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THE RAYLEIGH–RITZ SOLUTION TO ESTIMATE VIBRATION CHARACTERISTICS OF BUILDING FLOORS

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An approximate solution for floor vibration is proposed as an aid to structural design. The building floor is approximated as a thin rectangular plate in order to use the Rayleigh–Ritz method in the analysis. To represent various structural types of building floors, the plate may be simultaneously subjected to many different factors, including orthotropy of the plate material, the presence of in-plane forces, uniform elastic edge supports, elastic point supports, reinforcement by flexural and torsional beams and vibration control by tuned mass dampers (TMDs). Beam functions, for which free edge conditions of the plate are generally difficult to represent, are used in practice as admissible functions in the Rayleigh–Ritz method under various boundary conditions, in consideration of the approximate approach for only free edge conditions of the plate. The accuracy and applicability of the approximate solution are confirmed in comparison with the results obtained by earlier studies and the finite element method (FEM).

1. INTRODUCTION

From the viewpoint of environmental problems concerning the habitability of residential space and mechanical manipulation under vibration isolation, the estimation of floor vibration is important in structural design. To comprehend the vibration characteristics of building floors, the FEM has often been used. However, it is troublesome for structural engineers to handle the FEM and its pre- and post-processors. Such a simplified analysis as the Rayleigh–Ritz method is helpful in the structural design of building floors.

A building floor is approximated as a thin rectangular plate in order to use the Rayleigh–Ritz method in the analysis. However, there are various structural types of building floors; for example, the flat slab, the void slab, the prestressed concrete slab, the slab using a steel deck, the slab reinforced by beams and columns, and the slab with TMDs to control vibration. Furthermore, various boundary conditions exist due to structural members surrounding the floor. To examine the vibration characteristics of such floors, the following requirements are considered in the analysis: (1) the orthotropy of the plate material; (2) the presence of in-plane forces; (3) uniform elastic edge supports against translation and rotation; (4) elastic point supports against translation and rotation at arbitrary locations; (5) reinforcement by flexural and torsional beams; and (6) vibration control by TMDs at arbitrary locations.

A number of important research papers on vibration analyses of a rectangular plate, can be found in the literature. For various boundary conditions implying uniform elastic

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edge supports and clamped, simply supported and free edge conditions, beam functions are often used as admissible functions in the Rayleigh–Ritz method. In 1959, Chrichael [1] analysed a plate with uniform elastic edge supports against rotation. In 1973, Leissa [2] showed accurate analytical solutions using exact characteristic equations and the Rayleigh–Ritz method for classical boundary conditions containing clamped, simply supported and free edges. Clamped and simply supported edge conditions were exactly satisfied by the use of beam functions, but free edge conditions were only approximated, making the approach usually less accurate when a free edge was involved. In 1975, Bassily and Dickinson [3] solved problems involving free edge conditions using degenerated beam functions. In 1984, Warburton and Edney [4] studied the application of beam functions that satisfied classical boundary conditions to represent uniform elastic edge supports. As for problems containing point supports, in 1979, Kerstens [5] showed solutions using Lagrangian multipliers. The fully free edge condition was assumed and beam functions were adopted in addition to special rigid body deflection modes for the first and second modes. On the other hand, several different polynomials [6–9] have also been proposed as admissible functions, since the use of beam functions is difficult for representing free edge conditions. Static beam functions [10] were also proposed for the various boundary conditions. Therefore, there is no example of beam functions that can be practically applied to the various boundary conditions; nevertheless, beam functions are superior to any other function as admissible functions.

For items (1)–(4) above, in 1990, Kim and coworkers [9] represented accurate natural frequencies up to the sixth mode using simple polynomials in the Rayleigh–Ritz method. For point supports, the simple technique of taking the strain energy of springs into account without Lagrangian multipliers was shown. In contrast, Gorman [11–13] developed the superposition method for items (1), (3) and (4).

For item (5), in 1970, Kirk [14] dealt with a plate reinforced by a single integral stiffener using the Rayleigh–Ritz method. The plate was fully simply supported. In 1981, Laura and Gutiérrez [15] added the condition of edges elastically restrained against rotation to the problem examined by Kirk. In those papers, the effects of torsional deflection in stiffeners was not treated. In 1995, Lee [16] studied the effects of torsional and bending restraints of intermediate stiffeners using the Rayleigh–Ritz method.

For item (6), in 1970, Sundara Raja Iyengar and Jagadish [17] studied a spring–mass system to simulate the dynamic response of highway bridges to moving loads. The orthotropic plate theory was used under the classical boundary conditions. However, the damping effect was not considered.

The primary purpose of this work is to apply the Rayleigh–Ritz solution to the problems in structural design concerning floor vibration by integrating those techniques mentioned above and taking TMDs into consideration. The secondary purpose is to show that beam functions can be used in practice for the various boundary conditions as admissible functions. For only free edge conditions, an approximate approach is introduced, because the rigid body deflection modes beneficial for the analysis of plates are produced in the primary modes by attaching extremely small restraints against translation along free edges. From a designer's viewpoint, the valid value of the translational spring constant for the extremely small restraints is shown as the dimensionless value.

It is confirmed that the approximate free edge conditions used here almost equal the exactly free edge conditions, in comparison with the results of earlier studies. Furthermore, the results of more realistic examples for building floors are also shown, compared with the results obtained by the FEM. As a consequence, it is clarified that the present solutions are sufficiently applicable to structural design of building floors.

2. THE RAYLEIGH–RITZ SOLUTION

2.1. EIGENVALUE EQUATION

The Rayleigh–Ritz method is employed under the Kirchhoff–Love hypotheses. The mechanical system investigated here is shown in Figure 1, using the Cartesian co-ordinate system x, y, z. The plate is subjected to in-plane forces per unit width N_x and N_y (tensile force positive) in the middle plane of the plate in the x- and y-directions, respectively. Translational restraints (having spring constants K_A, K_B, K_C and K_D) and rotational restraints (having spring constants R_A, R_B, R_C and R_D) are along the edges of the plate. Here, four boundaries are denoted as A ($x = 0$), B ($x = a$), C ($y = 0$) and D ($y = b$). The locations of point supports (having spring constants K_r against translation, R_{yr} against rotation about the y-direction and R_{xr} against rotation about the x-direction) are set at points (x_r, y_r). Beams in the y-direction are placed along the line $x = x_p$ with flexural rigidities E_pI_p and torsional rigidity G_pJ_p. Likewise, beams in the x-direction are located along the line $y = y_q$ with flexural rigidities E_qI_q and torsional rigidity G_qJ_q. The mid-point of the beam is placed so that it coincides with the centroid of the plate. TMDs are set at points (x_s, y_s) with spring constants k_s and masses m_s, where p, q, r and $s = 1, 2, 3, \ldots$.

The maximum strain energy U for the stiffened plate with TMDs is given as

$$
\begin{aligned}
U = {} & \tfrac{1}{2}\int_0^a \int_0^b [D_x w_{,xx}^2 + 2D_{xy} w_{,xx} w_{,yy} + D_y w_{,yy}^2 + 4H_{xy} w_{,xy}^2)\, \mathrm{d}y\, \mathrm{d}x \\
& + \tfrac{1}{2}\int_0^a \int_0^b (N_x w_{,x}^2 + N_y w_{,y}^2)\, \mathrm{d}y\, \mathrm{d}x \\
& + \tfrac{1}{2}\sum_{p=1} \left(E_pI_p \int_0^b w_{,yy}^2\Big|_{x=x_p} \mathrm{d}y + G_pJ_p \int_0^b w_{,xy}^2\Big|_{x=x_p} \mathrm{d}y \right) \\
& + \tfrac{1}{2}\sum_{q=1} \left(E_qI_q \int_0^a w_{,xx}^2\Big|_{y=y_q} \mathrm{d}x + G_qJ_q \int_0^a w_{,xy}^2\Big|_{y=y_q} \mathrm{d}x \right) \\
& + \tfrac{1}{2}K_C \int_0^a w^2\Big|_{y=0} \mathrm{d}x + \tfrac{1}{2}R_C \int_0^a w_{,y}^2\Big|_{y=0} \mathrm{d}x + \tfrac{1}{2}K_D \int_0^a w^2\Big|_{y=b} \mathrm{d}x + \tfrac{1}{2}R_D \int_0^a w_{,y}^2\Big|_{y=b} \mathrm{d}x \\
& + \tfrac{1}{2}K_A \int_0^b w^2\Big|_{x=0} \mathrm{d}y + \tfrac{1}{2}R_A \int_0^b w_{,x}^2\Big|_{x=0} \mathrm{d}y + \tfrac{1}{2}K_B \int_0^b w^2\Big|_{x=a} \mathrm{d}y + \tfrac{1}{2}R_B \int_0^b w_{,x}^2\Big|_{x=a} \mathrm{d}y \\
& + \tfrac{1}{2}\sum_{r=1} \left(K_r w^2\Big|_{(x_r, y_r)} + R_{yr} w_{,y}^2\Big|_{(x_r, y_r)} + R_{xr} w_{,x}^2\Big|_{(x_r, y_r)} \right) + \tfrac{1}{2}\sum_{s=1} k_s w_s^2, \qquad (1)
\end{aligned}
$$

where $D_x = E_x h^3/12(1 - \nu_{xy}\nu_{yx})$, $D_y = D_x E_y/E_x$, $H_{xy} = G_{xy}h^3/12$ and $D_{xy} = \nu_{xy} D_y$ are the flexural rigidities of the orthotropic plate, in which h is the plate thickness, E_x and E_y are Young's moduli in the x- and y-directions respectively, G_{xy} is the shear modulus, and ν_{xy} and ν_{yx} are Poisson ratios. w is the deflection of the stiffened plate. w_s indicates the

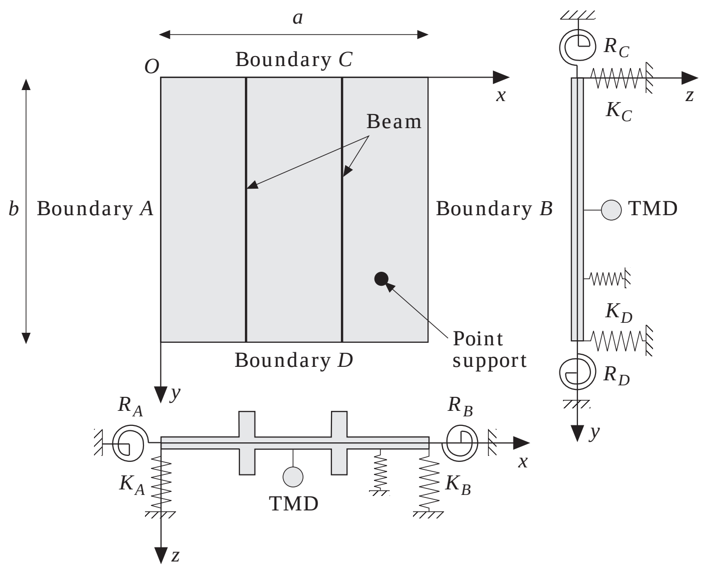

Figure 1. The mechanical system investigated in the present study.

displacement relative to the plate; that is, the deflection of the spring in a TMD. The symbol $(\,)_{,s}$ denotes differentials in the s-direction.

The kinetic energy T is given as,

$$T = \tfrac{1}{2}(\rho h + m_0)\omega^2 \int_0^a \int_0^b w^2 \, \mathrm{d}y \, \mathrm{d}x$$

$$+ \tfrac{1}{2} \sum_{p=1} \rho_p A_p \omega^2 \int_0^b w^2 \bigg|_{x=x_p} \mathrm{d}y + \tfrac{1}{2} \sum_{q=1} \rho_q A_q \omega^2 \int_0^a w^2 \bigg|_{y=y_q} \mathrm{d}x$$

$$+ \tfrac{1}{2} \sum_{p=1} \rho_p A_p \omega^2 \int_0^b w^2 \bigg|_{x=x_p} \mathrm{d}y + \tfrac{1}{2} \sum_{q=1} \rho_q A_q \omega^2 \int_0^a w^2 \bigg|_{y=y_q} \mathrm{d}x, \qquad (2)$$

where ω is the radian natural frequency. ρ, ρ_p and ρ_q are the mass densities of the plate and the y- and x-directional beams, respectively. m_0 is the distributed attached mass. A_p and A_q are the cross-sectional areas of the y- and x-directional beams, respectively.

Then, the deflection w may be expressed as

$$w(x, y) = \sum_{m=1} \sum_{n=1} w_{mn} f_{xm}(x) f_{yn}(y), \qquad (3)$$

in which $f_{xm}(x)$ and $f_{yn}(y)$ denote the assumed admissible functions in the x- and y-directions, respectively. w_{mn} are the undetermined constants.

Substituting equation (3) into equations (1) and (2) and applying the minimization of the Rayleigh quotient with respect to the coefficients w_{mn} leads to the following eigenvalue equation in matrix form:

$$\begin{bmatrix} K_{mn,ij} & 0 \\ 0 & K_{s,s} \end{bmatrix} \begin{Bmatrix} w_{ij} \\ w_s \end{Bmatrix} - \omega^2 \begin{bmatrix} M_{mn,ij} & M_{mn,s} \\ M_{s,ij} & M_{s,s} \end{bmatrix} \begin{Bmatrix} w_{ij} \\ w_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{4}$$

Here

$$K_{mn,ij} = D_x F^{(2,2)}_{xmi} F^{(0,0)}_{ynj} + D_y F^{(0,0)}_{xmi} F^{(2,2)}_{ynj} + D_{xy} (F^{(2,0)}_{xmi} F^{(0,2)}_{ynj} + F^{(0,2)}_{xmi} F^{(2,0)}_{ynj}) + 4H_{xy} F^{(1,1)}_{xmi} F^{(1,1)}_{ynj}$$

$$+ N_x F^{(1,1)}_{xmi} F^{(0,0)}_{ynj} + N_y F^{(0,0)}_{xmi} F^{(1,1)}_{ynj} + \sum_{p=1} (E_p I_p E^{(0,0)}_{x_p mi} F^{(2,2)}_{ynj} + G_p J_p E^{(1,1)}_{x_p mi} F^{(1,1)}_{ynj})$$

$$+ \sum_{q=1} (E_q I_q F^{(2,2)}_{xmi} E^{(0,0)}_{y_q nj} + G_q J_q F^{(1,1)}_{xmi} E^{(1,1)}_{y_q nj}) + K_C F^{(0,0)}_{xmi} E^{(0,0)}_{0nj} + R_C F^{(0,0)}_{xmi} E^{(1,1)}_{0nj}$$

$$+ K_D F^{(0,0)}_{xmi} E^{(0,0)}_{bnj} + R_D F^{(0,0)}_{xmi} E^{(1,1)}_{bnj} + K_A E^{(0,0)}_{0mi} F^{(0,0)}_{ynj} + R_A E^{(1,1)}_{0mi} F^{(0,0)}_{ynj}$$

$$+ K_B E^{(0,0)}_{ami} F^{(0,0)}_{ynj} + R_B E^{(1,1)}_{ami} F^{(0,0)}_{ynj} + \sum_{r=1} (K_r E^{(0,0)}_{x_r mi} E^{(0,0)}_{y_r nj} + R_{yr} E^{(0,0)}_{x_r mi} E^{(1,1)}_{y_r nj}$$

$$+ R_{xr} E^{(1,1)}_{x_r mi} E^{(0,0)}_{y_r nj}), \qquad K_{s,s} = k_s,$$

$$M_{mn,ij} = (\rho h + m_0) F^{(0,0)}_{xmi} F^{(0,0)}_{ynj} + \sum_{p=1} \rho_p A_p E^{(0,0)}_{x_p mi} F^{(0,0)}_{ynj} + \sum_{q=1} \rho_q A_q F^{(0,0)}_{xmi} E^{(0,0)}_{y_q nj}$$

$$+ \sum_{s=1} m_s E^{(0,0)}_{x_s mi} E^{(0,0)}_{y_s nj},$$

$$M_{mn,s} = m_s f_{xm}(x_s) f_{yn}(y_s), \qquad M_{s,ij} = m_s f_{xi}(x_s) f_{yj}(y_s), \qquad M_{s,s} = m_s,$$

$$F^{(\alpha,\beta)}_{xmi} = \int_0^a f^{(\alpha)}_{xm}(x) f^{(\beta)}_{xi}(x)\, \mathrm{d}x, \qquad F^{(\alpha,\beta)}_{ynj} = \int_0^b f^{(\alpha)}_{yn}(y) f^{(\beta)}_{yj}(y)\, \mathrm{d}y,$$

$$E^{(\alpha,\beta)}_{x_\gamma mi} = f^{(\alpha)}_{xm}(x_\gamma) f^{(\beta)}_{xi}(x_\gamma), \qquad E^{(\alpha,\beta)}_{y_\gamma nj} = f^{(\alpha)}_{yn}(y_\gamma) f^{(\beta)}_{yj}(y_\gamma),$$

in which the symbol $f^{(n)}$ denotes the number of differentials.

The solution of equation (4) yields the natural frequencies and coefficients for the mode shapes (3) of free vibration. For the case without TMDs, only the terms $K_{mn,ij}$, $M_{mn,ij}$ and w_{ij} are used in equation (4).

2.2. DYNAMIC RESPONSE ANALYSES

It is important to consider the effect of the viscous damper in TMDs for dynamic response analyses. Although there are some procedures for considering it, Rayleigh damping, which is proportional to both mass and stiffness, is adopted here, because the damping coefficients for TMDs can simply be added to the Rayleigh damping of the stiffened plate without TMDs. Similarly to the example of equation (4), the damping elements are written as

$$C_{mn,ij} = \alpha_M \left(M_{mn,ij} - \sum_{s=1} m_s E^{(0,0)}_{x_s mi} E^{(0,0)}_{y_s nj} \right) + \alpha_K K_{mn,ij}, \tag{5a}$$

$$C_{mn,s} = C_{s,ij} = 0, \qquad C_{s,s} = c_s, \tag{5b, c}$$

where c_s in equation (5c) is the damping coefficients for TMDs, and the two coefficients

$$\alpha_M = 2\omega_i \omega_j \frac{\xi_j \omega_i - \xi_i \omega_j}{\omega_i^2 - \omega_j^2} \qquad \text{and} \qquad \alpha_K = \frac{2(\xi_i \omega_i - \xi_j \omega_j)}{\omega_i^2 - \omega_j^2} \tag{6}$$

in equation (5a) are the Rayleigh damping coefficients, in which ω_i is the radian natural frequency and ξ_i is the modal critical damping ratio for the ith mode without TMDs.

The term concerning external forces is considered by introducing the potential energy of external forces in the z-direction. Then, the Newmark-β method is performed in the analysis, since the orthogonality is lost between modes of different degrees, because of the addition of equation (5c). The deflections of the plate can be obtained via equation (3). The velocity and acceleration for the deflections of the plate are also obtained in the same manner.

2.3. ADMISSIBLE FUNCTIONS

In the present work, the following beam function [1, 18] in the x-direction is used as an admissible function with respect to the various boundary conditions including free edges:

$$f_{xm}(x) = C_{1m} D_{1m} + C_{2m} D_{2m} + C_{3m} D_{3m} + C_{4m} D_{4m}, \tag{7}$$

where

$$D_{1m} = \cosh(\lambda_m x/a) + \cos(\lambda_m x/a), \qquad D_{2m} = \cosh(\lambda_m x/a) - \cos(\lambda_m x/a),$$

$$D_{3m} = \sinh(\lambda_m x/a) + \sin(\lambda_m x/a), \qquad D_{4m} = \sinh(\lambda_m x/a) - \sin(\lambda_m x/a),$$

in which λ_m is a root of the characteristic equation. C_{1m}, C_{2m}, C_{3m} and C_{4m} are the undetermined constants. Then, four boundary conditions [6] for the plate in the x-direction are given as

$$w|_{x=0} = -D_x w_{,xxx}|_{x=0}/K_A, \qquad w|_{x=a} = D_x w_{,xxx}|_{x=a}/K_B,$$

$$w_{,x}|_{x=0} = D_x w_{,xx}|_{x=0}/R_A, \qquad w_{,x}|_{x=\mathrm{a}} = -D_x w_{,xx}|_{x=a}/R_B. \tag{8}$$

Substitution of equation (7) into equation (8) leads to the value of λ_m and the ratio of the four undetermined constants for each mode through a simple iterative process of varying λ_m gradually as the determinant vanishes. The beam function in the y-direction is expressed in the same manner.

When this beam function is applied to free edge conditions, extremely small restraints against translation should be attached along free edges, because the beneficial rigid body deflection modes are added to the original mode. By means of our numerical examples, the dimensionless value of the translational spring constant attached along the free edges in the case of boundary A is given as

$$K_A a^3/D_x \equiv 0{\cdot}11. \tag{9}$$

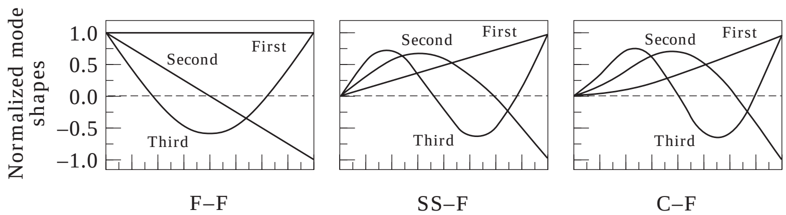

Figure 2. The first three mode shapes of a beam, including the approximate free edge conditions.

It can be given in the same manner in the cases of boundaries B, C and D. If the value of equation (9) is given to free edges as the extremely small restraint against translation, the rigid body deflection modes are produced as shown by Figure 2, in which C, SS and F denote clamped, simply supported and free edge conditions, respectively. The first and second modes in the case of F–F and the first mode in the case of SS–F cannot be obtained without the small restraints against translation along free edges. Such modes are effective for the analysis of plates. The other mode shapes correspond to shapes obtained by using just free edge conditions. On the other hand, the number of terms in series is limited, since equation (8) is solved numerically. The maximum number of terms is 14 for the case of F–F in our calculation.

3. NUMERICAL RESULTS

3.1. FREE VIBRATION ANALYSES OF A RECTANGULAR PLATE WITH CLASSICAL BOUNDARY CONDITIONS

In order to examine the applicability of the beam functions including the approximate approach for only free edge conditions, free vibration analyses of a rectangular plate with classical boundary conditions are performed using 6×6 terms in series. The present solutions are shown in Table 1, together with references [2] and [6]. In reference [2], exact characteristic equations are given for the six cases having two opposite sides simply supported. The Rayleigh–Ritz method is employed using the products of beam functions with 6×6 terms to analyze the remaining cases. In reference [6], the Rayleigh–Ritz method is also employed using polynomials with 2×2 terms. The solutions are compared by the dimensionless fundamental natural frequency parameter written as

$$\Omega = \omega_1 a^2 \sqrt{\rho h / D},$$

in which D denotes flexural rigidity of the isotropic plate.

In all cases, the present solutions indicate good agreement with the results of other studies.

3.2. FREE VIBRATION ANALYSES OF A SQUARE PLATE WITH TRANSLATIONAL AND ROTATIONAL EDGE RESTRAINT PARAMETERS

The performance of the beam functions is examined for uniform elastic edge supports. The fundamental natural frequency parameter for a square plate with the same restraint parameters along each edge is plotted in Figure 3. There are the following three cases: case A—$R_A = R_B = R_C = R_D = S_R$ and $K_A = K_B = K_C = K_D = \infty$ (from a fully simply supported to a fully clamped plate), case B—$R_A = R_B = R_C = R_D = 0$ and $K_A = K_B = K_C = K_D = S_K$ (from a fully free to a fully simply supported plate), case C—$R_A = R_B = R_C = R_D = S_R$ and $K_A = K_B = K_C = K_D = S_K$ (from a fully free to a fully clamped plate), $S_R = R_A a/D$ and $S_K = K_A a^3/D$ are dimensionless restraint parameters varying from 10^{-2} to 10^4, respectively. This problem has been solved by several researchers. For example, in reference [4], the Rayleigh–Ritz method was used by combining beam functions that satisfied classical boundary conditions. In reference [9], the Rayleigh–Ritz method was employed by using simple polynomials with 8×8 terms.

The present solutions obtained by using the beam functions with 6×6 terms are compared with results in references [4] and [9] and solutions obtained by the FEM (using the commercial package MARC). The present solutions show good agreement with the results in reference [9] and by the FEM. In the present solutions, the value of equation (9) was not used, since the stable solutions were obtained up to $S_K = 10^{-2}$.

TABLE 1

Comparison of dimensionless fundamental natural frequency parameter Ω

Boundary conditions	a/b	Present study	Reference [2]	Reference [6]
SS–SS–SS–SS	1·0	19·74	19·73	19·74
	1·5	32·08	32·07	32·09
SS–C–SS–C	1·0	29·00	28·95	28·95
	1·5	56·45	56·34	56·40
SS–C–SS–SS	1·0	23·66	23·64	23·67
	1·5	42·56	42·52	42·55
SS–C–SS–F	1·0	12·76	12·68	12·92
	1·5	16·93	16·82	17·00
SS–SS–SS–F	1·0	11·75	11·68	11·79
	1·5	13·82	13·71	13·81
SS–F–SS–F	1·0	9·72	9·63	9·98
	1·5	9·70	9·55	10·12
C–C–C–C	1·0	36·07	35·99	35·99
	1·5	60·89	60·77	60·85
C–C–C–SS	1·0	31·88	31·82	31·86
	1·5	48·23	48·16	48·23
C–C–C–F	1·0	24·07	24·02	24·20
	1·5	26·81	26·73	26·89
C–C–SS–SS	1·0	27·08	27·05	27·11
	1·5	44·93	44·89	45·02
C–C–SS–F	1·0	17·64	17·61	17·79
	1·5	21·10	21·03	21·17
C–C–F–F	1·0	7·00	6·94	7·00
	1·5	11·33	11·21	11·48
C–SS–C–F	1·0	23·51	23·46	23·54
	1·5	24·85	24·77	24·84
C–SS–SS–F	1·0	16·89	16·86	16·92
	1·5	18·60	18·54	18·59
C–SS–F–F	1·0	5·43	5·36	5·42
	1·5	7·07	6·93	7·13
C–F–C–F	1·0	22·32	22·27	22·50
	1·5	22·28	22·21	22·58
C–F–SS–F	1·0	15·31	15·28	15·53
	1·5	15·27	15·21	15·62
C–F–F–F	1·0	3·58	3·49	3·59
	1·5	3·69	3·47	3·64
SS–SS–F–F	1·0	3·46	3·36	3·43
	1·5	5·21	5·02	5·17

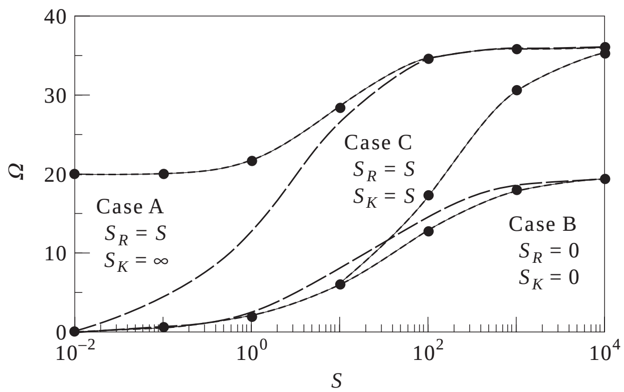

Figure 3. The variation of the dimensionless fundamental frequency parameter of a square plate with translational and rotational restraint parameter. ——, Present study; — —, references [4]; - - - - , reference [9]; ●, FEM.

3.3. FREE VIBRATION ANALYSES OF A SQUARE PLATE WITH POINT SUPPORTS

The applicability of the beam functions, including the approximate approach concerning free edge conditions, is examined for the higher modes. Dimensionless frequency parameters for a fully free isotropic plate point supported (a) at all four corners, (b) at the mid-point of each side and (c) at all four corners and at the mid-point of each side are calculated by beam functions containing some different terms in series.

The values of dimensionless frequency parameters up to the sixth mode are compared with the results in references [8] and [19] and by the FEM (using the MARC) in Table 2. In reference [8], the Rayleigh–Ritz method is employed using polynomials as admissible

TABLE 2

Comparison of dimensionless natural frequency parameters of rectangular plates with point supports

Problem type	Mode number	Present study			Reference [8]	Reference [19]	FEM
		6 × 6	8 × 8	10 × 10			
(a)	1	7·18	7·17	7·16	—	7·11	7·13
	2	15·86	15·84	15·83	—	15·77	15·74
	3	15·86	15·84	15·83	—	15·77	15·74
	4	19·78	19·72	19·69	—	19·58	19·39
	5	38·87	38·73	38·68	—	38·43	38·54
	6	44·49	44·37	44·35	—	44·37	43·98
(b)	1	13·52	13·51	13·51	13·47	13·47	13·41
	2	18·29	18·16	18·03	18·03	17·85	17·87
	3	19·35	19·19	19·05	18·93	18·79	18·96
	4	19·35	19·19	19·05	18·93	18·79	18·96
	5	27·71	27·47	27·26	27·05	26·92	26·73
	6	52·04	51·51	51·38	51·44	51·13	50·74
(c)	1	18·30	18·16	18·03	18·03	17·85	17·87
	2	35·87	35·62	35·32	35·17	34·89	35·04
	3	35·87	35·62	35·32	35·17	34·89	35·04
	4	38·87	38·73	38·68	38·43	38·43	38·54
	5	62·74	61·96	61·06	60·58	60·12	60·39
	6	69·53	69·31	69·12	69·14	68·51	67·75

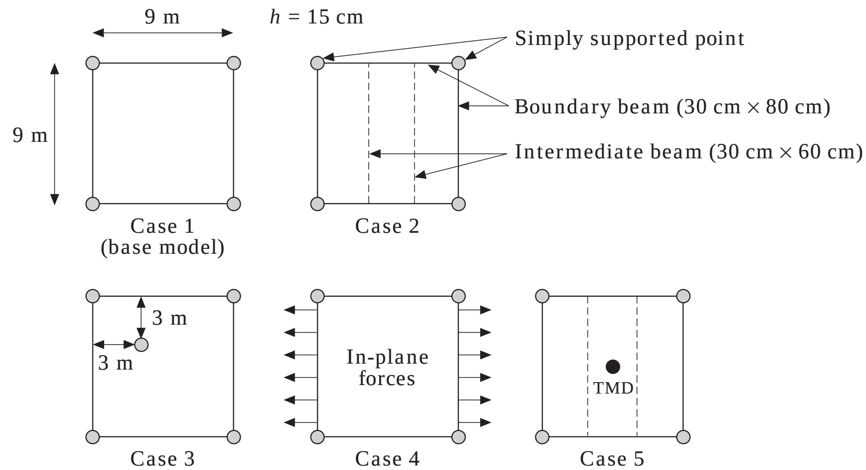

Figure 4. Examples to investigate the effects of various analytical conditions.

functions. The solutions using polynomials with 6 × 6 terms are presented in Table 2. In reference [19], the spline finite strip method is used. From the viewpoint of designers, the present solutions up to the sixth mode show good agreement with the other results.

3.4. VIBRATION ANALYSES OF A SQUARE PLATE UNDER VARIOUS ANALYTICALCONDITIONS

More realistic problems are treated here. As the base model, a square isotropic plate simply supported at four corner points and restrained along four edges against only rotation, having infinite spring constants, is used. The plate is reinforced by the same four beams along each boundary. The span, thickness, Poisson ratio, Young's modulus and mass density of the plate are 9 m, 0·15 m, 0·167, 2·1e7 Pa and 4·31 Ns^2/m, respectively. The cross-section of the beams having the same material constants as the plate, but a mass density of 2·35 Ns^2/m, is rectangular. The width and depth are 0·3 m and 0·8 m, respectively. The following cases shown in Figure 4 are investigated: case 1, the base model; case 2, adding two parallel beams in the same interval to case 1; case 3, adding a simply supported point near the centre of the plate to case 1; case 4, adding in-plane forces ($N_x = 9{\cdot}8e5$ N/m, $N_y = 0$ N/m) to case 1; case 5; adding a TMD at the centre of the plate to case 2. The material constants and dimensions of the intermediate beams in cases 2 and 5 are the same as those of the boundary beams, but the depth is 0·6 m. For cases 1–4, free

TABLE 3

Comparison of natural frequencies up to 6th mode

	Case 1		Case 2		Case 3		Case 4	
Mode number	Present study (Hz)	FEM (Hz)	Present study (Hz)	FEM (Hz)	Present study (Hz)	FEM (Hz)	Present study (Hz)	FEM (Hz)
1	6·28	6·36	8·84	8·95	8·10	8·24	6·61	6·53
2	12·00	12·34	14·04	14·42	12·00	12·34	12·20	12·44
3	12·00	12·34	14·90	15·14	16·02	16·42	12·48	12·57
4	17·40	17·91	18·48	19·15	17·40	17·91	17·63	18·06
5	18·34	18·86	22·56	23·27	18·34	18·86	18·63	18·97
6	19·62	20·41	22·81	23·60	21·65	22·48	20·10	20·64

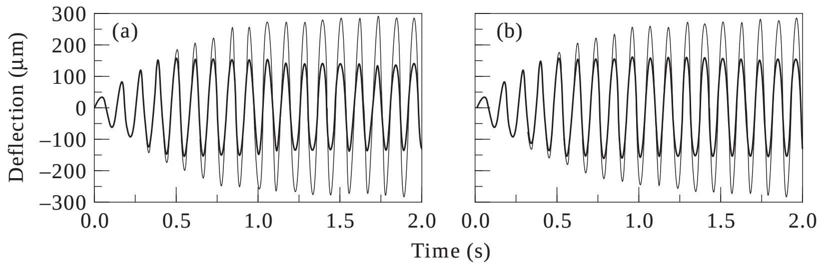

Figure 5. A comparison of time history curves for deflection at the centre of the plate. ——, Without TMD; ——, with TMD. (a) Present study; (b) FEM.

vibration analysis is performed. For cases 2 and 5, the dynamic response analysis is carried out using a sine curve of 9 Hz as a dynamic concentrated load at the centre of the plate. The Rayleigh damping coefficients are calculated by equation (6) using the values $\xi_1 = 0{\cdot}03$ and $\xi_2 = 0{\cdot}03$. The values of the spring constant, mass and modal critical damping ratio for the TMD are given as 5·0e5 N/m, 1·6e2 s2/m and 0·061, respectively, in order optimally to control the vibration.

The natural frequencies up to the sixth mode are given in Table 3, and the time history curves for deflection at the centre of the plate are shown in Figure 5, compared with the results obtained by the FEM (using the MARC). Here, 10×10 terms in series were used in all cases, since fully convergent solutions can be obtained. In the FEM, 12×12 meshes were used for the plate with rectangular shell elements. Beam elements were located along the meshes.

Considering that these are complicated problems with some analytical conditions, the agreement is good, especially for the fundamental natural frequency.

4. CONCLUSIONS

In order to estimate the vibration characteristics of building floors in a simple manner, the Rayleigh–Ritz solution was adopted for the transverse vibration analysis of a thin rectangular plate. For representing various structural types of building floors, the plate may be simultaneously subjected to many different factors, including the orthotropy of the plate material, the presence of in-plane forces, uniform elastic edge supports, elastic point supports, reinforcement by flexural and torsional beams and vibration control by TMDs. In the analysis, beam functions, for which free edge conditions are generally difficult to represent, were applied in practice to various boundary conditions, implying uniform elastic edge supports and clamped, simply supported and free edge conditions as admissible functions. The approximate approach was introduced for only free edge conditions, because the rigid body deflection modes that were beneficial for the analysis of plates were produced in the primary modes when extremely small restraints against translation were attached along free edges. Then, the valid value of the translational spring constant for the extremely small restraints was shown from a designer's viewpoint.

It was confirmed that, in comparison with the results of earlier studies the approximate free edge conditions almost equal the exactly free edge conditions. The more realistic numerical examples were calculated and compared with the results obtained by the FEM. As a consequence, it was clarified that the present solutions are sufficiently applicable to structural design of building floors.

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