}{3712000}x_1^2x_2 - \frac{23169}{92800}x_1^2x_3 \\
& - \frac{102613}{371200}x_1^2x_4 + \frac{231}{128000}x_1x_2^2 + \frac{17799}{92800}x_2^2x_3 - \frac{125837}{371200}x_2^2x_4 + \frac{4731}{2320}x_1x_3^2 \\
& - \frac{1087}{2320}x_2x_3^2 + \frac{4587}{232}x_3^2x_4 + \frac{1185}{928}x_1x_4^2 + \frac{305}{928}x_2x_4^2 + \frac{3725}{116}x_3x_4^2 \\
& + \frac{3213}{46400}x_1x_2x_3 + \frac{3041}{185600}x_1x_2x_4 + \frac{14987}{46400}x_1x_3x_4 + \frac{3841}{4640}x_2x_3x_4, \tag{A1}
\end{aligned}$$

$$\begin{aligned}
g_2 = & -\frac{16811}{5568000}x_1^3 - \frac{6179}{1856000}x_2^3 - \frac{71}{87}x_3^3 - \frac{221}{87}x_4^3 - \frac{18753}{1856000}x_1^2x_2 - \frac{4831}{46400}x_1^2x_3 \\
& - \frac{22077}{185600}x_1^2x_4 - \frac{111}{64000}x_1x_2^2 - \frac{999}{46400}x_2^2x_3 - \frac{13173}{185600}x_2^2x_4 + \frac{749}{1160}x_1x_3^2 \\
& - \frac{1273}{1160}x_2x_3^2 - \frac{637}{116}x_3^2x_4 + \frac{5}{464}x_1x_4^2 - \frac{635}{464}x_2x_4^2 - \frac{415}{58}x_3x_4^2 \\
& - \frac{413}{23200}x_1x_2x_3 - \frac{5911}{92800}x_1x_2x_4 + \frac{1443}{2320}x_1x_3x_4 - \frac{5751}{2320}x_2x_3x_4, \tag{A2}
\end{aligned}$$

$$\begin{aligned}
g_3 = & -\frac{1123}{1024000}x_1^3 + \frac{2077}{3072000}x_2^3 - \frac{7}{16}x_3^3 - \frac{35}{24}x_4^3 - \frac{3787}{1024000}x_1^2x_2 - \frac{973}{25600}x_1^2x_3 \\
& - \frac{4469}{102400}x_1^2x_4 - \frac{801}{1024000}x_1x_2^2 - \frac{357}{25600}x_2^2x_3 - \frac{2221}{102400}x_2^2x_4 + \frac{143}{640}x_1x_3^2 \\
& - \frac{291}{640}x_2x_3^2 - \frac{181}{64}x_3^2x_4 - \frac{7}{256}x_1x_4^2 - \frac{151}{256}x_2x_4^2 - \frac{123}{32}x_3x_4^2 \\
& - \frac{119}{12800}x_1x_2x_3 - \frac{1407}{51200}x_1x_2x_4 + \frac{47}{256}x_1x_3x_4 - \frac{275}{256}x_2x_3x_4, \tag{A3}
\end{aligned}$$

$$\begin{aligned}
g_4 = & \frac{16811}{11136000}x_1^3 + \frac{6179}{3712000}x_2^3 + \frac{71}{174}x_3^3 + \frac{221}{174}x_4^3 + \frac{18753}{3712000}x_1^2x_2 - \frac{4831}{92800}x_1^2x_3 \\
& + \frac{22077}{371200}x_1^2x_4 + \frac{111}{128000}x_1x_2^2 + \frac{999}{92800}x_2^2x_3 + \frac{13173}{371200}x_2^2x_4 - \frac{749}{2320}x_1x_3^2 \\
& + \frac{1273}{2320}x_2x_3^2 + \frac{637}{232}x_3^2x_4 - \frac{5}{928}x_1x_4^2 + \frac{635}{928}x_2x_4^2 + \frac{415}{116}x_3x_4^2 \\
& + \frac{413}{46400}x_1x_2x_3 + \frac{5911}{185600}x_1x_2x_4 - \frac{1443}{4640}x_1x_3x_4 + \frac{5751}{4640}x_2x_3x_4. \tag{A4}
\end{aligned}$$