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LETTERS TO THE EDITOR

FREE VIBRATIONS OF UNIFORM TIMOSHENKO BEAMS WITH LUMPED ATTACHMENTS

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In a recent paper [1], a novel approach was presented to analyze the free vibration of a combined dynamical system which consists of a continuous structure onto which various lumped elements are attached. Using the commonly assumed-modes method [2] with N component modes, the free vibration of such a combined system corresponds to the solution of an $N \times N$ generalized eigenvalue problem, whose stiffness and mass matrices consist of diagonal matrices modified by the sum of rank one matrices. Algebraically manipulating this generalized eigenvalue problem, the free vibration natural frequencies can be calculated instead by solving a reduced characteristic determinant of size $R \times R$, where R corresponds to the number of constraints of lumped elements, resulting in substantial computational savings.

In a recent published work [3], Posiadała analyzed the free vibrations of uniform Timoshenko beams with lumped attachments (see Figure 1 of reference [3]). He derived the frequency equation governing free response for the combined system by means of the Lagrange multiplier approach. While the final results are concise, the scheme he used to derive the frequency equation is quite complicated, because R Lagrange multipliers and R constraint equations need to be introduced. In this note, the authors intend to show that the same results can be obtained by using the approach outlined in reference [1].

Consider the free vibration of a Timoshenko beam with the lumped attachments as shown in Figure 1, which consists of linear translational springs at x_1 and x_6, linear rotational springs at x_3 and x_7, a concentrated mass at x_2, an element with rotary inertia at x_4, and a linear undamped spring–mass oscillator at x_5. The total kinetic energy of the system is

$$T = \tfrac{1}{2}\int_0^L \rho A(x)\dot{w}^2(x,t)\,\mathrm{d}x + \tfrac{1}{2}\int_0^L \rho I(x)\dot{\psi}^2(x,t)\,\mathrm{d}x$$

$$+\tfrac{1}{2}m\dot{w}^2(x_2,t) + \tfrac{1}{2}J\dot{\psi}^2(x_4,t) + \tfrac{1}{2}M\dot{z}^2(t), \tag{1}$$

where $w(x,t)$ represents the transverse displacement of the beam, $\psi(x,t)$ is the angle of rotation due to bending, $z(t)$ is the displacement of the undamped oscillator, ρ is the mass density, $A(x)$ is the cross sectional area of the beam, $I(x)$ is the moment of inertia of the

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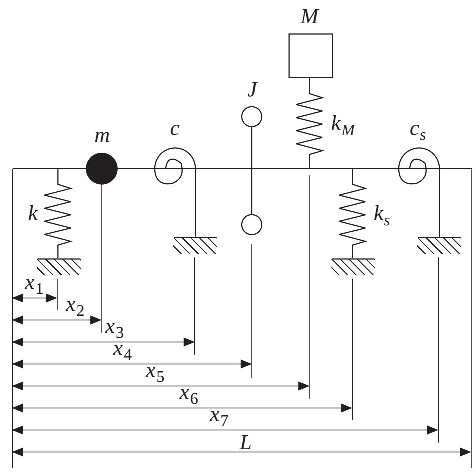

Figure 1. A uniform Timoshenko beam with lumped attachments.

cross section, and an overdot denotes the partial derivative with respect to time. The total potential energy of the system is

$$V = \tfrac{1}{2}\int_0^L EI(x)\psi'^2(x, t)\,\mathrm{d}x + \tfrac{1}{2}\int_0^L \kappa GA(x)[w'(x, t) - \psi(x, t)]^2\,\mathrm{d}x$$
$$+ \tfrac{1}{2}kw^2(x_1, t) + \tfrac{1}{2}c\psi^2(x_3, t) + \tfrac{1}{2}k_M[z(t) - w(x_5, t)]^2$$
$$+ \tfrac{1}{2}k_s w^2(x_6, t) + \tfrac{1}{2}c_s\psi^2(x_7, t), \qquad (2)$$

where E is the Young's modulus, G is the shear modulus, κ is a shape factor which depends on the cross-section of the beam, and a prime represents the partial derivative with respect to space.

Using the Rayleigh–Ritz method [2], one writes

$$w(x, t) = \sum_{i=1}^{N} W_i(x)\eta_i(t) \qquad \text{and} \qquad \psi(x, t) = \sum_{i=1}^{N} \Psi_i(x)\eta_i(t), \qquad (3)$$

where $W_i(x)$ and $\Psi_i(x)$ are the ith transverse and rotational eigenfunctions, respectively, of the unconstrained Timoshenko beam (or the Timoshenko beam without any attachments), that serve as the basis functions for this approximate solution. Substituting equation (3) into equations (1) and (2), one obtains the following discretized total kinetic and potential energies:

$$T = \tfrac{1}{2}\sum_{i=1}^{N} M_i\dot{\eta}_i^2 + \tfrac{1}{2}m\left[\sum_{i=1}^{N} W_i(x_2)\dot{\eta}_i\right]^2 + \tfrac{1}{2}J\left[\sum_{i=1}^{N} \Psi_i(x_4)\dot{\eta}_i\right]^2 + \tfrac{1}{2}M\dot{z}^2, \qquad (4)$$

$$V = \tfrac{1}{2}\sum_{i=1}^{N} K_i\eta_i^2 + \tfrac{1}{2}k\left[\sum_{i=1}^{N} W_i(x_1)\eta_i\right]^2 + \tfrac{1}{2}c\left[\sum_{i=1}^{N} \Psi_i(x_3)\eta_i\right]^2$$

$$+ \tfrac{1}{2}k_M\left[z - \sum_{i=1}^{N} W_i(x_5)\eta_i\right]^2 + \tfrac{1}{2}k_s\left[\sum_{i=1}^{N} W_i(x_6)\eta_i\right]^2 + \tfrac{1}{2}c_s\left[\sum_{i=1}^{N} \Psi_i(x_7)\eta_i\right]^2, \qquad (5)$$

where

$$M_i = \int_0^L \rho A(x) W_i^2(x)\,\mathrm{d}x + \int_0^L \rho I(x) \Psi_i^2(x)\,\mathrm{d}x, \tag{6}$$

$$K_i = \int_0^L EI(x) \Psi_i'^2(x)\,\mathrm{d}x + \int_0^L \kappa GA(x)[W_i'(x) - \Psi_i(x)]^2\,\mathrm{d}x. \tag{7}$$

Applying Lagrange's equations and assuming simple harmonic motion,

$$\eta_i(t) = \bar{\eta}_i\,\mathrm{e}^{\mathrm{j}\omega t}, \qquad z(t) = \bar{z}\,\mathrm{e}^{\mathrm{j}\omega t}, \tag{8}$$

where $\mathrm{j} = \sqrt{-1}$ and ω is the natural frequency, the eigenvalue equation for the system of Figure 1 is given by

$$\begin{bmatrix} [\mathscr{K}] & -k_M \mathbf{W}(x_5) \\ -k_M \mathbf{W}^{\mathrm{T}}(x_5) & k_M \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{\mathrm{z}} \end{bmatrix} = \omega^2 \begin{bmatrix} [\mathscr{M}] & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & M \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{\mathrm{z}} \end{bmatrix}, \tag{9}$$

where the vector of normal co-ordinates $\bar{\boldsymbol{\eta}} = [\bar{\eta}_1 \;\; \bar{\eta}_2 \ldots \bar{\eta}_N]^{\mathrm{T}}$, and the $N \times N$ matrices $[\mathscr{M}]$ and $[\mathscr{K}]$ are

$$[\mathscr{M}] = [\mathbf{M}^d] + m\mathbf{W}(x_2)\mathbf{W}^{\mathrm{T}}(x_2) + J\boldsymbol{\Psi}(x_4)\boldsymbol{\Psi}^{\mathrm{T}}(x_4), \tag{10}$$

$$\begin{aligned}[\mathscr{K}] = [\mathbf{K}^d] &+ k\mathbf{W}(x_1)\mathbf{W}^{\mathrm{T}}(x_1) + c\boldsymbol{\Psi}(x_3)\boldsymbol{\Psi}^{\mathrm{T}}(x_3) + k_M\mathbf{W}(x_5)\mathbf{W}^{\mathrm{T}}(x_5) \\ &+ k_s\mathbf{W}(x_6)\mathbf{W}^{\mathrm{T}}(x_6) + c_s\boldsymbol{\Psi}(x_7)\boldsymbol{\Psi}^{\mathrm{T}}(x_7). \end{aligned} \tag{11}$$

In equations (10) and (11), $[\mathbf{M}^d]$ and $[\mathbf{K}^d]$ are diagonal matrices whose ith diagonal elements are M_i and K_i, respectively, and

$$\mathbf{W}(x_i) = [W_1(x_i) \;\; W_2(x_i) \ldots W_N(x_i)]^{\mathrm{T}}, \qquad \boldsymbol{\Psi}(x_i) = [\Psi_1(x_i) \;\; \Psi_2(x_i) \ldots \Psi_N(x_i)]^{\mathrm{T}}. \tag{12}$$

Note that both $[\mathscr{M}]$ and $[\mathscr{K}]$ are diagonal matrices modified by a sequence of rank one matrices.

Solving for $\bar{z}$ using the second equation of (9), one obtains

$$\bar{z} = -k_M \mathbf{W}^{\mathrm{T}}(x_5)\bar{\boldsymbol{\eta}}/(M\omega^2 - k_M). \tag{13}$$

Substituting equation (13) into the first equation of (9) yields

$$\{[\mathscr{K}] + k_M^2 \mathbf{W}(x_5)\mathbf{W}^{\mathrm{T}}(x_5)/(M\omega^2 - k_M)\}\bar{\boldsymbol{\eta}} = \omega^2[\mathscr{M}]\bar{\boldsymbol{\eta}}. \tag{14}$$

Rearranging equation (14), one can write it alternatively as

$$\left\{[\mathbf{K}^d] + \sum_{i=1}^{R} \sigma_i \boldsymbol{\phi}(x_i)\boldsymbol{\phi}^{\mathrm{T}}(x_i) - \omega^2[\mathbf{M}^d]\right\}\bar{\boldsymbol{\eta}} = \mathbf{0}, \tag{15}$$

where $R = T$ and

$$\begin{aligned} &\sigma_1 = k, \qquad \sigma_2 = -m\omega^2, \qquad \sigma_3 = c, \qquad \sigma_4 = -J\omega^2, \\ &\sigma_5 = k_M M\omega^2/(M\omega^2 - k_M), \qquad \sigma_6 = k_s, \qquad \sigma_7 = c_s \end{aligned} \tag{16}$$

and $\boldsymbol{\phi}(x_i)$ is a vector of length N whose elements are given by

$$\phi_r(x_i) = \begin{Bmatrix} W_r(x_i), & i = 1, 2, 5, 6, \\ \Psi_r(x_i), & i = 3, 4, 7. \end{Bmatrix} \tag{17}$$

For non-trivial $\bar{\boldsymbol{\eta}}$, the eigenvalues, ω^2, must satisfy the zeros of the following $N \times N$ characteristic determinant:

$$\det\left\{[\mathbf{K}^d] + \sum_{i=1}^{R} \sigma_i \boldsymbol{\phi}(x_i)\boldsymbol{\phi}^{\mathrm{T}}(x_i) - \omega^2[\mathbf{M}^d]\right\} = 0. \tag{18}$$

Upon rearranging, equation (18) becomes

$$\det\{[\mathbf{K}^d] - \omega^2[\mathbf{M}^d]\}\det\left\{[\mathbf{I}] + \sum_{i=1}^{R} \sigma_i([\mathbf{K}^d] - \omega^2[\mathbf{M}^d])^{-1}\boldsymbol{\phi}(x_i)\boldsymbol{\phi}^{\mathrm{T}}(x_i)\right\} = 0. \tag{19}$$

After some algebra, equation (19) can be shown to be identical to

$$\det\{[\mathbf{K}^d] - \omega^2[\mathbf{M}^d]\}\det[\mathbf{B}] = \prod_{i=1}^{N}(K_i - \omega^2 M_i)\det[\mathbf{B}] = 0, \tag{20}$$

where the (i, j)th element of $[\mathbf{B}]$, of size $R \times R$, is given by

$$b_{ij} = \sum_{r=1}^{N} \frac{\phi_r(x_i)\phi_r(x_j)}{K_r - \omega^2 M_r} + \frac{1}{\sigma_i}\delta_i^j, \qquad i, j = 1, 2, \ldots, R, \tag{21}$$

and δ_i^j represents the Kronecker delta. Note that each element of $[\mathbf{B}]$ consists of a sum of N terms. Finally, in the limit as k_s and c_s approach infinity, the supports against beam translation and rotation analyzed in reference [3] are recovered.

Comparing equation (20) and equation (19) of reference [3], one notices the absence of the product terms. When the constraint locations of the lumped attachments do not coincide with the nodes of the unconstrained component modes, the eigenvalues of the constrained and unconstrained systems must be distinct; thus $K_i \neq \omega^2 M_i$, and equation (20) reduces to equation (19) of reference [3]. However, when all the constraint locations coincide wtih the nodes of a given unconstraint component mode, equation (19) of reference [3] (obtained via Lagrange multipliers formalism) fails to generate all the eigenvalues of the constrained system, because some of the $\phi_r(x_i)$'s in the summation of equation (21) become zero. In this case, equation (20) (obtained via the assumed-modes method) can still be used to extract all of the eigenvalues of the contrained system, though some of the eigenvalues will correspond to those of the unconstrained system.

In summary, the concise eigenvalue equation for the free vibration of a combined dynamical system, obtained by means of the Lagrange multipliers formalism, can also be extracted by using the more straightforward and simpler assumed-modes method. The latter approach also has the distinct advantage that it can be used when the constraint locations are located at the nodes of the unconstrained component mode.

REFERENCES

1. P. D. Cha and W. Wong *Journal of Sound and Vibration*. (submitted). A novel approach to determine the frequency equations of combined dynamical systems.
2. L. Meirovitch 1967 *Analytical Methods in Vibrations*. New York: The Macmillan Company.
3. B. Posiadala 1997 *Journal of Sound and Vibration* **204**, 359–369. Free vibrations of uniform Timoshenko beams with attachments.