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SMALL AMPLITUDE VIBRATIONS OF AN INTERNALLY CONSTRAINED ELASTIC LAYER

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Small amplitude vibrations, in the form of infinitesimal harmonic waves, in an incompressible elastic layer are considered. When the layer is additionally subject to the constraint of restricted shear and the associated preferred planes are parallel to the free surface, it is shown that the order of the governing equations of motion is reduced. The implication is that traction free boundary conditions at the upper and lower surfaces of the plate cannot be satisfied. Unlike previous studies, involving fibre inextensibility, it seems that no simple physical interpretations is possible. In an attempt to resolve this anomaly and satisfy the boundary conditions, the constraint is relaxed slightly and the strain energy function expanded as a Taylor series. However, it is found that even in this case the dispersion relation has no solutions above a certain wave speed, this usually occurring in the low wavenumber regime. It is verified analytically that no low wavenumber phase speed limit exists. It is postulated that the reason for this rather unusual behaviour is attributable to the rate of shearing. In the high wavenumber regime asymptotic expansions are obtained which give phase speed as a function of wavenumber and harmonic number. These expansions are shown to provide excellent agreement with the numerical solution over a remarkably large wavenumber region.

1. INTRODUCTION

Internal constraints often provide far more mathematically tractable models to help elucidate the properties of complex, sometimes highly anisotropic elastic solids. The two most commonly encountered constraints in solid mechanics are undoubtedly those of incompressibility and inextensibility; these being commonly employed in the modelling of fibre-reinforced composites. In addition to the use of internal constraints in specific material models, considerable research effort has also been focused on the problem of determining the properties of materials subject to an arbitrary function of the deformation gradient being constrained to zero. This motivated the dimensional classification of constraints proposed by Pipkin [1], in which the dimension of a constraint is equal to the number of non-zero eigenvalues of the associated reaction stress. Inextensibility and incompressibility are of dimensions one and three, respectively and exemplify constraints of their specific dimension.

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In comparison with the two classes of constraint already mentioned two-dimensional constraints have received comparatively little attention in the literature. The first discussion of a two-dimensional constraint was seemingly a brief remark by Chen and Gurtin [2] in the context of wave transmission in unbounded media and specifically involved the constraint of restricted shear; this constraint typifying those of dimension two. Several years later the problem of surface waves in an incompressible elastic semi-infinite body with restricted shear was discussed by Whitworth and Chadwick [3]. There then followed two papers [4, 5] in which it was noted that various crystal classes exhibit preferred planes near phase transformations. Within these planes the shear modulus is generally an order of magnitude larger than other material parameters; the constraint of restricted shear therefore being both relevant and offering considerable simplification. A specific example of these crystals are the A-15 superconductors which, near certain critical temperatures, change from cubic to tetragonal symmetry. There then followed several papers in which the problem of waves propagating in unbounded media subject to the restricted shear constraint was discussed. The problem of just one preferred plane was addressed by Rogerson [6] and specific problems involving three preferred planes by Rogerson [7] and Scott [8].

It is with the aforementioned motivations in mind that we embark on the present study, in which we aim to examine the effects of restricted shear on dispersive wave propagation. In order to have dispersive waves, a natural length scale is required, and so we examine the simplest problem leading to dispersive waves; namely, that of a layer of finite thickness and infinite lateral extent, the surfaces of which are taken to be traction free. We assume that this layer is subject to both the incompressibility constraint and that of restricted shear.

In section 2 the basic governing equations and boundary conditions are given, while in section 3 we briefly look at the propagation properties in an unbounded medium. Section 4 deals with the infinite layer problem in which the embedded fibre directions, between which the shear is restricted, are orthogonal. When the restricted shear constraint is applied exactly, we show that, in general, no Lamb-type waves can be propagated in any given direction. So-called exceptional waves (see reference [3]) can, however, propagate but only along the direction bisecting the embedded fibre directions. The specific reason that no Lamb-type waves exist is that all six of the traction free boundary conditions cannot be satisfied. This is a direct consequence of the constraint of restricted shear, and in particular the fact that it acts to reduce the order of the governing equations from six to four. A similar phenomenon occurs in materials reinforced by inextensible fibres, in which case a discontinuity in the component of traction along the fibre direction is usually postulated. Such a postulate is physically well motivated and indicated by a rapid variation in stress in the neighbourhood of the fibre direction for highly anisotropic materials (see reference [9]).

In the case of restricted shear, no simple physical interpretation akin to that discussed in the previous paragraph seems possible. Accordingly, in section 5 the restricted shear constraint is relaxed in the sense that while shearing is now possible between the embedded fibre directions, the associated shear modulus is much greater than in the other planes. Such nearly constrained materials have been discussed in reference [10]. With this relaxation, it is now possible for Lamb waves to propagate within the layer and the dispersion relations for both symmetric and antisymmetric modes are derived. These dispersion relations are then studied numerically in section 6, where it is shown that even the relaxed restricted shear constraint introduced in section 5 acts to inhibit the propagation of waves for large enough wave speeds. Asymptotics are then carried out in the short and long wavelength limits, and their predictions are plotted against the purely numerical solutions and are shown to provide excellent agreement in a remarkably large wavenumber region.

In the low wavenumber regime it is shown that even the fundamental mode does not have a finite limiting wave speed as $kh \to 0$. This reaffirms our earlier remarks concerning the non-existence of solutions of the dispersion relation in the low wavenumber regime. It is conjectured that the reason for this phenomenon is the increase in the rate of shearing associated with the harmonics. This is further elucidated in the cases in which the angle of propagation intersects the embedded fibre directions or the effects of restriction in shear are totally removed. In either of these two cases, the effects of restriction in shear, either in the exact or nearly constrained sense, are inactive and all harmonics have a phase speed which tends to infinity as $hk \to 0$. Numerical solutions of these two special cases are shown for the purposes of comparison.

2. BASIC EQUATIONS

Consider an incompressible elastic body which is subject to the additional constraint of restricted shear. A body subject to the latter constraint may only be deformed in such a way that the angle between two embedded fibre directions remains constant. It should be emphasized that the fibres themselves are extensible, and it is only the angle between the tangents of each material curve that is invariable. Suppose that in an initial unstressed state $\mathscr{B}_u$ the directions of those line elements are defined by the unit vectors **A** and **B**. The same line elements, in a subsequent homogeneously stressed equilibrium state $\mathscr{B}_e$, are denoted by **a** and **b**, which are related to **A** and **B** through the component relations

$$a_i = F_{ip} A_p, \qquad b_i = F_{ip} B_p, \tag{2.1}$$

within which **F** is the deformation gradient associated with the homogeneous deformation $\mathscr{B}_u \to \mathscr{B}_e$. If the unit vectors along **a** and **b** are denoted by $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, an appropriate constraint function associated with the restriction in shear is

$$\lambda(\mathbf{F}) \equiv \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} - \mathbf{A} \cdot \mathbf{B} = 0. \tag{2.2}$$

The constraint function associated with incompressibility is expressed in the usual form; namely,

$$\lambda(\mathbf{F}) = (J - 1) = 0, \qquad J = \det \mathbf{F}. \tag{2.3}$$

For problems involving wave propagation in constrained elastic solids it is usual to introduce a pseudo strain-energy function. In the case of an incompressible elastic solid with restricted shear, this function is expressible in the form

$$W(\mathbf{F}) = W_0(\mathbf{F}) + p(J - 1) + s(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} - \mathbf{A} \cdot \mathbf{B}). \tag{2.4}$$

In equation (2.4), $W_0(\mathbf{F})$ is that part of the strain-energy function that generates the constitutive stress, while the other two linear terms each generate a workless reaction stress associated with each constraint. Equation (2.4) may now be used to generate both the constitutive and reaction stresses using the Cauchy stress relation

$$\sigma_{ij} = F_{ip} \frac{\partial W}{\partial F_{jp}}, \tag{2.5}$$

see, for example [11]. If (2.4) is inserted into (2.5) it is observed that

$$\sigma_{ij} = \sigma^0_{ij} + N^S_{ij} + N^I_{ij}, \tag{2.6}$$

within which σ^0_{ij} is the constitutive stress, and the superscripts S and I refer to the reaction stresses associated with restricted shear and incompressibility, respectively. These latter two stress components are given explicitly by the component forms

$$N^S_{ij} = s[a_i b_j + b_i a_j - (\mathbf{a}\cdot\mathbf{b})(\hat{a}_i\hat{a}_j + \hat{b}_i\hat{b}_j)], \qquad N^I_{ij} = -p\delta_{ij}, \tag{2.7}$$

where δ_{ij} is the Kronecker delta. In equation (2.7), s and p are interpreted as a shear stress, in the plane of $\mathbf{a}$ and $\mathbf{b}$, and a hydrostatic pressure, respectively.

In order to greatly simplify the governing equations, two assumptions will be made. First, the vectors $\mathbf{A}$ and $\mathbf{B}$ are assumed to be orthogonal and, second, the equilibrium state $\mathscr{B}_e$ is assumed to be isotropic and stress-free. Motions will now be considered which are universally small, in the sense that the displacement gradient is small. In this linearized approximation the constraint functions (2.2) and (2.3) reduce to

$$a_i(u_{i,j} + u_{j,i})b_j = 0, \qquad u_{i,i} = 0, \tag{2.8}$$

where $\mathbf{u}$ is the small amplitude displacement, a comma indicates differentiation with respect to the position $\mathbf{x}$, and the two vectors $\mathbf{a}$ and $\mathbf{b}$ are, to this order of approximation, indistinguishable from $\mathbf{A}$ and $\mathbf{B}$. For further details concerning the linearization of constraint functions, see reference [11].

The linear stress–strain relationship may now be written in a form first employed in reference [12]; namely,

$$\sigma_{ij} = -p\delta_{ij} + s(a_i b_j + b_i a_j) + \mu(u_{i,j} + u_{j,i}), \tag{2.9}$$

where the constant μ is a shear modulus. Equations (2.8) and (2.9), in conjunction with the equations of motion

$$\sigma_{ij,j} = \rho\ddot{u}_i, \tag{2.10}$$

where ρ is the material density and the superimposed dot denotes differentiation with respect to time, govern the displacement $\mathbf{u}$, the pressure p and the shear stress s, which are all regarded as functions of position $\mathbf{x}$ and time t. In equation (2.10) and throughout this paper, the summation convention will be taken to apply.

3. AN UNBOUNDED ELASTIC SOLID

Suppose that an unbounded, incompressible elastic solid with restricted shear is subject to a small amplitude sinusoidal vibration $\mathbf{u}$ in the form of the plane wave

$$\mathbf{u} = \mathbf{e}\exp(\mathrm{i}k(\mathbf{x}\cdot\mathbf{n} - vt)), \tag{3.1}$$

where $\mathbf{e}$ denotes the constant wave polarization vector, k is the wavenumber, $\mathbf{n}$ is the wave normal and v is the phase speed. The arbitrary pressure and shear stress appearing in equation (2.9) are assumed to take the similar forms

$$p = \mathrm{i}kP\exp(\mathrm{i}k(\mathbf{x}\cdot\mathbf{n} - vt)), \qquad s = \mathrm{i}kS\exp(\mathrm{i}k(\mathbf{x}\cdot\mathbf{n} - vt)). \tag{3.2}$$

If equations (3.1) and (3.2) are substituted into the equation of motion (2.10), we obtain the propagation condition

$$(\rho v^2 - \mu)\mathbf{e} + P\mathbf{n} - S((\mathbf{a}\cdot\mathbf{n})\mathbf{b} + (\mathbf{b}\cdot\mathbf{n})\mathbf{a}) = \mathbf{0}. \tag{3.3}$$

Solutions of equation (3.3) must be sought that satisfy the appropriate forms of the two constraint conditions (2.8); namely,

$$(\mathbf{a}\cdot\mathbf{e})(\mathbf{b}\cdot\mathbf{n}) + (\mathbf{b}\cdot\mathbf{e})(\mathbf{a}\cdot\mathbf{n}) = 0, \qquad \mathbf{e}\cdot\mathbf{n} = 0, \tag{3.4}$$

where use has been made of equation (3.1). In general, the only solution of (3.3) that satisfies the constraints (3.4) is

$$\rho v^2 = \mu, \qquad \mathbf{e} = \zeta\{(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\times\mathbf{n}) + (\mathbf{b}\cdot\mathbf{n})(\mathbf{a}\times\mathbf{n})\}, \tag{3.5}$$

where ζ is a constant chosen to ensure that $\mathbf{e}$ is a unit vector. It is usual that the shear modulus μ is positive and so we conclude that, in general, one homogeneous plane wave propagates with speed $v = \sqrt{\mu/\rho}$ and polarization vector given by equation $(3.5)_2$.

There are, however, two exceptions to the above general conclusions. The first concerns the case of waves propagating normal to the plane containing $\mathbf{a}$ and $\mathbf{b}$; i.e., when $\mathbf{a}\cdot\mathbf{n} = \mathbf{b}\cdot\mathbf{n} = 0$. It is then possible to cast the propagation condition (3.3) into the simpler form

$$(\rho v^2 - \mu)\mathbf{e} + P\mathbf{n} = \mathbf{0}. \tag{3.6}$$

In this case equation $(3.4)_1$ is satisfied without it placing any restrictions on $\mathbf{e}$, the only restriction on $\mathbf{e}$ now being imposed by $(3.4)_2$. This now means that for this direction of propagation, two homogeneous shear waves can propagate with speed $v = \sqrt{\mu/\rho}$ and with polarization vectors satisfying $(3.4)_2$. These conclusions are exactly those associated with wave propagation in an incompressible elastic solid, with the restricted shear constraint being rendered inactive in this exceptional direction.

The second exceptional case arises when the wave normal is in the plane of $\mathbf{a}$ and $\mathbf{b}$ and such that $\mathbf{a}\cdot\mathbf{n} = \mathbf{b}\cdot\mathbf{n}$; i.e., the direction in the $(\mathbf{a}, \mathbf{b})$ plane that bisects the embedded fibre directions. Again, in this direction the material response, regarding plane wave propagation, is that of an incompressible elastic solid. However, this time both the constraints (3.4) are equivalent in this direction and so two homogeneous plane waves can propagate with phase speed $v = \sqrt{\mu/\rho}$ and polarization vector satisfying either of equations (3.4).

4. AN INFINITE ELASTIC LAYER

The propagation of waves in an infinite, incompressible, isotropic elastic layer of width $2h$ within which shear is also restricted is now considered. We take the Cartesian co-ordinate system $Ox_1x_2x_3$ with origin in the mid-plane of the layer, Ox_1 normal to the plane of the layer, and Ox_2 and Ox_3 coincident with the two orthogonal embedded fibre directions $\mathbf{a}$ and $\mathbf{b}$. The equations of motion (2.9) are now to be solved in conjunction with traction free boundary conditions on the upper and lower surfaces of the layer; namely,

$$\sigma_{1i} = 0, \qquad i = 1, 2, 3 \text{ at } x_1 = \pm h. \tag{4.1}$$

Solutions of the equations of motion are now sought which represent waves propagating in the plane of the layer with phase speed v in a direction at angle γ to Ox_3. The associated displacement field may then be represented in the form

$$u_1 = U(x_1)\cos\phi, \qquad u_2 = V(x_1)\sin\phi, \qquad u_3 = W(x_1)\sin\phi, \tag{4.2}$$

within which

$$\phi = k(x_2\sin\gamma + x_3\cos\gamma - vt), \tag{4.3}$$

where k is the wavenumber. The arbitrary pressure and shear stress associated with the internal constraints also have the assumed forms

$$p = P(x_1)\cos\phi, \qquad s = S(x_1)\cos\phi. \tag{4.4}$$

Equations (4.2)–(4.4) may now be used together with the stress–displacement relation (2.9) to obtain the stress components

$$\sigma_{11} = (-P + 2\mu U')\cos\phi, \qquad \sigma_{22} = (-P + 2\mu ksV)\cos\phi,$$
$$\sigma_{33} = (-P + 2\mu kcW)\cos\phi, \qquad \sigma_{12} = \mu(-ksU + V')\sin\phi,$$
$$\sigma_{13} = \mu(-kcU + W')\sin\phi, \qquad \sigma_{23} = (\mu k(cV + sW) + S)\cos\phi, \tag{4.5}$$

in which $s = \sin\gamma$, $c = \cos\gamma$ and a prime denotes differentiation with respect to x_1.

If the stress components shown in equation (4.5) are inserted into the equation of motion (2.10), the three component equations are given by

$$-P' + 2\mu U'' + k^2(\rho v^2 - \mu)U + \mu ksV' + \mu kcW' = 0,$$
$$ksP - kcS + \mu V'' + k^2(\rho v^2 - 2\mu s^2)V - \mu ksU' = 0,$$
$$kcP - ksS + \mu W'' + k^2(\rho v^2 - 2\mu c^2)W - \mu kcU' = 0, \tag{4.6}$$

where use has been made of the incompressibility and restricted shear constraints (2.8), which now have the forms

$$U' + ksV + kcW = 0, \qquad cV + sW = 0. \tag{4.7}$$

Solutions of equations (4.6) and constraints (4.7) are now sought in the form

$$(U, V, W, P, S) = (U_0, V_0, W_0, kP_0, kS_0)\,\mathrm{e}^{(kqx_1)}, \tag{4.8}$$

within which all quantities with a subscript zero are constants. If equation (4.8) is substituted into (4.6) and (4.7) a set of five linear homogeneous equations is obtained; namely,

$$(\mu q^2 + \rho v^2 - \mu)U_0 - qP_0 = 0,$$
$$-\mu qsU_0 + (\mu q^2 + \rho v^2 - 2\mu s^2)V_0 + sP_0 - cS_0 = 0,$$
$$-\mu qcU_0 + (\mu q^2 + \rho v^2 - 2\mu c^2)W_0 + cP_0 - sS_0 = 0,$$
$$qU_0 + sV_0 + cW_0 = 0, \qquad cV_0 + sW_0 = 0. \tag{4.9}$$

A non-trivial solution of this system of equations exists provided that the determinant of the coefficients vanishes. This yields the following four possible solutions for q

$$q_1^2 = \frac{\mu - \rho v^2}{\mu} = 1 - \Omega^2, \qquad q_2^2 = (c^2 - s^2)^2 = \cos^2 2\gamma, \tag{4.10}$$

where $\Omega^2 = \rho v^2/\mu$.

The solutions of equations (4.9) may now be written as a linear summation of the four, in general, independent solutions arising from equation (4.10). Thus

$$(U, V, W, P, S) = \sum_{i=1}^{4} \{1, \bar{F}(q_i), \bar{H}(q_i), k\bar{M}(q_i), k\bar{N}(q_i)\}\, U_0^{(i)}\,\mathrm{e}^{(kq_i x_1)}, \tag{4.11}$$

where

$$\bar{F}(q) = \frac{qs}{(c^2 - s^2)}, \qquad \bar{H}(q) = \frac{-qc}{(c^2 - s^2)},$$
$$\bar{M}(q) = \frac{\mu q^2 + \rho v^2 - \mu}{q}, \qquad \bar{N}(q) = 2sc\bar{M}(q), \tag{4.12}$$

and the q's are ordered so that $q_{i+2} = -q_i$ $(i = 1, 2)$. The traction free boundary conditions (4.1) may now be satisfied at the upper surface provided that

$$\sum_{i=1}^{4} \bar{\alpha}_i U_0^{(i)} \, \mathrm{e}^{khq_i} = \sum_{i=1}^{4} \bar{\beta}_i U_0^{(i)} \, \mathrm{e}^{khq_i} = \sum_{i=1}^{4} \bar{\gamma}_i U_0^{(i)} \, \mathrm{e}^{khq_i} = 0, \tag{4.13}$$

and at the lower surface provided that

$$\sum_{i=1}^{4} \bar{\alpha}_i U_0^{(i)} \, \mathrm{e}^{-khq_i} = \sum_{i=1}^{4} \bar{\beta}_i U_0^{(i)} \, \mathrm{e}^{-khq_i} = \sum_{i=1}^{4} \bar{\gamma}_i U_0^{(i)} \, \mathrm{e}^{-khq_i} = 0, \tag{4.14}$$

where $\bar{\alpha}_i$, $\bar{\beta}_i$ and $\bar{\gamma}_i$ are defined through the relations

$$\bar{\alpha}_i = 2\mu q_i - \bar{M}(q_i), \qquad \bar{\beta}_i = q_i \bar{F}(q_i) - s, \qquad \bar{\gamma}_i = q_i \bar{H}(q_i) - c. \tag{4.15}$$

After a little algebraic manipulation, it is possible to represent the six homogeneous equations shown in equations (4.13) and (4.14) as two sets of three homogeneous linear equations in $(U_0^{(i)} - U_0^{(i+2)})$ and $(U_0^{(i)} + U_0^{(i+2)})$ $(i = 1, 2)$, yielding

$$\sum_{i=1}^{2} \bar{\alpha}_i (U_0^{(i)} - U_0^{(i+2)}) \cosh(khq_i) = \sum_{i=1}^{2} \bar{\beta}_i (U_0^{(i)} - U_0^{(i+2)}) \sinh(khq_i)$$
$$= \sum_{i=1}^{2} \bar{\gamma}_i (U_0^{(i)} - U_0^{(i+2)}) \sinh(khq_i) = 0, \tag{4.16}$$

and

$$\sum_{i=1}^{2} \bar{\alpha}_i (U_0^{(i)} + U_0^{(i+2)}) \sinh(khq_i) = \sum_{i=1}^{2} \bar{\beta}_i (U_0^{(i)} + U_0^{(i+2)}) \cosh(khq_i)$$
$$= \sum_{i=1}^{2} \bar{\gamma}_i (U_0^{(i)} + U_0^{(i+2)}) \cosh(khq_i) = 0, \tag{4.17}$$

where use has been made of the fact that $\bar{M}(q)$, $\bar{F}(q)$ and $\bar{H}(q)$ are odd functions. In passing we note that the system (4.16) relates to antisymmetric motions, or bending waves (for which U is an even function of x_1), while equation (4.17) relates to extensional or symmetric waves (where U is an odd function of x_1).

In view of the fact that each of the two sets of equations (4.16) and (4.17) is a system of three equations in only two unknowns, it is in general impossible to satisfy all the boundary conditions. This situation arises directly as a consequence of the imposition of the constraint of restricted shear and will be resolved in the next section by relaxing this constraint slightly. Such a situation has previously been observed for similar problems involving the constraint of inextensibility (see, e.g., references [9] and [13]).

Although it has clearly been established that all the boundary conditions (4.1) cannot in general be satisfied, a special case occurs when $s = c$; i.e., when the direction of wave propagation bisects the two embedded fibre directions **a** and **b**. In this case, the solutions

(4.11) are clearly not valid and we therefore return to equations (4.9), which can now be written in the slightly simpler forms

$$(\mu q^2 + \rho v^2 - \mu)U_0 - qP_0 = 0,$$

$$-\mu qsU_0 + (\mu q^2 + \rho v^2 - \mu)V_0 + s(P_0 - S_0) = 0,$$

$$-\mu qsU_0 + (\mu q^2 + \rho v^2 - \mu)W_0 + s(P_0 - S_0) = 0,$$

$$qU_0 + s(V_0 + W_0) = 0, \qquad s(V_0 + W_0) = 0. \tag{4.18}$$

The final two equations of this set may now be used to deduce that $qU_0 = 0$, and then equations $(4.18)_{2,3,5}$ are invoked to conclude that $P_0 = S_0$ and $V_0 + W_0 = 0$.

In the case $q = 0$, $U_0 \neq 0$ the equations (4.18) correspond to shear waves propagating with a wave normal of the form $\mathbf{n} = (0, \pm 1/\sqrt{2}, \pm 1/\sqrt{2})$ and having a *constant* speed $v = \sqrt{\mu/\rho}$. Furthermore, such waves will always leave the traction unchanged on *any* plane $x_1 =$ constant. Such body waves are usually referred to as exceptional (see reference [3]) and have fundamental importance in the theory of elastic surface wave propagation and body was reflections at traction free boundaries. It is of interest to note that these directions of propagation associated with exceptional waves coincide with the exceptional directions associated with the doubly constrained wave propagation problem in unbounded media. In conclusion, we note that in the case $U_0 = 0$, $q \neq 0$ the traction free boundary conditions afford only the trivial solution.

5. RELAXATION OF THE RESTRICTED SHEAR CONSTRAINT

In this section, the mathematical idealization of restricted shear is removed and a material with a high shear modulus in the plane of the two embedded fibre directions is considered. This is done in an attempt to try to satisfy the boundary conditions, which in the previous section were shown to be incompatible with the exact constraint. The isotropic constraint of incompressibility is still included in our analysis as it does not affect the satisfying of the boundary conditions and also simplifies the problem. Following the approach developed by Rogerson and Scott [10] for nearly constrained materials, the pseudo strain-energy functions (2.4) is replaced by

$$W(\mathbf{F}) = W_0(\mathbf{F}) + p(J - 1) + \tfrac{1}{2}G(\mathbf{a} \cdot \mathbf{b})^2, \tag{5.1}$$

where G is the large shear modulus in the plane of the two embedded fibre directions.

At this point we recall that the two embedded fibre directions are taken to be orthogonal in $\mathscr{B}_u$, and in using the form (5.1) it is tacitly assumed that in all subsequent material deformations $(\mathbf{a} \cdot \mathbf{b})$ is small. The limit to the exact constraint of restricted shear may then be formalized through the requirements

$$G \to \infty, \qquad (\mathbf{a} \cdot \mathbf{b}) \to 0, \qquad G(\mathbf{a} \cdot \mathbf{b}) \to s, \tag{5.2}$$

where s is interpreted as the shear stress associated with the exact constraint, see equation $(2.7)_1$.

In order to derive the appropriate linearized stress–displacement relations, it is first noted that the first two terms of equation (5.1) are exactly those of equation (2.4). It is then only the counterpart of the second term of equation (2.9) which need be derived. To this end, we first obtain the elasticity tensor associated with the third term of equation (5.1); namely,

$$B^*_{ijkl} = F_{ip}F_{kq}\frac{\partial^2}{\partial F_{jp}\partial F_{lq}}\{\tfrac{1}{2}G(\mathbf{a}\cdot\mathbf{b})^2\}|_{\mathbf{F}=\mathbf{I}}$$
$$= G(\delta_{i2}\delta_{j3}+\delta_{i3}\delta_{j2})(\delta_{k2}\delta_{l3}+\delta_{k3}\delta_{l2}), \quad (5.3)$$

where, as previously, the embedded fibre directions are taken to be along Ox_2 and Ox_3. Equations (5.3) may now be employed to yield the stress–displacement relations

$$\sigma_{ij} = \mu(u_{i,j}+u_{j,i}) + p\delta_{ij} + G(u_{3,2}+u_{2,3})(\delta_{i2}\delta_{j3}+\delta_{i3}\delta_{j2}). \quad (5.4)$$

Equations (4.2), (4.3) and $(4.4)_1$ may now be used to yield exactly the same stress components as equation (4.5), except for the shear stress in the preferred plane, which now takes the form

$$\sigma_{23} = k(G+\mu)(cV+sW)\cos\phi. \quad (5.5)$$

The three components of the governing equations of motion may now, with the use of the appropriate subset of equation (4.8), be written in the form

$$(2\mu q^2+\rho v^2-\mu)U_0 + \mu sqV_0 + \mu cqW_0 - qP_0 = 0,$$
$$(\mu q^2+\rho v^2-\mu-c^2G)V_0 - scGW_0 + sP_0 = 0,$$
$$-scGV_0 + (\mu q^2+\rho v^2-\mu-s^2G)W_0 + cP_0 = 0, \quad (5.6)$$

which must be solved together with the incompressibility condition $(4.9)_4$.

A non-trivial solution of the equations of motion (5.6), subject to equation $(4.9)_4$, exists provided that

$$\mu^2q^6 - \mu(G+3\mu-2\rho v^2)q^4 - ((\rho v^2-\mu)(3\mu+G-\rho v^2) - G\mu(s^2-c^2)^2)q^2$$
$$+ (\rho v^2-\mu)((s^2-c^2)^2G-\rho v^2+\mu) = 0, \quad (5.7)$$

which can be recast into the non-dimensionalized form

$$q^6 - (\bar{G}+3-2\bar{\Omega}^2)q^4 - ((\bar{\Omega}^2-1)(\bar{G}+3-\bar{\Omega}^2) - \bar{G}\cos^2 2\gamma)q^2$$
$$+ (\bar{\Omega}^2-1)(\bar{G}\cos^2 2\gamma + 1 - \bar{\Omega}^2) = 0, \quad (5.8)$$

within which $\bar{G} = G/\mu$ and $\bar{\Omega}^2 = \rho v^2/\mu$. Equation (5.8) has six roots which are given by

$$q_1^2 = 1-\bar{\Omega}^2, \qquad q_2^2 = \tfrac{1}{2}(\bar{G}+2-\bar{\Omega}^2+\sqrt{\Delta}) \qquad \text{and} \qquad q_3^2 = \tfrac{1}{2}(\bar{G}+2-\bar{\Omega}^2-\sqrt{\Delta}), \quad (5.9)$$

where

$$\Delta = (\bar{\Omega}^2-\bar{G})^2 + 4\bar{G}\sin^2 2\gamma. \quad (5.10)$$

It is easily verified that, on taking the limit $G\to\infty$ in equation (5.7), the resulting quadratic in q^2 is exactly that which yielded the two solutions (4.10) associated with the corresponding doubly constrained problem. At this point, it is also worth noting that $\gamma = \pi/4$ is a special case, which is discussed later.

Before continuing with the derivation, we briefly comment on how this case relates to the purely incompressible case; that is, when the restriction on the shear is completely removed by setting $G = 0$. It can be simply seen from equation (5.9) and (5.10) that we now have $q_2^2 = 1$ with $q_1^2 = q_3^2 = 1-\bar{\Omega}^2$, that is we have only two distinct roots with one of the roots now repeated. This multiplicity means that when the system (5.6) is solved with $(4.9)_4$ we obtain the following quadratic in q^2:

$$\mu q^4 + (\rho v^2-2\mu)q^2 + (\mu-\rho v^2) = 0, \quad (5.11)$$

which, when substituted into the boundary conditions, gives rise to the classical Rayleigh–Lamb equations for the incompressible layer (see equations (1.1) of reference [14]. This reduction from a cubic to a quadratic in q^2 when G is set equal to zero, results from the decoupling of the V and W components when all restriction on the shearing is removed.

We now return to solving the general case resulting from equation (5.8). The solutions of the governing equations may now be written as a linear combination of the six independent solutions of equation (5.8), which are expressible in the form

$$(U, V, W, P) = \sum_{i=1}^{6} \{1, F(q_i), H(q_i), kM(q_i)\} U_0^{(i)} \mathrm{e}^{(kq_i x_1)}, \tag{5.12}$$

within which

$$F(q) = \frac{c^2 - s^2 - q^2}{2sq}, \qquad H(q) = -\frac{c^2 - s^2 + q^2}{2cq}, \qquad M(q) = \frac{\mu q^2 + \rho v^2 - \mu}{q}, \tag{5.13}$$

and the q_i's have again been ordered so that $q_{i+3} = -q_i$ $(i = 1, 2, 3)$. The solutions (5.12) may now be used in the appropriate stress components of equation (4.5), to deduce that the upper surface of the elastic layer is traction free provided that

$$\sum_{i=1}^{6} \alpha_i U_0^{(i)} \mathrm{e}^{khq_i} = \sum_{i=1}^{6} \beta_i U_0^{(i)} \mathrm{e}^{khq_i} = \sum_{i=1}^{6} \gamma_i U_0^{(i)} \mathrm{e}^{khq_i} = 0, \tag{5.14}$$

with the corresponding lower surface conditions yielding

$$\sum_{i=1}^{6} \alpha_i U_0^{(i)} \mathrm{e}^{-khq_i} = \sum_{i=1}^{6} \beta_i U_0^{(i)} \mathrm{e}^{-khq_i} = \sum_{i=1}^{6} \gamma_i U_0^{(i)} \mathrm{e}^{-khq_i} = 0, \tag{5.15}$$

with

$$\alpha_i = 2\mu q_i - M(q_i), \qquad \beta_i = q_i F(q_i) - s, \qquad \gamma_i = q_i H(q_i) - c. \tag{5.16}$$

As in the previous section, the boundary conditions (5.14) and (5.15) can be manipulated into the more convenient forms

$$\sum_{i=1}^{3} \alpha_i (U_0^{(i)} - U_0^{(i+3)}) \cosh(khq_i) = \sum_{i=1}^{3} \beta_i (U_0^{(i)} - U_0^{(i+3)}) \sinh(khq_i)$$
$$= \sum_{i=1}^{3} \gamma_i (U_0^{(i)} - U_0^{(i+3)}) \sinh(khq_i) = 0, \tag{5.17}$$

and

$$\sum_{i=1}^{3} \alpha_i (U_0^{(i)} + U_0^{(i+3)}) \sinh(khq_i) = \sum_{i=1}^{3} \beta_i (U_0^{(i)} + U_0^{(i+3)}) \cosh(khq_i)$$
$$= \sum_{i=1}^{3} \gamma_i (U_0^{(i)} + U_0^{(i+3)}) \cosh(khq_i) = 0, \tag{5.18}$$

where use was made of the fact that $F(q)$, $H(q)$ and $M(q)$ are all odd functions of their argument. A non-trivial solution is now observed to occur if either of the determinants of the coefficients of equation (5.17) or equation (5.18) vanish; that is, either

$$\sum_{i,m,n=1}^{3} \varepsilon_{imn}\alpha_i\beta_m\gamma_n \tanh(khq_m)\tanh(khq_n) = 0, \tag{5.19}$$

or

$$\sum_{i,m,n=1}^{3} \varepsilon_{imn}\alpha_i\beta_m\gamma_n \tanh(khq_i) = 0, \tag{5.20}$$

where ε_{imn} is the usual alternating tensor.

On substituting from equations (5.13) and (5.16) into the determinants (5.19) and (5.20), we obtain the simpler forms

$$\begin{aligned}(2q_1)(q_2^2 - q_3^2)\tanh(khq_2)\tanh(khq_3) \\ -(q_2^2 + q_1^2)(q_1^2 - q_3^2)\frac{\tanh(khq_1)\tanh(khq_3)}{q_2} \\ +(q_3^2 + q_1^2)(q_1^2 - q_2^2)\frac{\tanh(khq_1)\tanh(khq_2)}{q_3} = 0\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}(2q_1)(q_2^2 - q_3^2)\tanh(khq_1) - (q_2^2 + q_1^2)(q_1^2 - q_3^2)\frac{\tanh(khq_2)}{q_2} \\ +(q_3^2 + q_1^2)(q_1^2 - q_2^2)\frac{\tanh(khq_3)}{q_3} = 0\end{aligned} \tag{5.22}$$

respectively, where the q_i are the positive roots of the expressions (5.9) and use has been made of the fact that $q_1^2 = 1 - \bar{\Omega}^2$.

It is worth noting at this point that both of q_1^2 and q_3^2, as defined by equation (5.9) with equation (5.10), can become negative for certain values of the parameters. When this happens the dispersion relations (5.21) and (5.22) can be written in terms of the variable $\bar{q}_i$ given by $q_i = i\bar{q}_i$, which causes the hyperbolic terms to turn into trigonometric ones. As was noted earlier, the above is not valid when $\gamma = \pi/4$ and so now consider this as a special case. When $\gamma = \pi/4$ the system (5.6) with $(4.9)_4$ reduces to

$$\begin{aligned}(2\mu q^2 + \rho v^2 - \mu)U_0 + \frac{\mu}{2}(V_0 + W_0) - qP_0 = 0, \\ \left(\mu q^2 + \rho v^2 - \mu - \frac{G}{2}\right)V_0 - \frac{G}{2}W_0 + sP_0 = 0, \\ -\frac{G}{2}V_0 + \left(\mu q^2 + \rho v^2 - \mu - \frac{G}{2}\right)W_0 + sP_0 = 0, \\ qU_0 + s(V_0 + W_0) = 0,\end{aligned} \tag{5.23}$$

where $s = \sin(\pi/4) = 1/\sqrt{2}$. From $(5.23)_2$ and $(5.23)_3$, we obtain that either $V_0 = W_0$ or $\mu q^2 + \rho v^2 - \mu = 0$. When $\mu q^2 + \rho v^2 - \mu = 0$ it is easy to show that for waves to propagate we must have $q = 0$, which implies that $v = \sqrt{\mu/\rho}$ and correspond to the exceptional shear wave that was observed in the exactly constrained problem.

However, on setting $V_0 = W_0$, we require that

$$q^4 + q^2(\bar{\Omega}^2 - 2 - \bar{G}) - (\bar{\Omega}^2 - 1) = 0, \tag{5.24}$$

which is a quadratic in q^2, the roots of which we label as $q_3 = -q_1$ and $q_4 = -q_2$. The solution can then be written in a form similar to equation (5.12), as

$$(U, V, W, P) = \sum_{i=1}^{4} \{1, F(q_i), F(q_i), kM(q_i)\} U_0^{(i)} \mathrm{e}^{(kq_i x_1)}, \tag{5.25}$$

where

$$F(q_i) = -\frac{q_i}{\sqrt{2}}, \qquad M(q_i) = q_i(\mu q_i^2 + \rho v^2 - \mu - G). \tag{5.26}$$

The deformation (5.25) can then be substituted into the traction free boundary conditions as above to yield the relations

$$\frac{\tanh(\eta q_1)}{\tanh(\eta q_2)} = \left[\frac{q_1(q_1^2 + \bar{\Omega}^2 - 3 - \bar{G})(q_2^2 + 1)}{q_2(q_2^2 + \bar{\Omega}^2 - 3 - \bar{G})(q_1^2 + 1)}\right]^{\pm 1}, \quad \eta = kh, \tag{5.27}$$

where the positive sign corresponds to symmetric modes and the negative sign to antisymmetric modes. We note in passing that on setting $\bar{G} = 0$ in (5.24) yields the two roots $q_1^2 = 1$ and $q_2^2 = 1 - \bar{\Omega}^2$ which, when substituted into equation (5.27), yields the standard incompressible Rayleigh–Lamb equations (see equations (1.1) in reference [14]); namely,

$$\frac{\tanh(\eta)}{\tanh(\eta\sqrt{1 - \bar{\Omega}^2})} = \left[\frac{4\sqrt{1 - \bar{\Omega}^2}}{(2 - \bar{\Omega}^2)^2}\right]^{\pm 1}. \tag{5.28}$$

6. ANALYSIS OF THE RELAXED CONSTRAINT EQUATIONS

While a numerical approach must be used to produce graphs of the dispersion relations (5.21) and (5.22), it is possible to obtain some asymptotic results that can be used to study the short and long wavelength limits. We now consider both these approaches in turn.

6.1. NUMERICAL RESULTS

In this section, we look at the numerical solutions of the equations (5.21) and (5.22) when $\gamma \neq \pi/4$ and of equation (5.27) when $\gamma = \pi/4$. In each case, we plot the dispersion curves; that is, we plot $\bar{\Omega}$ against $\eta = kh$ for given values of $\bar{G}$ and γ. In the regions in which q_1 and q_3 are purely imaginary, that is, for large values of $\bar{\Omega}$ when $\gamma \neq \pi/4$, the associated hyperbolic terms in equations (5.21) and (5.22) become trigonometric, and so possess a hierarchy of zeros and poles. This complex structure frustrates attempts to study the solutions in this region using either a commercial package such as MAPLE or by means of a simple bisection type method. The approach adopted was to use an algebraic curve following code to trace out the solution branches. By evaluating the left sides of either

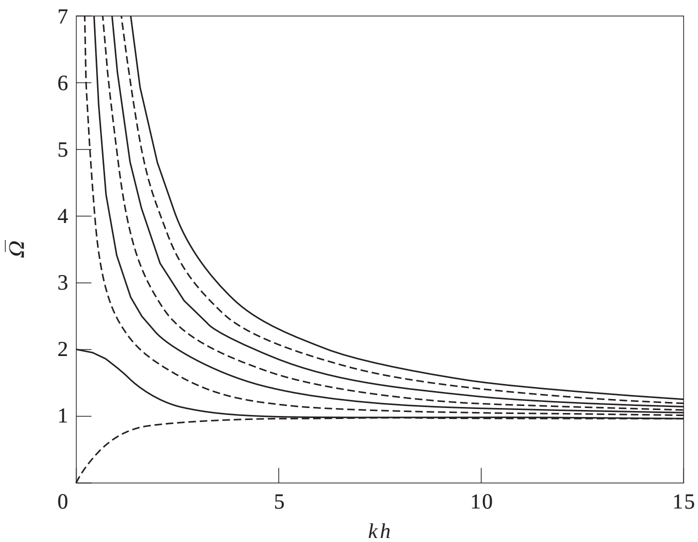

Figure 1. Dispersion curves for the classical Rayleigh–Lamb equations. ——, Symmetric modes; ----, antisymmetric modes.

equation (5.21) or equation (5.22) along a line $\overline{\Omega}$ = constant, it was possible to verify that the curves so obtained correspond to zeros and not poles. When $\gamma = \pi/4$ the situation is much simpler and a bisection method approach was adopted to find the solutions of equation (5.27).

In order to facilitate the study on the effect of the restricted shear constraint, in Figure 1 we reproduce the classical result for the incompressible Rayleigh–Lamb equations when no restriction on the shear is made. This is the same as in reference [14] and can be obtained from equation (5.27) on setting $G = 0$. In all the figures in this section, the dashed lines will correspond to antisymmetric (flexural) modes and the solid lines to symmetric (barrelling) modes.

In Figure 1 it is clear that there are two distinct types of mode. The lowest solution branches of both the symmetric and antisymmetric modes have a finite limit as $kh \to 0$, and are called the fundamental modes, while all other branches tend to infinity as kh becomes small, and are called the harmonics.

In Figure 2, we look at the numerical solutions to equations (5.21) and (5.22) where the shear modulus $G = 10$ and the direction of propagation γ are taken to be 0 and $\pi/6$ respectively. The nature of these figures is typical and the value of $G = 10$ was chosen for demonstrative purposes; as the value of G increases, the behaviour of the solution branches remains the same, but they are packed more closely together.

It is clear that the dispersion curves displayed in Figure 2 are qualitatively different from those shown in Figure 1. Perhaps the most striking difference is in the pairing off of a symmetric and an antisymmetric mode. This is a result of a loss of the monotonicity of the individual solution branches. While it is still possible to make a distinction between fundamental modes and the harmonics, they both behave somewhat differently. The behaviour for large values of kh allows us to differentiate between these modes as the fundamental modes approach a limit $\overline{\Omega} \to \Omega_c < 1$ as $kh \to \infty$, while all of the harmonic modes have $\overline{\Omega} \to 1^+$ as kh becomes large. This latter observation will be used in the subsequent section to obtain an asymptotic approximation for the curves in the large kh region. It can also be observed for the fundamental modes that the limiting value $\Omega_c \to 1^-$ as $\gamma \to \pi/4$.

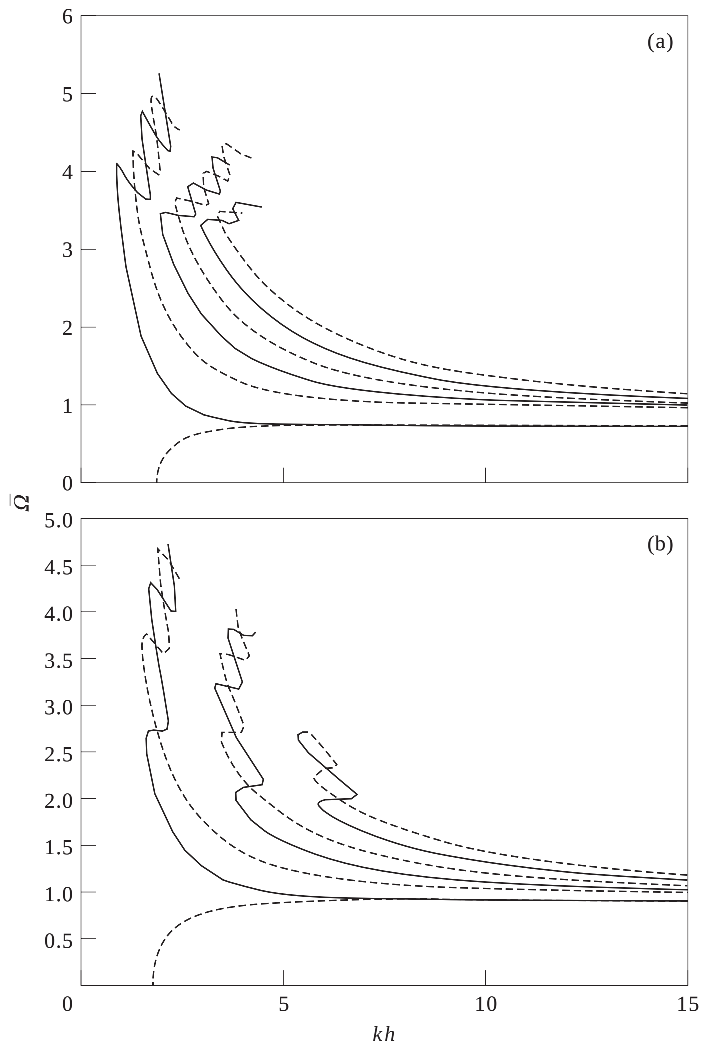

Figure 2. Dispersion curves of equations (5.21) and (5.22), when $G = 10$ in both cases (a) $\gamma = 0$ and (b) $\gamma = \pi/6$. ——, Symmetric modes; ----, antisymmetric modes.

One point which is not immediately obvious from Figure 2 is that, given any value of kh, there is an upper bound on the value of $\bar{\Omega}$ above which no dispersion curves exist for either symmetric or antisymmetric modes. This is in direct contrast to the unrestricted case shown in Figure 1. The most probable reason for this unusual behaviour is that the quantity $\bar{\Omega}$ is directly related to the wave speed which, since we are dealing with the time harmonic case, will be related to the rate of shearing associated with the wave motion. As the wave speed, and hence $\bar{\Omega}$, is increased, so will the rate of shearing associated with the wave. The inclusion of the restricted shear constraint introduced in section 5, even in its relaxed form, would then act to preclude the propagation of waves with infinite wave speeds, which would require an infinite rate of shearing. We note in passing that the behaviour of the dispersion curve branches in the low wavenumber region should not be confused with the osculation previously seen in plots of *frequency* against wavenumber (see, e.g., Green and Baylis [15]). For the corresponding incompressible elastic plate the phase speed associated with each harmonic is a monotonically decreasing function of the wavenumber, as shown in Figure 1.

For the special case in which $\gamma = \pi/4$, then we have $V_0 = W_0$ and the wave motion induces no shearing between the embedded fibre directions, and so we expect the restricted shearing, in either the exact or relaxed form, to act differently in this direction. This distinction in behaviour is displayed in Figure 3, where we plot the dispersion curves for $G = 10$ along $\gamma = \pi/4$. In this case, the solution branches behave very similarly to the classical case shown in Figure 1. It is again possible to distinguish the fundamental mode and the harmonics, where the fundamental modes have finite limits as $kh \to 0$ and harmonic modes infinite limits. When kh is large, then all the modes then approach the limit $\bar{\Omega} = 1$. We note at this point that the fundamental modes intercept the $\bar{\Omega}$-axis at $\bar{\Omega} = 0$ for the antisymmetric mode and $\bar{\Omega} \approx 3{\cdot}74 \approx \sqrt{14} = \sqrt{G+4}$ for the symmetric mode. Both of the results will be confirmed asymptotically in the following section.

6.2. ASYMPTOTIC RESULTS

6.2.1. *Short wavelength limit*

Here we consider the short wavelength limit by considering the limit as $\eta = kh \to \infty$. Provided that $\gamma \neq \pi/4$, then there are two possible asymptotic regimes dealing with either the fundamental modes or the harmonic modes. From the numerical studies of the preceding section, it is clear that in the short wavelength limit the fundamental modes of both the symmetric and antisymmetric modes approach the same limit lying in the interval $0 < \bar{\Omega} < 1$ for which each of the q_i $(i = 1, 2, 3)$ are real and finite. This limit represents the Rayleigh surface wave speed for the material. Likewise, the numerical evidence shows that all of the symmetric and antisymmetric harmonic modes approach the limit $\bar{\Omega} = 1$ from above as $\eta \to \infty$. As $\bar{\Omega} \to 1^+$ both q_2 and q_3 will be real and finite; however, q_1 will be purely imaginary with modulus that tends to zero as $\eta \to \infty$ and $\bar{\Omega} \to 1$. Thus we write $q_1 = \mathrm{i}q$, where q is real and positive and $q \to 0$ as $\eta \to \infty$.

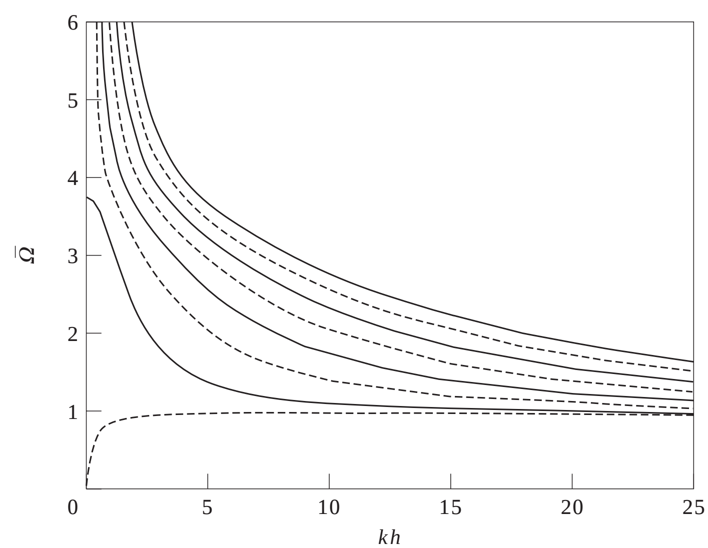

Figure 3. Dispersion curves for equations (5.27) along $\gamma = \pi/4$ when $G = 10$. ——, Symmetric modes; ----, antisymmetric modes.

We consider first the fundamental mode of both types of wave. As was noted above, here, all the q_i ($i = 1, 2, 3$) are real and finite, so that each of the terms $\tanh(\eta q_i) \to 1$ as $\eta \to \infty$. Using this result, it is clear that both (5.21) and (5.22) yield the same limiting form

$$2q_1(q_2^2 - q_3^2) - \frac{(q_2^2 + q_1^2)(q_1^2 - q_3^2)}{q_2} + \frac{(q_3^2 + q_1^2)(q_1^2 - q_2^2)}{q_3} = 0, \tag{6.1}$$

with the q_i ($i = 1, 2, 3$) given by equation (5.9). This can in fact be shown to be the Rayleigh surface wave speed equation, corresponding to surface waves propagating parallel to the surface of a half-space composed of linear incompressible elastic material subject to restricted shear. To see this, we consider such a half-space occupying the region $x_1 \leq 0$. In this case, the solutions (5.12) are replaced by

$$(U, V, W, P) = \sum_{i=1}^{3} \{1, F(q_i), H(q_i), kM(q_i)\} U_0^{(i)} \mathrm{e}^{(kq_i x_1)}, \tag{6.2}$$

in which $F(q)$, $H(q)$ and $M(q)$ are given by equation (5.13) and q_i, $i = (1, 2, 3)$, are the solutions of equation (5.8) chosen with positive real part to ensure decay of stress and displacement as $x_1 \to -\infty$. The condition that the traction vanishes on $x_1 = 0$ yields the three homogeneous equations

$$\sum_{i=1}^{3} \alpha_i U_0^{(i)} = \sum_{i=1}^{3} \beta_i U_0^{(i)} = \sum_{i=1}^{3} \gamma_i U_0^{(i)} = 0. \tag{6.3}$$

The three homogeneous equations (6.3) will yield a non-trivial solution provided that

$$\alpha_1(\beta_2\gamma_3 - \beta_3\gamma_2) - \alpha_2(\beta_1\gamma_3 - \beta_3\gamma_1) + \alpha_3(\beta_1\gamma_2 - \beta_2\gamma_1) = 0. \tag{6.4}$$

Equation (6.4) is the equation for the speed of propagation of Rayleigh surface waves and after a little algebraic manipulation and use of equation (5.16) is readily shown to be identical to equation (6.1). This is then the dispersion relation for the Rayleigh surface wave on this type of material.

For the harmonics, we must make substitution $q_1 = \mathrm{i}q$ outlined above. Since q_2 and q_3 are again real and finite, the $\tanh(\eta q_i)$ terms behave as for the fundamental modes; however, $\tanh(\eta q_1) = \mathrm{i}\tan(\eta q)$ on making the above substitution. This complicates matters, as the periodicity of the tan together with the fact that $q \to 0$ means that the asymptotic behaviour is no longer clear as $\eta \to \infty$. This means that the symmetric and antisymmetric asymptotics must be considered separately.

We first consider the symmetric harmonic modes governed by equation (5.21). On inspection of equation (5.21), it is immediately obvious that for solutions to exist

$$\tan(\eta q) = O(q), \tag{6.5}$$

which in turn suggests that we write

$$\eta q = n\pi + \phi_1/\eta, \tag{6.6}$$

which, from equation $(5.9)_1$, implies that

$$\bar{\Omega}^2 = 1 + \left(\frac{n\pi}{\eta}\right)^2 + \frac{2n\pi\phi_1}{\eta^3} + O(\eta^{-4}) \tag{6.7}$$

and

$$\tan(\eta q) = \frac{\phi_1}{\eta} - \frac{\phi_1^3}{3\eta^3} + O(\eta^{-4}). \tag{6.8}$$

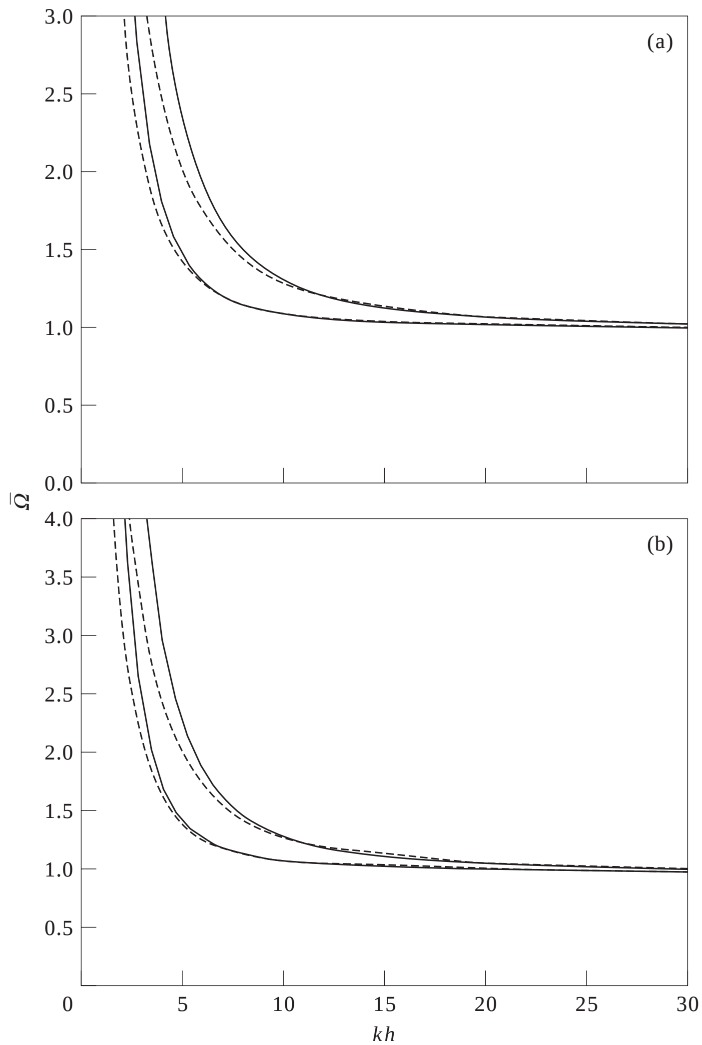

Figure 4. A comparison of the asymptotic results (——) with the numerical results (----) for symmetric modes along $\gamma = 0$, with (a) $G = 10$ and (b) $G = 100$.

On substituting from equation (6.7) into equations $(5.9)_2$ and $(5.9)_3$, expressions can be found for q_2^2 and q_3^2 respectively. These expressions can then be substituted into equation (5.21), together with equations (6.6) and (6.8), to yield the $O(\eta^{-1})$ term

$$\frac{2An\pi}{\eta} + \frac{1}{2\sqrt{2}} (\sqrt{(G+1)^2 - A^2} (\sqrt{G+1+A} - \sqrt{G+1-A})) \frac{\phi_1}{\eta} = 0, \qquad (6.9)$$

where $A = \sqrt{(G-1)^2 + 4G \sin^2(2\gamma)}$. Equation (6.9) can be rearranged to yield

$$\phi_1 = \frac{4\sqrt{2}An\pi}{\sqrt{(G+1)^2 - A^2}(\sqrt{G+1+A} - \sqrt{G+1-A})}, \qquad (6.10)$$

which can then be substituted directly into equation (6.7) to give the expression for $\bar{\Omega}$.

A similar procedure can be carried out looking for the next term in the expansion, by writing

$$\eta q = n\pi + \frac{\phi_1}{\eta} + \frac{\psi_1}{\eta^2}, \tag{6.11}$$

and on inspecting the $O(\eta^{-2})$ terms in the resulting expression for equation (5.21) it can be seen that

$$\psi_1 = \frac{4\sqrt{2}A\phi_1}{\sqrt{(G+1)^2 - A^2}(\sqrt{G+1+A} - \sqrt{G+1-A})}. \tag{6.12}$$

In Figure 4 is shown a comparison between the numerical solution of equation (5.21) and the expansion (6.11) with (6.10) and (6.12), in which only the first two harmonics are displayed for clarity.

A similar procedure can also be carried out for the antisymmetric modes described by equation (5.22). On letting $\eta \to \infty$ in equation (5.22) we can see that we must have

$$\tan(\eta q) = O(q^{-1}), \tag{6.13}$$

for solutions to arise. This suggests that we write

$$\eta q = (n + \tfrac{1}{2})\pi + \frac{\phi_2}{\eta}, \tag{6.14}$$

in the first instance. From equation (6.14), we obtain that

$$\tan(\eta q) = -\frac{\eta}{\phi_2} + \frac{\phi_2}{3\eta} + O(\eta^{-3}), \tag{6.15}$$

which can be substituted, together with the relevant expressions for q_2^2 and q_3^2, into equation (5.22). The leading order, $O(1)$, term then yields the following expression for ϕ_2:

$$\phi_2 = \frac{4\sqrt{2}A(n+\frac{1}{2})\pi}{\sqrt{(G+1)^2 - A^2}(\sqrt{G+1+A} - \sqrt{G+1-A})}, \tag{6.16}$$

where $A = \sqrt{(G-1)^2 + 4G\sin^2(2\gamma)}$ as above.

Furthermore, on writing

$$\eta q = (n + \tfrac{1}{2})\pi + \frac{\phi_2}{\eta} + \frac{\psi_2}{\eta^2}, \tag{6.17}$$

and examining the $O(\eta^{-1})$ terms in the resulting expression for equation (5.22), we find that

$$\psi_2 = \frac{1}{(2n+1)\pi}. \tag{6.18}$$

In Figure 5 is displayed the comparison between the numerical solution and the asymptotic solution (6.17) of the equation (5.22) governing the antisymmetric modes. Here only the first two harmonic modes are shown.

6.2.2. *Long wavelength limit*

As can be seen from the numerical results, for the case in which the shear is restricted, even in the relaxed form discussed here, the dispersion curves do not meet the $\bar{\Omega}$-axis; that is, we have no solutions in the long wavelength limit. There is, of course, the special case in which $\gamma = \pi/4$, in which the restricted shear constraint does not act, as we will see shortly.

For the general case in which $\gamma \neq \pi/4$, we consider the limits of equations (5.21) and (5.22) as $\eta = kh \to 0$. Looking first at equation (5.21) for symmetric modes, then on taking $\eta = kh$ to be small, the lowest order term will be $O(\eta^2)$, and takes the form

$$q_1^2 q_2^2 (q_1^2 - q_2^2) + q_2^2 q_3^2 (q_2^2 - q_3^2) + q_1^2 q_3^2 (q_3^2 - q_1^2) = 0, \tag{6.19}$$

irrespective of the signs of q_1^2 and q_3^2, which we recall can be negative. Turning now to equation (5.22) and the antisymmetric modes, we see that the lowest order term $O(\eta)$ is identically zero. We thus consider the first non-zero term, in $O(\eta^3)$, which turns out to have the identical form as equation (6.19) for symmetric modes. We note that equation (6.19) is obtained irrespective of the signs of q_1^2 and q_3^2, which can be negative.

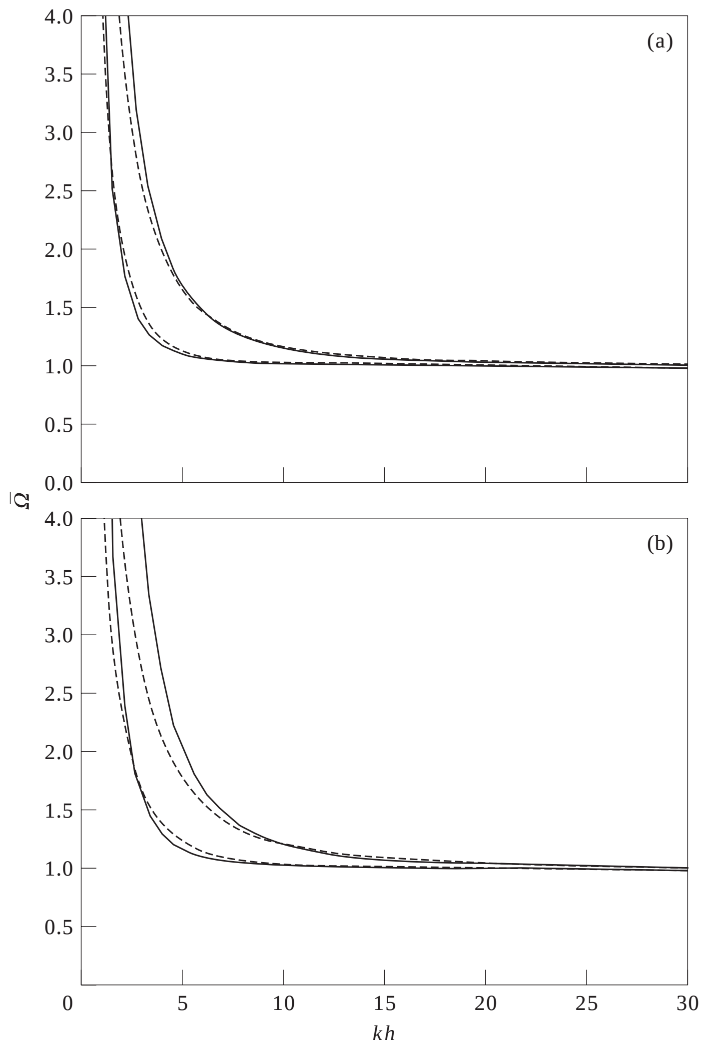

Figure 5. A comparison of the asymptotic results (——) with the numerical results (----) for antisymmetric modes with $G = 100$, along (a) $\gamma = 0$ and (b) $\gamma = \pi/6$.

Equations (5.9) and (5.10) can then be substituted into equation (6.19) to yield the relation

$$G\sqrt{(\bar{\Omega}^2 - G)^2 + 4G\sin^2(2\gamma)}[\bar{\Omega}^2 - \sin^2(2\gamma)] = 0. \tag{6.20}$$

Since we are looking for solutions for which G and $\bar{\Omega}^2$ are positive, then we see that there are only two possible solutions; one in which $\gamma = 0$ and $\bar{\Omega}^2 = G$ and the other in which $\bar{\Omega}^2 = \sin^2(2\gamma)$. However, both of these solutions turn out to be degenerate cases of the full dispersion relations, and do not correspond to long wavelength asymptotics solutions. This can be seen by first substituting $\gamma = 0$ and $\bar{\Omega}^2 = G$ into equations (5.9) and (5.10), which yields $q_2^2 = q_3^2 = 1$ and $q_1^2 = 1 - G$. These values then identically satisfy both equation (5.21) and equation (5.22). It is possible to solve the system formed by equations (5.6) and $(4.9)_4$ for this special case and then substitute into the boundary conditions to yield the appropriate conditions for the existence of solutions. However, this approach only yields a discrete solution set for η and G corresponding to the points on the graphs where the solution curves of equations (5.21) and (5.22) cross the line $\bar{\Omega} = \sqrt{G}$. The second of the special cases arises when $\bar{\Omega}^2 = \sin^2(2\gamma)$, and this time substituting into equations (5.9) and (5.10) yields $q_1^2 = q_3^2 = \cos^2(2\gamma)$ and $q_2^2 = 1 + G$. Again, these values identically satisfy both equation (5.21) and equation (5.22), and correspond to a transitional state of the full system, when two of the solutions of equation (5.9) are equal, and not to any long wavelength asymptotic solution.

Since these two cases were the only solutions to equation (6.19), this equation then indicates that no long wavelength asymptotics are possible, which is in agreement with the numerical results presented in the previous section, where no solution branches were found, to intercept the $\eta = 0$ axis.

For the case in which $\gamma = \pi/4$, things are more straightforward. On taking $\eta = kh$ to be small and looking at the symmetric form of equation (5.27), it is easy to show that the $O(\eta)$ term yields the condition

$$\bar{\Omega} = \sqrt{G + 4}, \tag{6.21}$$

for symmetric modes in the long wavelength limit. Similarly, the $O(\eta)$ term for the antisymmetric form of (5.27) yields

$$\bar{\Omega} = 0, \tag{6.22}$$

as $\eta \to 0$. Both equation (6.21) and equation (6.22) agree with the numerical results depicted in Figure 3 of the previous section.

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