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A GENERALIZED APPROACH FOR THE ACOUSTIC RADIATION FROM A BAFFLED OR UNBAFFLED PLATE WITH ARBITRARY BOUNDARY CONDITIONS, IMMERSED IN A LIGHT OR HEAVY FLUID

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The present paper is dedicated to the study of the radiation of a baffled and/or unbaffled plate with arbitrary boundary conditions. The proposed work deals with a complete and rigorous fluid–structure coupling. A Rayleigh–Ritz approach is used to develop the plate deflection and, in the unbaffled case, the sound pressure jump function. A suitable set of trial functions is proposed. It allows to obtain only one generic numerical code for all the boundary conditions and—a key aspect of the formulation—for both the baffled and the unbaffled plate. An efficient numerical scheme is proposed for the computation of the radiation impedance coefficients. A good agreement is found with a numerical approach based on the boundary element method. Validation with experimental results is also presented. The present formulation provides a useful tool to perform a clear and systematic comparison between the baffled and the unbaffled plate.

1. INTRODUCTION

Throughout the past 20 years, the acoustic radiation from baffled structures has been a subject of intense research, at least for simple structures such as beams and plates. Many efforts have been concentrated on the calculation of the radiation efficiency to qualify the radiation behaviour of these structures. Early works on plane structures have been mainly devoted to the study of simply supported plates [1–4], for which modal radiation efficiencies are calculated. Generally, only the resistive parts of the modal radiation efficiencies are evaluated, since it governs the radiated power. Moreover, the intermodal radiation impedance coefficients are usually neglected, an assumption that remains valid only when the fluid is considered to be "light".

From a practical point of view, realistic plates prove to be more complicated: (1) the boundary conditions are not restricted to the simply supported case, so the effect of other boundary conditions has to be examined; (2) common plates are generally not inserted in an infinite rigid baffle; (3) many applications require the fluid–structure coupling to be rigorously taken into account—as is the case, for example, for acoustic excitation or for "heavy" fluid (e.g., water).

Many authors have proposed different approaches to the study of the above considerations. The effect of cross-coupling terms has been discussed by Keltie and Peng [5] for baffled simply supported and clamped plates and by Davies [6]. Sound radiation from plates with general boundary conditions has been studied by Gomperts [7, 8] and

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more recently by Berry [9, 10]. The latter formulation presents the advantage that only one set of trial functions is needed to develop the plate deflection, so only one generic numerical code can be used to simulate plates with arbitrary boundary conditions. Despite the useful information and physical tendencies that these studies can provide, they have all been dedicated to the case of plates inserted in a rigid baffle. Moreover, the calculation of modal radiation impedance coefficients for baffled plates remains a difficult mathematical task.

Few works have been devoted to the study of unbaffled plates, particulary when the coupling between the fluid and the plate is important. Atalla *et al*. [11] have treated the radiation in the air of an unbaffled plate excited by a point force. Their method uses a two-step procedure for which the fluid–structure coupling is neglected when calculating the plate deflection. Comparisons with experimental results have shown that this procedure was quite valid in the case of a plate that radiates in air. The authors have also presented a brief literature review for the unbaffled plate. The radiation of an oscillating unbaffled disk has been studied by Williams [12] and Wu *et al*. [13]. In the study by Wu *et al*., a variational formulation is used to avoid the calculation of highly singular integrals that are classically encountered when using double layer potential formulations [14, 15]. An equivalent formulation has been also developed by Hamdi [16]. This variational formulation is implemented in commercial FEM–BEM codes [17, 18]. The numerical interface between two commercial numerical codes has been developed to study the vibrations of unbaffled tridimensional structures [19] but, once again it seems, to the authors' knowledge, that fluid–structure coupling is neglected. Moreover, there is a lack of realistic results published for unbaffled structures.

This short literature review has made it clear that, while radiation by baffled plates is well mastered, the case of an unbaffled plate remains a subject of interest, in particular for strong fluid–structure coupling. Interpretations based on physical tendencies observed for radiation by baffled plates cannot simply be transposed to the case of unbaffled plates. It is well known, for low frequency, that monopolar behaviour characterizes the baffled plate, while dipolar behaviour is observed for the unbaffled plate, leading to an acoustic "short-circuit" effect. If one makes an exception for the works of Atalla *et al*. [11] for weak coupling, there are no clear and systematic comparisons between baffled and unbaffled plates. It is thought that such a study would be of great interest in helping to understand the physical mechanisms that govern the acoustic radiation of both baffled and unbaffled plates.

In the present paper a new semi-analytical approach, denoted as TRM, is presented; this is designed to perform, in a simple way, a systematic comparison between the baffled and the unbaffled plate with complete fluid–structure coupling. The paper draws together the theoretical framework for such a study, and numerical and experimental validations are presented. The present paper is mainly divided into a vibrational part and an acoustic part. First, the equation of motion of a thin plate in the fluid is introduced. A Rayleigh–Ritz solution is proposed as an approximation for the plate deflection. Using the stationary condition for the Hamiltonian of the plate, a standard linear set of equations is obtained for the Rayleigh–Ritz expansion coefficients. One main original feature of the approach lies in the choice of a suitable set of trial functions that allows arbitrary boundary conditions on the plate contour. A set of trigonometric functions is used, leading to what one can call the Trigonometric Ritz Method (TRM). The basis equations of the fluid are then presented, to lead to the complete fluid–structure coupling relations. The quantity of importance in the fluid–structure coupling is the surface acoustic pressure acting on the plate, detailed calculations of which are then given. These calculations require different approaches depending on whether or not the plate is baffled. To compute the surface acoustic pressure in the baffled case, the well-known Rayleigh integral is used. This allows us to relate the surface acoustic pressure to the radiation impedance coefficients with few manipulations. For the unbaffled plate, one has to identify the sound pressure jump over the plate. To do so, a variational formulation associated to the Kirchhoff integral equation is used [16]. Once again, a Rayleigh–Ritz expansion is used to approximate the sound pressure jump function. This finally allows us to relate this latter to a set of coefficients that can be interpreted as the radiation impedance coefficients. A key aspect of the formulation is the introduction of an efficient numerical scheme to compute these coefficients without requiring many recurrences, as is the case in the work of Atalla *et al.* [11] or Berry [10]. Despite the fact that the baffled and the unbaffled plate are treated using two different approaches, it is shown here that the coefficient of the radiation impedances can be computed using the same numerical scheme in both cases. This is a direct consequence of the choice of trial functions in the Rayleigh–Ritz expansions. This represents a significant improvement over other semi-analytical methods. The main originality of the proposed approach is that it provides, probably for the first time, a simple semi-analytical tool that includes all of the important features for the radiation of plates: (1) arbitrary boundary conditions can be achieved with only one set of trial functions; (2) both baffled and unbaffled plates can be simulated; (3) the fluid–structure coupling is rigorously taken into account; and (4) an efficient computation and interpolation scheme for the radiation impedance coefficients is proposed.

The remainder of this paper is organized as follows. In the next section the basis equations for both the plate motion and the fluid motion are presented. The choice of a suitable set of trial functions for the plate deflection is then discussed, followed by the detailed calculations of the surface acoustic pressure for both the baffled and unbaffled cases. The forced vibration response is discussed and, finally, numerical validations as well as experimental validations are presented.

2. BASIC EQUATIONS

Let us consider an homogeneous thin rectangular plate of dimensions $L_x \times L_y$ and uniform thickness h, excited by a force density $f_{exc}(x, y, t)$. The plate is characterized by a mass density ρ_s, a Young's modulus E_Y and a Poisson ratio ν. Two cases will be considered in this paper, as shown in Figures 1 and 2. In Figure 1 is shown a baffled plate for which the $z < 0$ half-space is filled with a fluid (characterized by its density ρ and the celerity c), while the $z > 0$ half-space is considered to be the vacuum. The unbaffled plate, shown in Figure 2, is immersed entirely in the fluid. In both cases, the plates are considered to be strictly identical. Throughout the paper, the baffled problem will be denoted **BF**, while the unbaffled one will be **UBF**.

2.1. FLUID-LOADED PLATE MOTION

A variational formulation is used for the fluid-loaded plate motion. The Hamilton functional for the present problem is the action integral given by

$$\mathscr{H}(w) = \int_{t_1}^{t_2} [T(\dot{w}) - V(w) + W(w)]\,\mathrm{d}t, \tag{1}$$

where t_1 and t_2 are arbitrary times and $w(x, y, t)$ is the plate deflection.

The term $T(\dot{w})$ represents the kinetic energy of the plate and is given by

$$T(\dot{w}) = \int_{S_p} \tfrac{1}{2}\rho_s(\partial_t w)^2\,\mathrm{d}S. \tag{2}$$

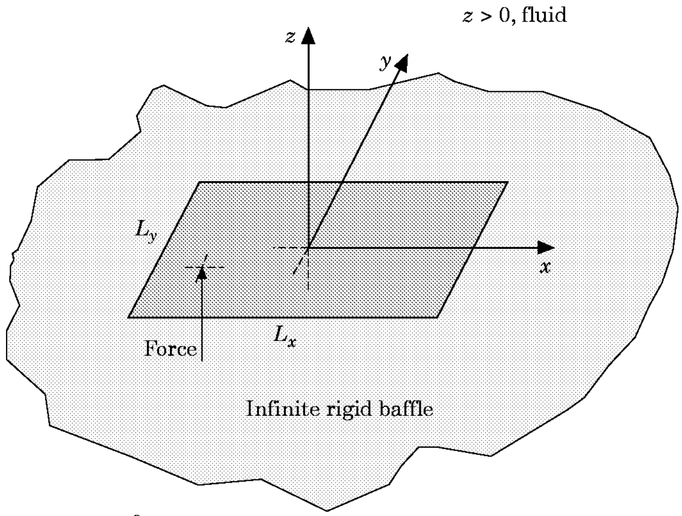

Figure 1. The geometry of a baffled plate.

The second term of equation (1) is the potential energy,

$$V(w) = \int_{S_p} \frac{D}{2} \{(\partial^2_{xx} w)^2 + (\partial^2_{yy} w)^2 + 2\nu \partial^2_{xx} w \partial^2_{yy} w + 2(1-\nu)(\partial^2_{xy} w)^2\} \, \mathrm{d}S, \tag{3}$$

where D is the flexural rigidity, defined by $D = E_Y h^3/12(1-\nu^2)$.

The last term is the work associated with the external forces $f(x, y, t)$:

$$W(w) = \int_{S_p} f(x, y, t) w(x, y, t) \, \mathrm{d}S. \tag{4}$$

The force density $f(x, y, t)$ acting on the plate is the sum of the excitation force density $f_{exc}(x, y, t)$ and the surface acoustic pressure $f^P(x, y, t)$. This surface pressure is induced

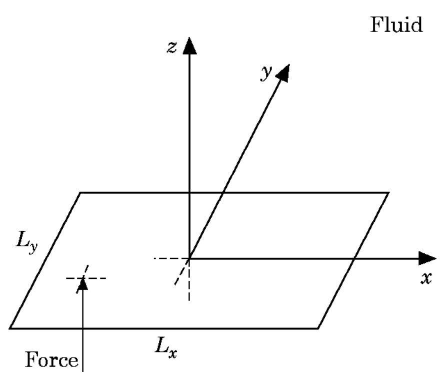

Figure 2. The geometry of an unbaffled plate.

by both the plate motion and the incident acoustic field if such a field is introduced in the analysis.

According to Hamilton's principle, the true plate deflection $w(x, y, t)$ renders the functions $\mathscr{H}(w)$ stationary. The deflection $w(x, y, t)$ that satisfies both the geometric boundary conditions and the plate equation is thus found by calculating the extremum of the functional $\mathscr{H}(w)$. The eigenvalue problem of the *in-vacuo* plate is found by setting up $f(x, y, t) = 0$ in the functional (1). Thus, in this case, the stationary condition leads to the *in-vacuo* eigenvalues and eigenmodes of the plate.

2.2. RAYLEIGH–RITZ SOLUTION

To find the extremum of the functional $\mathscr{H}(w)$ the Rayleigh–Ritz method is used. It consists of finding an approximate solution of $w(x, y, t)$ by expanding it over basis functions satisfying the geometric boundary conditions. Thus, the following linear combination is used:

$$w(x, y, t) = \sum_{mn} w_{mn}(t)\phi_{mn}(x, y). \tag{5}$$

The functions $\phi_{mn}(x, y)$ are linearly independent and depend only on the spatial co-ordinates. The set $\{w_{mn}(t)\}$ forms a vector of unknown coefficients to be determined.

With this expansion and assuming a harmonic state $e^{i\omega t}$ for the time dependence, the stationary condition gives

$$\sum_{pq} (-\omega^2 M_{mnpq} + K_{mnpq}) w_{pq} = \begin{cases} 0, & \text{free vibration,} \\ f_{mn}, & \text{forced vibration,} \end{cases} \tag{6a, b}$$

where the right-hand term in equation (6) is the generalized-force vector and $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices.

Using non-dimensional variables $u = 2x/L_x$ and $v = 2y/L_y$, the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ are given by

$$M_{mnpq} = \rho_s \frac{L_x L_y}{4} \int_{-1}^{1} \int_{-1}^{1} \phi_{mn}(u, v)\phi_{pq}(u, v)\, \mathrm{d}u\, \mathrm{d}v \tag{7}$$

and

$$K_{mnpq} = D\frac{L_x L_y}{4} \int_{-1}^{1} \int_{-1}^{1} \left\{ \left(\frac{2}{L_x}\right)^4 \frac{\partial^2 \phi_{mn}}{\partial u^2} \frac{\partial^2 \phi_{pq}}{\partial u^2} + \left(\frac{2}{L_y}\right)^4 \frac{\partial^2 \phi_{mn}}{\partial v^2} \frac{\partial^2 \phi_{pq}}{\partial v^2} \right.$$

$$\left. + \frac{32}{(L_x L_y)^2} \left[\nu \frac{\partial^2 \phi_{mn}}{\partial u^2} \frac{\partial^2 \phi_{pq}}{\partial v^2} + \nu \frac{\partial^2 \phi_{mn}}{\partial v^2} \frac{\partial^2 \phi_{pq}}{\partial u^2} + (1 - \nu) \frac{\partial^2 \phi_{mn}}{\partial u\, \partial v} \frac{\partial^2 \phi_{pq}}{\partial u\, \partial v} \right] \right\} \mathrm{d}u\, \mathrm{d}v, \tag{8}$$

these matrices being symmetric.

The solution of equation (6a) give the decompositions $\{w_{mn}\}$ over the basis functions $\{\phi_{mn}\}$ of the *in-vacuo* mode shapes and also give the natural frequencies associated with these modes. The resolution of equation (6b) gives the solution of the forced vibrations. In this case, dissipation within the plate can be introduced by the use of a complex bending stiffness $D = D(1 + i\eta_s)$, η_s being the structural damping.

2.3. FLUID EQUATIONS

The pressure field $P(\vec{r})$ must satisfy the following Helmholtz equation and the Sommerfeld radiation condition if an harmonic state $e^{i\omega t}$ is assumed:

$$\nabla_{\vec{r}}^2 P(\vec{r}) + k^2 P(\vec{r}) = \begin{cases} 0, & \text{for } z > 0; \quad \text{baffled plate,} \\ 0, & \forall\, \vec{r} \text{ in } V; \quad \text{unbaffled plate,} \end{cases} \tag{9}$$

$$\lim_{R\to\infty} R\left[\frac{\partial P}{\partial r} + \mathrm{i}kP\right]_{r=R} = 0, \tag{10}$$

where R is the radius of a hemisphere (**BF**) or a sphere (**UBF**) and $k = \omega/c$.

The pressure field must also respect the boundary condition at the interface between the fluid and the plate, which states that the plate and acoustic velocities must be equal. This velocity continuity on S_p reads

$$\mathrm{i}\omega w(x, y) = -\frac{1}{\mathrm{i}\omega\rho}\frac{\partial P}{\partial z}\bigg|_{z=0}. \tag{11}$$

The two main challenges in this study are as follows. (1) It is necessary to find a suitable set of functions $\{\phi_{mn}\}$ that satisfy the geometric boundary conditions. Moreover, the functions ϕ_{mn} must be linearly independent and regular enough in order to ensure that the integrals involving these functions are well-behaved. (2) The surface acoustic pressure induced by the plate motion must be computed. Calculations of this surface pressure lead to the well-known radiation impedance. Computations of the radiation impedance coefficients represent a challenging task, particularly for the unbaffled plate, for which no straightforward calculations and approximations can be performed.

The remainder of the paper is mainly dedicated to the study of a suitable set of $\{\phi_{mn}\}$ and to the computation of the radiation impedance for both the baffled and unbaffled plates.

3. CHOICE OF THE BASIS FUNCTIONS

3.1. TRIGONOMETRIC FUNCTIONS

Our goal is to find only one set of functions that fits arbitrary boundary conditions on the edge of the plate without the use of a contour spring [9, 10, 20], since in this case some ill-conditioned matrices can be obtained. By arbitrary conditions, one means that the same boundary condition prevails along a distinct edge but can differ from one edge to another. The Leissa [20] convention will be used to characterize the different boundary conditions: F (free), C (clamped), S (simply supported) and G (guided).

In a previous work [21] the authors have shown that, for a simply supported plate, the use of sine functions allows one to avoid numerical instabilities for the radiation impedances due to the use of polynomials in a Ritz expansion. In the same spirit, a set of trigonometric functions, originally developed by Beslin [22], has then been used to expand the plate deflection. This allows one to simulate arbitrary conditions on the plate contour without the use of contour springs, and leads to a stable algorithm for the computation of the radiation impedance coefficients. These types of functions were found to be particularly suitable for both the mechanical and the acoustical parts of the problem addressed in this paper.

TABLE 1

Coefficients for the basis function $\phi_m(u)$

m	a_m	b_m	c_m	d_m
1	$\pi/4$	$3\pi/4$	$\pi/4$	$3\pi/4$
2	$\pi/4$	$3\pi/4$	$\pi/2$	$3\pi/2$
3	$\pi/4$	$-3\pi/4$	$\pi/4$	$-3\pi/4$
4	$\pi/4$	$-3\pi/4$	$\pi/2$	$-3\pi/2$
$m \geqslant 5$	$(m-4)\pi/2$	$(m-4)\pi/2$	$\pi/2$	$\pi/2$

The trial function ϕ_{mn} is written as a product of a u-dependent function $\phi_m(u)$ with a v-dependent one, $\phi_n(v)$:

$$\phi_{mn}(u, v) = \phi_m(u)\phi_n(v). \tag{12}$$

The proposed set consists of a product of trigonometric functions of the form

$$\phi_m(u) = \sin(a_m u + b_m)\sin(c_m u + d_m), \tag{13}$$

where the coefficients a_m, b_m, c_m and d_m are listed in Table 1.

The first four functions are used to set the boundary conditions at $u = \pm 1$, since they allow non-zero displacement and slope at the edges, as prescribed by particular boundary conditions. The higher order functions ($m \geqslant 5$) possess both zero displacement and zero slope at the edges. So that they do not affect the displacements along the free edges. The first six basis functions are sketched in Figure 3. Arbitrary conditions such as mixed support conditions can be achieved with few operations using the appropriate functions suitable for the boundary conditions.

With this choice of basis functions, the calculation of M_{mnpq} and K_{mnpq} presents no major difficulties and leads to well-behaved matrices. It has been shown by Beslin *et al*. [22] that these trigonometric functions present greater numerical stability (no ill-conditioning) and faster convergence than polynomials for the mechanical part (mass and stiffness matrices) of the problem; therefore, very high order functions can be used. This suggests that the medium frequency range can be approached using such a set of functions. For the acoustical part, the next sections are devoted to the computation of the surface acoustic pressure for the **BF** and **UBF** case using the above set of functions.

4. CALCULATION OF THE SURFACE ACOUSTIC PRESSURE

One of the major difficulties for a radiation problem is to compute the surface acoustic pressure for both the baffled and unbaffled cases. Knowledge of this surface pressure allows to obtain the near field, a quantity of importance in radiation problems. This surface acoustic pressure acting on the plate is defined by

$$f^p(x, y) = \begin{cases} -P(x, y, 0), & \textbf{BF}, \\ -\overline{P}(x, y) = -\lim\limits_{\varepsilon \to 0} (P(x, y, \varepsilon) - P(x, y, -\varepsilon)), & \textbf{UBF}. \end{cases} \tag{14a, b}$$

The function $\overline{P}(x, y)$ is defined as the "pressure jump" function. In the literature, the pressure jump is more often called the "double-layer density".

To compute the surface acoustic pressure, the integral equation method is used. Using Green's identities, the pressure field $P(\vec{r}_0)$ can be expressed as

$$P(\vec{r}_0) = \begin{cases} -\iint_{S_p} \vec{\nabla}_{\vec{r}} P(\vec{r}) \cdot \vec{n}_p G(\vec{r}_0, \vec{r})\, \mathrm{d}S, & \textbf{BF}, \\ \iint_{S_p} \overline{P}(x, y) \vec{\nabla}_{\vec{r}} G(\vec{r}_0, \vec{r}) \cdot \vec{n}_p\, \mathrm{d}S, & \textbf{UBF}, \end{cases} \tag{15a, b}$$

where $G(\vec{r}_0, \vec{r})$ is the Green function for the semi-infinite free field (**BF**) or the infinite free field (**UBF**). It is given by

$$G(\vec{r}_0, \vec{r}) = \begin{cases} \dfrac{\mathrm{e}^{-\mathrm{i}kR}}{2\pi R}, & \textbf{BF}, \\ \dfrac{\mathrm{e}^{-\mathrm{i}kR}}{4\pi R}, & \textbf{UBF}, \end{cases} \tag{16}$$

with $R = |\vec{r} - \vec{r}_0|$, the distance between the points $\vec{r}_0$ and $\vec{r}$.

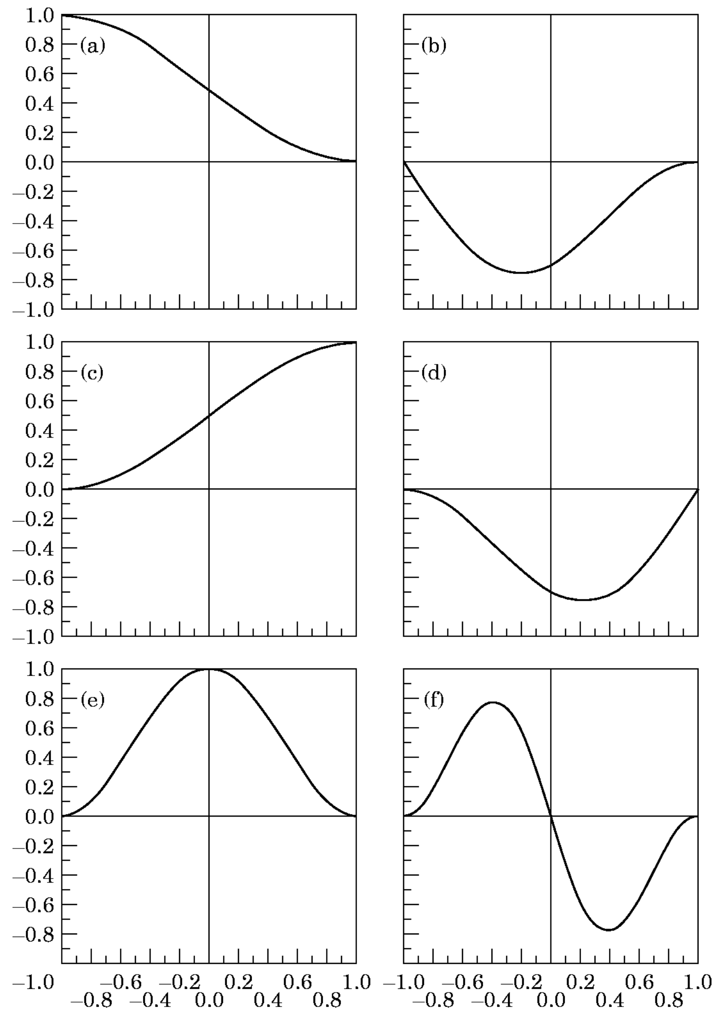

Figure 3. The basis functions $\phi_m(u)$ as a function of u for (a) $m = 1$, (b) $m = 2$, (c) $m = 3$, (d) $m = 4$, (e) $m = 5$ and (f) $m = 6$.

With these relations and the velocities continuity condition (11), the surface acoustic pressure can be computed as a function of the plate deflection. While being direct for **BF**, the computation of $f^P(x, y)$ is quite complicated for the unbaffled plate, since an integro-differential system has to be solved. Moreover, taking the gradient of the Green function lead to strong singularities that have to be overcome. In the next two sections, the calculations of the surface acoustic pressure for, respectively, the baffled and the unbaffled plates, are presented.

4.1. BAFFLED PLATE

As mentioned earlier, the calculation of the surface acoustic pressure is straightforward. It starts with the velocity continuity condition (11), which can be written in a more convenient way:

$$\rho\omega^2 w(x, y) = \left.\frac{\partial P}{\partial z}\right|_{z=0}. \tag{17}$$

Using this equation in equation (15), one obtains

$$P(x, y, z) = -\rho\omega^2 \iint_{S_p} w(x', y') G(x, y, z; x', y', 0)\, \mathrm{d}S', \tag{18}$$

which is often referred as the ''Rayleigh integral''. Particularizing this last one to $z = 0$, one then obtains the surface acoustic pressure $f^P = -P(x, y, 0)$ for the baffled plate.

To compute the forced vibration of the plate, one has to calculate instead the term

$$f^P_{mn} = \iint_{S_p} f^P(x, y)\phi_{mn}(x, y)\, \mathrm{d}S. \tag{19}$$

Using equation (18) and by expanding the deflection $w(x, y)$ over the $\{\phi_{mn}\}$ set, one finally obtains

$$f^P_{mn} = \rho\omega^2 \sum_{pq} \left(\iint_{S_p} \iint_{S_p} \phi_{mn}(x, y) G(x, y, 0; x', y', 0) \phi_{pq}(x', y')\, \mathrm{d}S\, \mathrm{d}S' \right) w_{pq}, \tag{20}$$

which can be rewritten as

$$f^P_{mn} = -\mathrm{i}\omega \sum_{pq} Z^{\mathbf{BF}}_{mnpq} w_{pq}, \tag{21}$$

with

$$Z^{\mathbf{BF}}_{mnpq} = \mathrm{i}\rho\omega \iint_{S_p} \iint_{S_p} \phi_{mn}(x, y) G(x, y, 0; x', y', 0) \phi_{pq}(x', y')\, \mathrm{d}S\, \mathrm{d}S'. \tag{22}$$

These coefficients are the radiation impedance coefficients. The real part of the $Z^{\mathbf{BF}}_{mnpq}$ is the radiation resistance and expresses the radiation damping of the plate, while the imaginary part is the radiation reactance and expresses the added mass of fluid on the plate.

Many authors have calculated the radiation resistance coefficients for simply supported panels [2, 3, 6, 12]. Generally, only the diagonal part of the radiation resistance matrix is considered, neglecting the intermodal coupling. A numerical integration scheme has been proposed by Sandman [23] but, once again, it is limited to the case of simply supported plates. More recently, Berry [9, 10] has proposed an approach based on a Taylor expansion to compute the radiation impedance coefficients. The use of polynomials allows one to deal with arbitrary boundary conditions, but it leads to strong instabilities when the frequency increases, particularly for the radiation impedance computation, which involves many recurrences.

The approach proposed in this paper to compute the radiation impedance coefficients is based, like that of Sandman [23], on a numerical integration scheme. Details of the calculations are given in section 4.3. First, let us derive the expressions for the surface acoustic pressure for the unbaffled plate.

4.2. UNBAFFLED PLATE

Our goal is to calculate the sound pressure jump function defined in equation (14), which is an unknown of the problem. Since the pressure field depends on this pressure jump (see section (14)), the computation of the pressure jump is not straightforward. Moreover, one has to take the gradient of the Green function leading to double-layer density singularities.

A variational approach, based on one initially developed by Hamdi [16], has been used with success by the present authors [21, 24] to treat the fluid-loaded simply supported plate. The main aspects of their calculations are stated here by extending it to the case of the plate with arbitrary boundary conditions.

As in the baffled case, one starts with the velocity continuity condition

$$w(\vec{r}') = \frac{1}{\rho\omega^2}\frac{\partial P}{\partial z}\bigg|_{z=0} = \frac{1}{\rho\omega^2}\nabla_{\vec{r}'} P \cdot \hat{n}_{\vec{r}}, \tag{23}$$

where the point $\vec{r}'$ lies on the surface S_p. The acoustic pressure defined in equation (15b) can then be written

$$w(\vec{r}') = \frac{1}{\rho\omega^2}\left\{\iint_{S_p} \overline{P}(\vec{r})\,\frac{\partial^2 G(\vec{r},\vec{r}')}{\partial n_{\vec{r}}\,\partial n_{\vec{r}'}}\,\mathrm{d}S\right\}, \tag{24}$$

where the kernel of the integral equation (24) is given by

$$\frac{\partial^2 G(\vec{r},\vec{r}')}{\partial n_{\vec{r}}\,\partial n_{\vec{r}'}}. \tag{25}$$

This kernel approaches infinity, as R^{-3} when $R \to 0$, since the Green function behaves like R^{-1}. Thus, it is singular for $\vec{r} = \vec{r}'$. However, physically, the acoustic velocity must remain continuous across the panel surface. To ensure that this condition is well established, the integral equation is replaced by

$$w(\vec{r}') = \frac{1}{\rho\omega^2}\,F.P.\left\{\iint_{S_p} \overline{P}(\vec{r})\,\frac{\partial^2 G(\vec{r},\vec{r}')}{\partial n_{\vec{r}}\,\partial n_{\vec{r}'}}\,\mathrm{d}S\right\}, \tag{26}$$

where $F.P.$ designates the finite part of the integral by the Hadamard mean [16].

To overcome the singularity and to solve equation (26), Hamdi [16] has suggested a variational formulation. He has shown that the solution $\overline{P}(\bar{r})$ of equation (26) must be such that it renders stationary the functional

$$\mathscr{F}(\overline{P}(\vec{r})) = \frac{1}{2}\frac{1}{\rho\omega^2} J(\overline{P}, \overline{P}) + \iint_{S_p} w(\vec{r})\overline{P}(\vec{r})\, \mathrm{d}S, \tag{27}$$

where $J(\overline{P}, \overline{P})$ is the quadratic functional

$$J(\overline{P}, \overline{P}) = -\iiiint_{S_p \times S_p} \left[k^2 \overline{P}(\vec{r})\overline{P}(\vec{r}') + \frac{\partial \overline{P}(\vec{r})}{\partial x}\frac{\partial \overline{P}(\vec{r}')}{\partial x'} + \frac{\partial \overline{P}(\vec{r})}{\partial y}\frac{\partial \overline{P}(\vec{r}')}{\partial y'} \right] G(\vec{r}, \vec{r}')\, \mathrm{d}S'\, \mathrm{d}S. \tag{28}$$

It is to be noted that this functional is only valid for a plane structure and for a vanishing pressure jump on the contour of the surface S_p.

To identify the pressure jump function $\overline{P}(x, y)$, an expansion over a set of admissible functions $\{\psi_{mn}\}$ is used:

$$\overline{P}(x, y) = \sum_{mn} \overline{P}_{mn}\psi_{mn}(x, y), \tag{29}$$

where the $\overline{P}_{mn}$ are unknown coefficients. The choice of the functions $\{\psi_{mn}\}$ will be discussed later.

With this expansion and the one for the plate deflection, equation (5), the stationary condition for the fluid functional $\mathscr{F}(\overline{P})$ reads

$$\frac{\partial \mathscr{F}(\overline{P}(\vec{r}))}{\partial \overline{P}_{mn}} = 0, \qquad \forall (mn), \tag{30}$$

and leads to

$$\frac{1}{\omega^2}\sum_{pq} A_{mnpq}\overline{P}_{pq} + \sum_{ij} S_{mnij} w_{ij} = 0. \tag{31}$$

The matrix $\mathbf{A}$, which behaves like an admittance, is given by

$$\mathbf{A} = -\omega^2 \mathbf{M}^A + \mathbf{K}^A, \tag{32}$$

with

$$M^A_{mnpq} = \frac{1}{\rho c^2}\iiiint_{S_p \times S_p} \psi_{mn}(\vec{r}) G(\vec{r}, \vec{r}')\psi_{pq}(\vec{r})\, \mathrm{d}S\, \mathrm{d}S'. \tag{33}$$

and

$$K^A_{mnpq} = \frac{1}{\rho}\iiiint_{S_p \times S_p} \left\{ \frac{\partial \psi_{mn}(\vec{r})}{\partial x}\frac{\partial \psi_{pq}(\vec{r}')}{\partial x'} + \frac{\partial \psi_{mn}(\vec{r})}{\partial y}\frac{\partial \psi_{pq}(\vec{r}')}{\partial y'} \right\} G(\vec{r}, \vec{r}')\, \mathrm{d}S\, \mathrm{d}S', \tag{34}$$

while $\mathbf{S}$ is a transform from a ψ to a ϕ basis matrix. It is defined by

$$S_{mnpq} = \iint_{S_p} \psi_{mn}(\vec{r})\phi_{pq}(\vec{r})\, \mathrm{d}S. \tag{35}$$

Details of the calculations of the matrix **A** elements, as well as the radiation impedance elements for the baffled plate, are presented in the next section.

Equation (31) now tells us how the sound pressure jump is related to the panel deflection. Thus, the surface acoustic pressure for the unbaffled plate reads

$$f^P(x, y) = -\bar{P}(x, y) = \omega^2 \sum_{ij} \left(\sum_{kl} \sum_{pq} A^{-1}_{ijkl} S_{klpq} w_{pq} \right) \psi_{ij}(x, y). \tag{36}$$

The generalized force coefficients f^P_{mn} associated with the surface acoustic pressure are then given, by using the definition of matrix **S** and denoting the transpose of **S** by $\mathbf{S}^\dagger$,

$$f^P_{mn} = \omega^2 \sum_{pq} \left(\sum_{ij} \sum_{kl} S^\dagger_{mnij} A^{-1}_{ijkl} S_{klpq} \right) w_{pq}, \tag{37}$$

which can be written in the same manner as in the baffled case:

$$f^P_{mn} = -\mathrm{i}\omega \sum_{pq} Z^{\mathbf{UBF}}_{mnpq} w_{pq}, \tag{38}$$

with

$$Z^{\mathbf{UBF}}_{mnpq} = \mathrm{i}\omega \sum_{ij} \sum_{kl} S^\dagger_{mnij} A^{-1}_{ijkl} S_{klpq} \tag{39}$$

or, in matrix form

$$\mathbf{Z}^{\mathbf{UBF}} = \mathrm{i}\omega \mathbf{S}^\dagger \mathbf{A}^{-1} \mathbf{S}. \tag{40}$$

It has to be noted that the radiation impedance coefficients for **BF** are quite similar to the coefficients M^A_{mnpq} appearing in the matrix **A** for **UBF**. In fact, if the same functions are used to expand both the plate deflection and the pressure jump ($\phi \equiv \psi$), the four-fold integrals involved in Z_{mnpq} for **BF** and M^A_{mnpq} for **UBF** are the same. The next section is devoted to the calculation of the radiation impedance **Z** for both the baffled and unbaffled cases.

4.3. RADIATION IMPEDANCE COEFFICIENTS

The main drawback of the above formulation for the radiation impedances is that it requires computation of four-fold integrals. Some formulations [3, 6] use a wavenumber transformation to compute the radiation impedance where only two-fold integrals over wavenumber variables are required. However, this procedure remains a quite difficult task.

A numerical method, based on an approach developed by Sandman [23], is proposed here to overcome the evaluation of four-fold integrals. Using an appropriate mapping of the co-ordinates, the four-fold integral can be converted to a two-fold one which can be evaluated with a Gaussian quadrature scheme. The present calculation is an extension of a previous work [21] in which the case of the simply supported plate has been considered.

One has to compute the coefficients of the radiation impedance $Z^{\mathbf{BF}}_{mnpq}$ and the coefficients of the matrix **A**, M^A_{mnpq} and K^A_{mnpq}. These three sets of coefficients involve integrals of the form

$$I_{mnpq} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} g_m(u)g_n(v)\frac{e^{ikL_xR/2}}{R}h_p(u')h_q(v')\,du\,dv\,du'\,dv', \tag{41}$$

where $g_m(u)$ and $h_m(u)$ are functions that depend on spatial variable u and integer m, and will be detailed later. The variable R is here given by

$$R = \left[(u-u')^2 + \frac{1}{r^2}(v-v')^2\right]^{1/2}, \tag{42}$$

with $r = L_x/L_y$. It is to be noted that the function $h_m(u)$ can be the same as $g_m(u)$, as it is the case for Z^{BF}_{mnpq} and M^A_{mnpq}.

The following mapping is then used.

$$\alpha = u - u', \qquad \alpha' = v - v', \qquad \beta = u', \qquad \beta' = v'$$

and leads, after a few mathematical manipulations, to

$$I_{mnpq} = \int_0^2\int_0^2 [G_{mp}(\alpha) + G_{pm}(\alpha)][H_{nq}(\alpha') + H_{qn}(\alpha')]\mathscr{K}(\alpha,\alpha')\,d\alpha\,d\alpha' \tag{43}$$

with

$$G_{mp}(\alpha) = \int_0^{2-\alpha} g_m(\alpha+\beta-1)g_p(\beta-1)\,d\beta,$$

$$H_{mp}(\alpha) = \int_0^{2-\alpha} h_m(\alpha+\beta-1)h_p(\beta-1)\,d\beta, \tag{44, 45}$$

and

$$\mathscr{K}(\alpha,\alpha') = \frac{e^{-ikL_x/2[\alpha^2+\alpha'^2/r^2]^{1/2}}}{[\alpha^2+\alpha'^2/r^2]^{1/2}}. \tag{46}$$

For the baffled plate, one has (see equation (13))

$$g_m(\alpha) \equiv h_m(\alpha) = \sin(a_m\alpha + b_m)\sin(c_m\alpha + d_m). \tag{47}$$

For the unbaffled plate one has to choose some admissible functions that must vanish on the plate contour to expand the pressure jump function. A natural choice is the normalized sine function, since it allows a non-vanishing slope but a vanishing value on the contour:

$$\psi_{ij}(u,v) = \frac{1}{\sqrt{N_{ij}}}\sin\left(\frac{i\pi}{2}(u+1)\right)\sin\left(\frac{j\pi}{2}(v+1)\right). \tag{48}$$

Using this set of functions, the functions g and h for the matrix $\mathbf{M}^A$ will be

$$g_m(\alpha) \equiv h_m(\alpha) = \sin\left(\frac{m\pi}{2}(\alpha+1)\right), \tag{49}$$

while for the matrix $\mathbf{K}^A$ it will be

$$g_m(\alpha) = \sin\left(\frac{m\pi}{2}(\alpha + 1)\right), \qquad h_m(\alpha) = \cos\left(\frac{m\pi}{2}(\alpha + 1)\right). \tag{50}$$

It is worth noting that the use of trigonometric functions for both the plate deflection and the pressure jump function allows one easily to perform analytically the integrals involved in equations (44) and (45). Moreover, the use of trigonometric functions avoids, contrary to polynomials basis functions, numerical instabilities due to recurrence relations [10]. The present approach leads to the same type of calculation for both the baffled and unbaffled cases so a generic computational routine can be used to compute the impedance coefficients. It offers the advantage to treat both the **BF** and **UBF** on an equal footing so interpretations can be done more easily.

With the above relations, the matrix coefficients Z^{BF}_{mnpq}, M^A_{mnpq} and K^A_{mnpq} finally reads

$$Z^{BF}_{mnpq} = \mathrm{i}\omega\rho\,\frac{L_x L_y^2}{16\pi}\int_0^2\!\!\int_0^2 [G_{mp}(\alpha) + G_{pm}(\alpha)][G_{nq}(\alpha') + G_{qn}(\alpha')]\mathscr{K}(\alpha, \alpha')\,\mathrm{d}\alpha\,\mathrm{d}\alpha', \tag{51}$$

$$M^A_{mnpq} = \frac{L_y}{8\pi\rho c^2}\int_0^2\!\!\int_0^2 [G_{mp}(\alpha) + G_{pm}(\alpha)][G_{nq}(\alpha') + G_{qn}(\alpha')]\mathscr{K}(\alpha, \alpha')\,\mathrm{d}\alpha\,\mathrm{d}\alpha', \tag{52}$$

and

$$K^A_{mnpq} = \frac{\pi L_y}{8\rho L_x^2}\int_0^2\!\!\int_0^2 [mpH_{mp}(\alpha)G_{nq}(\alpha') + r^2 nqG_{mp}(\alpha)H_{nq}(\alpha')]\mathscr{K}(\alpha, \alpha')\,\mathrm{d}\alpha\,\mathrm{d}\alpha'. \tag{53}$$

The two-fold integrals appearing in equations (51), (52) and (53) are computed numerically using a Gaussian quadrature scheme. Care must be taken when the variable $\alpha = \alpha'$, for which $\mathscr{K}(\alpha, \alpha)$ goes to infinity. However, this singularity is integrable, since the surface integral tends towards zero as R^2, while the singularity behaves as R^{-1}. Thus, the singularity can be overcome by a judicious choice of the discretization points in the Gaussian procedure.

It is also to be noted that the radiation coefficients presented in equations (51)–(53) exhibit a smooth behaviour as a function of the frequency, so an interpolation scheme can easily be used to compute these coefficients. Using the fact that the matrices are symmetric, and using some symmetries related to the G_{mp} and H_{mp} functions, a powerful numerical algorithm can then be developed to compute the various radiation impedance coefficients.

5. FORCED DYNAMIC RESPONSE

Knowledge of the radiation impedance coefficients allows one to compute the plate deflection unknowns w_{ij} for the forced vibrations by solving the linear system

$$[-\omega^2 M + K + \mathrm{i}\omega Z]\,[w] = [f_{exc}]. \tag{54}$$

This system can be solved with a standard linear algebra algorithm. Knowledge of the deflection unknowns and the radiation impedance coefficients thus allows one to compute different physical quantities of interest.

If one compares equation (39) for the radiation impedance coefficients of the unbaffled plate with the one for the baffled plate (equation (22)), it becomes clear that the computation of these coefficients is more complicated for the **UBF**. This requires the calculation of two four-fold integrals, a matrix inversion and two matrix multiplications.

For **BF**, one just has to compute one four-fold integral. Then, from a practical point of view, another set of linear equations is used to compute the plate deflection coefficients for the **UBF**. By combining equations (31) and (6), one obtains the plate deflection and the pressure jump function if the following system is solved:

$$\begin{bmatrix} \frac{1}{\omega^2} A & S \\ S & -\omega^2 M + K \end{bmatrix} \begin{bmatrix} \overline{P} \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ f_{exc} \end{bmatrix}. \tag{55}$$

Even if this system is larger than the one given in equation (54), the computation of the inverse of matrix **A** is avoided, and it provides direct access to the pressure jump coefficients $\overline{P}_{rs}$.

6. NUMERICAL AND EXPERIMENTAL VALIDATIONS

A first validation for the mass and stiffness matrices (equations (7) and (8)) has been proposed by Beslin *et al*. [22] for the basis functions given in equation (13). The authors have computed the eigenvalues and eigenvectors for different boundary conditions and compared them to well-known results. It has been verified that the same results prevail here.

To validate the forced vibration response in air, an experimental study has been performed. The set-up used for the validation consists of a free plate with dimensions $L_x = 0{\cdot}48$ m, $L_y = 0{\cdot}42$ m and thickness $h = 0{\cdot}00322$ m. The plate had the following properties: Young's modulus $E_y = 6{\cdot}7 \times 10^{10}$ Pa, density $\rho_s = 2680$ kg/m^3, Poisson ratio $\omega = 0{\cdot}3$ and structural damping $\eta_s = 3 \times 10^{-3}$. The mean squared velocity of the plate has been measured using a laser vibrometer. A total of 361 measurement points have been used to compute the quadratic velocity. The plate was excited by a shaker through a string at the excitation point (0·08 m, 0·07 m). For the present approach, a total of 14 functions in each direction (v and v) has been used to expand the plate deflection. The same number of functions has also been used to expand the pressure jump functions. The comparison between the present theory for the **UBF**, denoted TRM, and the experimental results is shown in Figure 4. The figure shows that exceptionally good agreements is obtained between the two approaches. In accordance with the validation proposed by Beslin *et al*. [22], this strongly suggests that the mass and stiffness matrices of the plate are then well computed.

On the other hand, since the fluid–structure coupling is weak in air, one cannot conclude using the above results that the coupling matrices (radiation impedance coefficients) are well computed. To do so, a numerical validation has been performed using a boundary element method (BEM). Using software developed in-house [25], an unbaffled all-clamped square plate of unitary surface (1 m × 1 m) and thickness $h = 0{\cdot}01$ m vibrating in water has been studied. The plate properties are as follows: Young's modulus $E_y = 2 \times 10^{11}$ Pa, density $\rho_s = 7800$ kg/m^3, Poisson ratio $v = 0{\cdot}3$ and structural damping $\eta_s = 1 \times 10^{-2}$. The BEM used a mesh of (14 × 14) eight-node square elements. Once again, a total of 14 functions in each directions (u and v) has been used to expand the plate deflection and the pressure jump function. The excitation point, a point force, is located at the point (0·7 m, 0·2 m). The radiation efficiency of the plate has been computed using the relation

$$\sigma = \frac{\mathscr{P}_{rad}}{2\rho_f c S \langle v^2 \rangle}, \tag{56}$$

where $\mathscr{P}_{rad}$, the radiated power, and $\langle v^2 \rangle$, the mean squared velocity, are given by

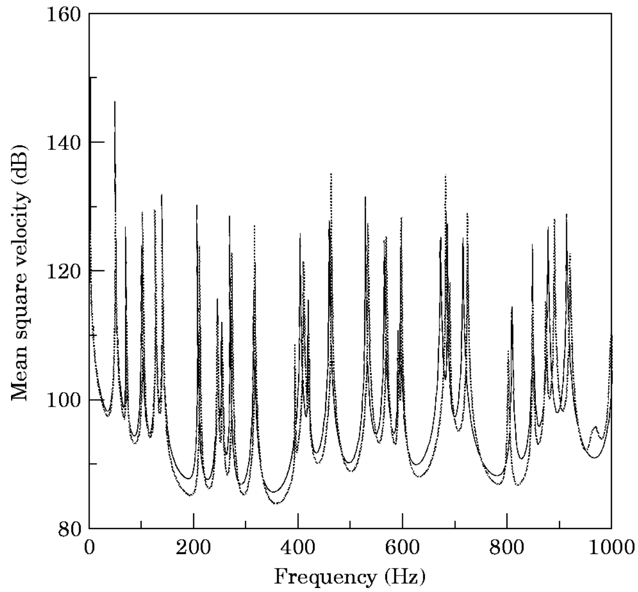

Figure 4. The mean squared velocity for a free plate: a comparison between experimental results (· · · · ·) and the present approach (——).

$$\mathscr{P}_{rad} = -\frac{\mathrm{i}\omega}{2}\,\mathrm{Re}\iint_{S_p} f^p(\vec{r})w^*(\vec{r})\,\mathrm{d}S, \qquad \langle v^2\rangle = \frac{\omega^2}{2S}\iint_{S_p} |w(\vec{r})|^2\,\mathrm{d}S. \qquad (57, 58)$$

A factor of 2 has been put on the denominator of equation (56) to take into account that both sides of the plate radiates in the fluid. For the configurations described above used in TRM and BEM, the TRM reveals to be faster by a factor of ~ 20 comparatively to the BEM simulation. The radiation efficiency for the frequency range 0–500 Hz is shown in Figure 5. The agreement between the two methods is excellent. This comparison allows us principally to validate the computation of the matrices $\mathbf{M}^A$ and $\mathbf{K}^A$ (equations (52) and (53)). Since the same technique is used to compute the radiation impedance coefficients for the baffled case, it is thought that the baffled case is giving correct results too. As an example, in Figure 6 the radiation efficiency is shown once again for the same configuration as above, but now also including the baffled plate compared to the unbaffled one. This figure clearly demonstrates that important differences are obtained between the two when strong fluid–structure coupling prevails. The expected result that a baffled plate radiates more at low frequencies (monopolar radiation versus dipolar one) is well observed. Moreover, one can see that frequency shifts induced by the fluid added mass are not the same whether or not the plate is baffled. In the authors' opinion, these results legitimate the work addressed in this paper.

7. CONCLUSIONS

This paper has been dedicated to the study of the radiation from baffled and/or unbaffled plates with arbitrary conditions. The present paper has focused on the theoretical framework of the formulation. A Rayleigh–Ritz procedure has been used to expand the quantity of interest. A suitable choice of trial functions for the plate deflection has been presented. It allows one simply to generate arbitrary boundary conditions on the plate contour using trigonometric functions. An efficient numerical scheme has been developed

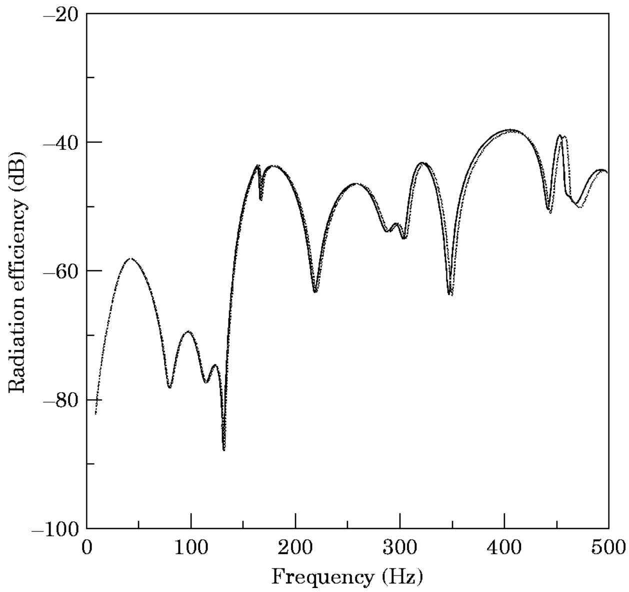

Figure 5. The radiation efficiency for a clamped plate in water: a comparison between the boundary element method (——) and the present approach (· · · · ·).

to compute the radiation impedance coefficients for both the baffled and the unbaffled plates. Good agreement has been found with experimental results and boundary element method predictions.

The proposed formulation presents many interesting novelties:

(1) It provides a direct comparison between the baffled and unbaffled plates, since the two cases are treated on the same basis.

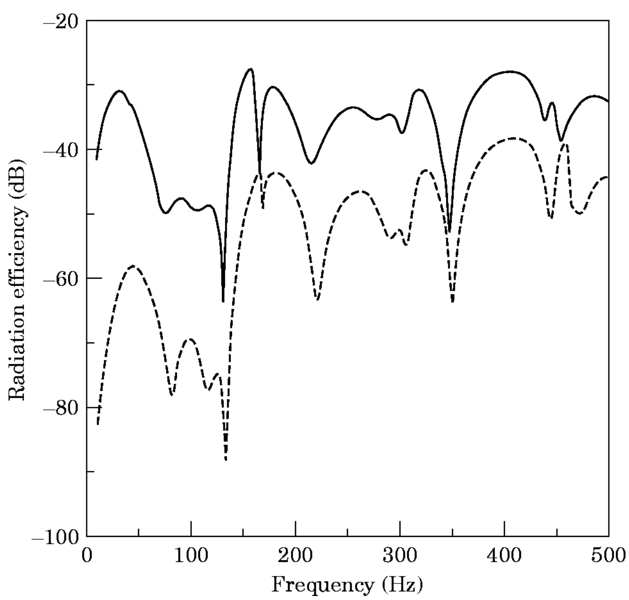

Figure 6. The radiation efficiency for a clamped plate in water: a comparison between the baffled (——) and the unbaffled plate (----).

(2) Only one generic numerical code is needed for all of the boundary conditions.

(3) An efficient numerical scheme has been developed to compute all of the four-fold integrals involved in the radiation impedance coefficients for both the baffled and unbaffled plates. The same scheme is used for the baffled and the unbaffled plate so, once again, only one generic code can be used, thanks to a judicious choice of the trial functions.

(4) The fluid–structure coupling is completely and rigorously taken into account.

(5) The numerical code based on the present formulation was found to be efficient and sufficiently fast to simulate numerous practical applications, and to perform parametric analysis. As an example, it was found that a plate with high modal density can be simulated (medium frequency range).

(6) It provides a useful tool with which to validate other techniques such as the BEM or the FEM, since the present formulation contains all of the main aspects related to the vibroacoustics of a plate.

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