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A UNIFIED MATRIX POLYNOMIAL APPROACH TO MODAL IDENTIFICATION

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One important current focus of modal identification is a reformulation of modal parameter estimation algorithms into a single, consistent mathematical formulation with a corresponding set of definitions and unifying concepts. Particularly, a matrix polynomial approach is used to unify the presentation with respect to current algorithms such as the least-squares complex exponential (LSCE), the polyreference time domain (PTD), Ibrahim time domain (ITD), eigensystem realization algorithm (ERA), rational fraction polynomial (RFP), polyreference frequency domain (PFD) and the complex mode indication function (CMIF) methods. Using this unified matrix polynomial approach (UMPA) allows a discussion of the similarities and differences of the commonly used methods. The use of least squares (LS), total least squares (TLS), double least squares (DLS) and singular value decomposition (SVD) methods is discussed in order to take advantage of redundant measurement data. Eigenvalue and SVD transformation methods are utilized to reduce the effective size of the resulting eigenvalue–eigenvector problem as well.

1. INTRODUCTION

Modal parameter estimation is a special case of system identification where the *a priori* model of the system is known to be in the form of modal parameters. Over the past twenty years, a number of algorithms have been developed to estimate modal parameters from measured frequency or impulse response function data. While most of these individual algorithms, summarized in Table 1, are well understood, the comparison of one algorithm to another has become one of the thrusts of current research in this area. Comparison of the different algorithms is possible when the algorithms are reformulated using a common mathematical structure.

This reformulation attempts to characterize different classes of modal parameter estimation techniques in terms of the structure of the underlying matrix polynomials rather than the physically based models used historically. Since the modal parameter estimation process involves a greatly over-determined problem (more data than independent equations), this reformulation is helpful in understanding the different numerical characteristics of each algorithm and, therefore, the slightly different estimates of modal parameters that each algorithm yields. As a part of this reformulation of the algorithms, the development of a conceptual understanding of modal parameter estimation technology has emerged. This understanding involves the ability to visualize the measured data in terms of the concept of *characteristic space*, the data domain (time, frequency, spatial), the evaluation of the model (polynomial) order of the problem, the condensation of the

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TABLE 1

Acronyms—modal parameter estimation algorithms

	Modal parameter estimation algorithms
CEA	Complex exponential algorithm [1, 2]
LSCE	Least squares complex exponential [1, 2]
PTD	Polyreference time domain [3, 4]
ITD	Ibrahim time domain [5, 6]
MRITD	Multiple reference Ibrahim time domain [5]
ERA	Eigensystem realization algorithm [7, 8]
PFD	Polyreference frequency domain [9, 10–12, 13, 14]
SFD	Simultaneous frequency domain [15]
MRFD	Multi-reference frequency domain [16]
RFP	Rational fraction polynomial [17]
OP	Orthogonal polynomial [18–20]
CMIF	Complex mode indication function [19, 21]

data, and a common parameter estimation theory that can serve as the basis for devloping any of the algorithms in use today. The following sections review these concepts as applied to the current modal parameter estimation methodology.

1.1. DEFINITION OF MODAL PARAMETERS

Modal identification involves estimating the modal parameters of a structural system from measured input–output data. Most current modal parameter estimation is based upon the measured data being the frequency response function or the equivalent impulse response function, typically found by inverse Fourier transforming the frequency response function. Modal parameters include the complex-valued modal frequencies λ_r, modal vectors $\{\boldsymbol{\psi}_r\}$ and modal scaling (modal mass or modal A). Additionally, most current algorithms estimate modal participation vectors $\{\mathbf{L}_r\}$ and residue vectors $\{\mathbf{A}_r\}$ as part of the overall process. Modal participation vectors are a result of multiple reference modal parameter estimation algorithms and relate how well each modal vector is excited from each of the reference locations included in the measured data. The combination of the modal participation vector $\{\mathbf{L}_r\}$ and the modal vector $\{\boldsymbol{\psi}_r\}$ for a given mode give the residue matrix $[\mathbf{A}_r]$ for that mode.

In general, modal parameters are considered to be global properties of the system. The concept of global modal parameters simply means that there is only one answer for each modal parameter and that the modal parameter estimation solution procedure enforces this constraint. Most of the current modal parameter estimation algorighms estimate the modal frequencies and damping in a global sense but very few estimate the modal vectors in a global sense.

2. SIMILARITIES IN MODAL PARAMETER ESTIMATION ALGORITHMS

The similarities in modal parameter estimation algorithms arise from the common theoretical basis of the algorithms. Fundamentally, each algorithm starts with a system that can be represented by a second order, linear, constant coefficient matrix equation. This fundamental equation depends upon several assumptions: linearity, time invariance, observability and reciprocity. Rather than working with this matrix equation directly,

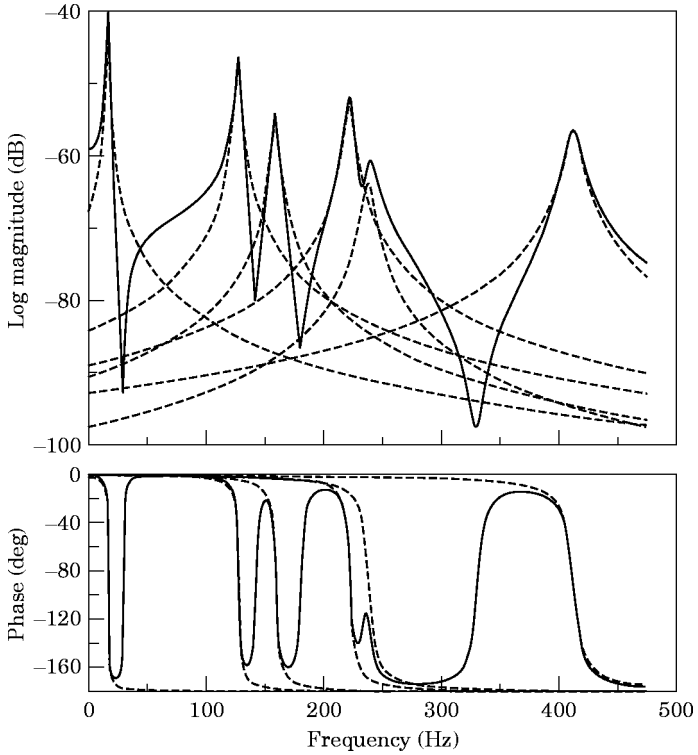

Figure 1. MDOF—superposition of SDOF (positive frequency poles).

most modal parameter estimation algorithms utilize measured frequency response functions (or the time domain equivalent, the impusle response functions) as the experimental database for the algorithm.

The current approach in modal identification involves using numerical techniques to separate the contributions of individual modes of vibration in measurements such as frequency response functions. The concept involves estimating the individual single-degree-of-freedom contributions (SDOF) to the multiple-degree-of-freedom measurement (MDOF):

$$[\mathbf{H}(\omega)]_{N_o \times N_i} = \sum_{r=1}^{N} \frac{[\mathbf{A}_r]_{N_o \times N_1}}{\mathrm{j}\omega - \lambda_r} + \frac{[\mathbf{A}_r]_{N_o \times N_i}}{\mathrm{j}\omega - \lambda_r^*}. \tag{1}$$

This concept is represented mathematically in equation (1) and graphically in Figure 1.

Equation (1) represents a mathematical problem that, at first observation, is non-linear in terms of the unknown modal parameters. Once the modal frequencies λ_r are known, the mathematical problem is linear with respect to the remaining unknown modal parameters $[\mathbf{A}_r]$. For this reason, the numerical approach in many algorithms involves two or more linear stages. Typically, the modal frequencies and modal participation vectors are found in a first stage and residues, modal vectors and modal scaling are determined in a second stage. This general approach is discussed in the sections 2.3 and 2.4.

2.1. DATA DOMAIN

Modal parameters can be estimated from a variety of different measurements that exist as discrete data in different data domains (time and/or frequency). These measurements can include free-decays, forced responses, frequency response functions (FRFs) or impulse response functions (IRFs). These measurements can be processed one at a time or in partial or complete sets simultaneously. The measurements can be generated with no measured inputs, a single measured input, or multiple measured inputs. The data can be measured individually or simultaneously. There is a tremendous variation in the types of measurements and in the types of constraints that can be placed upon the testing procedures used to acquire these data. For most measurement situations, FRFs are utilized in the frequency domain and IRFs are utilized in the time domain.

2.2. CHARACTERISTIC SPACE

From a conceptual viewpoint, the measurement space of a modal identification problem can be visualized as occupying a volume with the co-ordinate axes defined in terms of the three sets of characteristics. Two axes of the conceptual volume correspond to spatial information and the third axis to temporal information. The spatial axes are in terms of the input and output degrees of freedom (DOF) of the system. The temporal axis is either time or frequency depending upon the domain of the measurements. These three axes define a 3-D volume which is referred to as the *characteristic space*.

This space or volume represents all possible measurement data. This conceptual representation is very useful in understanding which data subset has been measured. Also, this conceptual representation is very useful in recognizing how the data are organized and utilized with respect to different modal parameter estimation algorithms. Information parallel to one axis consists of a superposition of the characteristics defined by that axis. The other two characteristics determine the scaling of each term in the superposition.

Any structural testing procedure measures a subset of the total possible data available. Modal parameter estimation algorithms may then use all of this subset or may choose to further limit the data to a more restrictive subset. It is theoretically possible to estimate the characteristics of the total space by measuring any subset which samples all three characteristics. Measurement data spaces involving many planes of measured data are the best possible modal identification situations since the data subset includes contributions from temporal and spatial characteristics. The particular subset which is measured and the weighting of the data within the subset in an algorithm are the main differences between the various modal idenitfication procedures which have been developed. This is discussed further in section 3.

2.3. FIRST STAGE MODAL IDENTIFICATION MODELS

Rather than using a physically based mathematical model, the common characteristics of different modal parameter estimation algorithms can be more readily identified by using a matrix polynomial model. One way of understanding the basis of this model can be developed from the polynomial model used historically for the frequency response function.

$$H_{pq}(\omega) = \frac{X_p(\omega)}{F_q(\omega)} = \frac{\beta_n(\mathrm{j}\omega)^n + \beta_{n-1}(\mathrm{j}\omega)^{n-1} + \cdots + \beta_1(\mathrm{j}\omega)^1 + \beta_0(\mathrm{j}\omega)^0}{\alpha_m(\mathrm{j}\omega)^m + \alpha_{m-1}(\mathrm{j}\omega)^{m-1} + \cdots + \alpha_1(\mathrm{j}\omega)^1 + \alpha_0(\mathrm{j}\omega)^0}. \tag{2}$$

This can be rewritten:

$$H_{pq}(\omega) = \frac{X_p(\omega)}{F_q(\omega)} = \sum_{k=0}^{n} \beta_k (\mathrm{j}\omega)^k \Big/ \sum_{k=0}^{m} \alpha_k (\mathrm{j}\omega)^k. \tag{3}$$

Further rearranging yields the following equation that is linear in the unknown α and β terms:

$$\sum_{k=0}^{m} \alpha_k (\mathrm{j}\omega)^k X_p(\omega) = \sum_{k=0}^{n} \beta_k (\mathrm{j}\omega)^k F_q(\omega). \tag{4}$$

This model can be generalized to represent the general multiple input, multiple output case as follows:

$$\sum_{k=0}^{m} [[\boldsymbol{\alpha}_k](\mathrm{j}\omega)^k]\{\mathbf{X}(\omega)\} = \sum_{k=0}^{n} [[\boldsymbol{\beta}_k](\mathrm{j}\omega)^k]\{F(\omega)\}. \tag{5}$$

Note that the size of the coefficient matrices $[\boldsymbol{\alpha}_k]$ will normally be $N_i \times N_i$ or $N_o \times N_o$ and the size of the coefficient matrices $[\boldsymbol{\beta}_k]$ will normally be $N_o \times N_i$ or $N_i \times N_\mathrm{o}$ when the equations are developed from experimental data.

Rather than developing the basic model in terms of force and response information, the models can be stated in terms of frequency response information. The response vector $\{\mathbf{X}(\omega)\}$ can be replaced by a matrix of frequency response functions $[\mathbf{H}(\omega)]$:

$$\sum_{k=0}^{m} [[\boldsymbol{\alpha}_k](\mathrm{j}\omega)^k][\mathbf{H}(\omega)] = \sum_{k=0}^{n} [[\boldsymbol{\beta}_k](\mathrm{j}\omega)^k]. \tag{6}$$

The above model, in the frequency domain, corresponds to an autoregressive–moving average (ARMA) model that is developed from a set of finite difference equations in the time domain. The general matrix polynomial model concept recognizes that both the time and frequency domain models generate functionally similar matrix polynomial models. For that reason, the UMPA terminology is used to describe both domains since the ARMA terminology has been connected primarily with the time domain. Equation (6) can be transposed and rearranged into a linear matrix equation. Additional equations can be developed by repeating equation (6) at many frequencies (ω_i) until all data or a sufficient overdetermination factor is achieved. Note that both positive and negative frequencies are required in order to accurately estimate conjugate modal frequencies.

Paralleling the development of equations (2)–(6), a time domain model representing the relationship between a single response degree of freedom and a single input degree of freedom can be stated as follows:

$$\sum_{k=0}^{m} \alpha_k x(t_{i+k}) = \sum_{k=0}^{n} \beta_k f(t_{i+k}). \tag{7}$$

In the time domain, this model is commonly known as the ARMA(m, n) model. For the general multiple input, multiple output case:

$$\sum_{k=0}^{m} [\boldsymbol{\alpha}_k]\{\mathbf{x}(t_{i+k})\} = \sum_{k=0}^{n} [\boldsymbol{\beta}_k]\{\mathbf{f}(t_{i+k})\}. \tag{8}$$

If the discussion is limited to the use of free decay or impulse response function data, the previous time domain equations can be simplified by noting that the forcing function can be assumed to be zero for all time greater than zero. If this is the case, the $[\boldsymbol{\beta}_k]$ coefficients can be eliminated from the equations:

$$\sum_{k=0}^{m} [\boldsymbol{\alpha}_k]\,[h(t_{i+k})] = 0. \tag{9}$$

Equation (9) can be transposed and rearranged into a linear matrix equation. Additional equations can be developed by repeating equation (9) at different time shifts into the data t_i until all data or a sufficient overdetermination factor is achieved. Note that at least one time shift is required in order to accurately estimate conjugate modal frequencies.

In light of the above discussion, it is now apparent that most of the modal parameter estimation processes available could have been developed by starting from a general matrix polynomial formulation that is justifiable based upon the underlying matrix differential equation. The general matrix polynomial formulation yields essentially the same characteristic matrix polynomial equation, for both time and frequency domain data. For the frequency domain data case, this yields

$$\left|[\boldsymbol{\alpha}_m]s^m + [\boldsymbol{\alpha}_{m-1}]s^{m-1} + [\boldsymbol{\alpha}_{m-2}]s^{m-2} + \cdots + [\boldsymbol{\alpha}_0]\right| = 0. \tag{10}$$

For the time domain data case, this yields

$$\left|[\boldsymbol{\alpha}_m]z^m + [\boldsymbol{\alpha}_{m-1}]z^{m-1} + [\boldsymbol{\alpha}_{m-2}]z^{m-2} + \cdots + [\boldsymbol{\alpha}_0]\right| = 0. \tag{11}$$

Note that the same nomenclature for the coefficient matrices $[\boldsymbol{\alpha}]$ was used in both equation (10) and equation (11). This is done to demonstrate the similarity of the two equations. While the two characteristic equations result in the same modal frequencies, clearly, the coefficient matrices are not the same in the two domains.

From a theoretical consideration, the number of characteristic values (number of modal frequencies, number of roots, number of poles, etc.) that can be determined depends upon the size of the matrix coefficients involved in the model and the order of the polynomial terms in the model. The characteristic matrix polynomial equation, equation (10) or equation (11) has a model order of m and the number of modal frequencies or roots that will be found from this characteristic matrix polynomial equation will be m times the size of the coefficient matrices $[\boldsymbol{\alpha}]$. In terms of sampled data, the time domain matrix polynomial results from a set of finite difference equations and the frequency domain matrix polynomial results from a set of linear equations where each equation is formulated at one of the frequencies of the measured data. This distinction is important to note since the roots of the matrix characteristic equation formulated in the time domain are in the *z domain* z_r and must be converted to the frequency domain λ_r while the roots of the matrix characteristic equation formulated in the frequency domain λ_r are already in the desired domain (reference [22]). Note that the roots that are estimated in the time domain are limited to maximum values determined by the Sampling Theorem relationship (discrete time steps):

$$z_r = e^{\lambda_r \Delta t}, \qquad \lambda_r = \sigma_r + j\omega_r, \qquad \sigma_r = \mathrm{Re}\,[\ln z_r/\Delta t], \qquad \omega_r = \mathrm{Im}\,[\ln z_r/\Delta t]. \tag{12, 13}$$

Much of the work concerned with modal parameter estimation since 1975 has involved methodology for determining the correct model order for the modal parameter model. The number of modal frequencies found will be equal to the modal order times the size of the matrix coefficients, normally N_o or N_i. For a given algorithm, the size of the matrix coefficients is normally fixed; therefore, determining the model order is directly linked to estimating N, the number of modal frequencies that are of interest in the measured data.

2.4. SECOND STAGE MODAL IDENTIFICATION MODELS

Most current modal parameter estimation algorithms utilize frequency or impulse response functions as the data, or known information, to solve for modal parameters. The general equation that can be used to represent the relationship between the measured frequency response function matrix and the modal parameters is shown in equation (1) or, in the more common matrix product form, in the following way:

$$[\mathbf{H}(\omega)]_{N_o \times N_i} = [\boldsymbol{\psi}]_{N_o \times 2N} \left[\frac{1}{\mathrm{j}\omega - \lambda_r} \right]_{2N \times 2N} [\mathbf{L}]^{\mathrm{T}}_{2N \times N_i}, \tag{14}$$

$$[\mathbf{H}(\omega)]^{\mathrm{T}}_{N_i \times N_o} = [\mathbf{L}]_{N_i \times 2N} \left[\frac{1}{\mathrm{j}\omega - \lambda_r} \right]_{2N \times 2N} [\psi]^{\mathrm{T}}_{2N \times N_o}. \tag{15}$$

Impulse response functions are rarely directly measured but are calculated from associated frequency response functions via the inverse FFT algorithm. The general relations that can be used to represent the relationship between the impulse response function matrix and the modal parameters are

$$[\mathbf{h}(t)]_{N_o \times N_i} = [\psi]_{N_o \times 2N} \lceil \mathrm{e}^{\lambda_r t} \rfloor_{2N \times 2N} [\mathbf{L}]^{\mathrm{T}}_{2N \times N_i}, \tag{16}$$

$$[\mathbf{h}(t)]^{\mathrm{T}}_{N_i \times N_o} = [\mathbf{L}]_{N_i \times 2N} \lceil \mathrm{e}^{\lambda_r t} \rfloor_{2N \times 2N} [\psi]^{\mathrm{T}}_{2N \times N_o}. \tag{17}$$

Many modal parameter estimation algorithms have been originally formulated from equations (14)–(17). However, a more general development for all algorithms is based upon relating the above equations to a general matrix polynomial model.

2.5. GENERAL (TWO-STAGE) SOLUTION PROCEDURE

Based upon equations (2)–(17), most modern modal identification algorithms can be outlined as follows:

First stage (*modal frequencies and modal participation vectors*).

- Load measured data into linear equation form (equations (6) or (9)).
- Find scalar or matrix autoregressive coefficients ($[\boldsymbol{\alpha}_k]$).
- Solve matrix polynomial for modal frequencies.
- Formulate companion matrix [23].
- Obtain eigenvalues of companion matrix (λ_r or z_r).
- Convert eigenvalues from z_r to λ_r (time domain only).
- Obtain modal participation vectors $\mathbf{L}_{qr}$ or modal vectors $\{\boldsymbol{\psi}\}_r$ from eigenvectors of the companion matrix.

Second stage (*modal vectors and modal scaling*).

- Find modal vectors and modal scaling from equations (14)–(17).

Further details concerning the mathematical structure and data configuration of these algorithms are given in Appendices A–E.

3. DIFFERENCES IN MODAL PARAMETER ESTIMATION ALGORITHMS

The differences in modal parameter estimation algorithms arise from the way in which different modal parameter estimation algorithms utilize the redundant information in the experimental database. Since the actual number of parameters being estimated is quite small compared to the amount of data available, the different numerical methods that are used to solve for the modal parameters can have a great influence on the results. These issues are discussed in the following sections.

3.1. DATA SIEVING/FILTERING/DECIMATION

For almost all cases of modal identification, a large amount of redundancy or overdetermination exists. This means that the number of equations available compared to the number required to form an exactly determined solution, defined as the *overdetermination factor*, will be quite large. Beyond some value of overdetermination factor, the additional equations contribute little to the result but may add significantly to the solution time. For this reason, the data space is often *filtered* (limited within minimum and maximum temporal axis values), *sieved* (limited to prescribed input DOFs and/or output DOFs) and/or *decimated* (limited number of equations from the allowable temporal data) in order to obtain a reasonable result in the minimum time.

3.2. COEFFICIENT CONDENSATION

For the low order modal identification algorithms, the number of physical co-ordinates (typically N_o), which dictates the size of the coefficient matrices $[\boldsymbol{\alpha}_k]$, is often much larger than the number of desired modal frequencies N. For this situation, the numerical solution procedure is constrained to solve for N_o or $2N_o$ modal frequencies. This can be very time consuming and is unnecessary. The number of physical co-ordinates N_o can be reduced to a more reasonable size ($N_e \approx N$ or $N_e \approx 2N$) by using a decomposition transformation from physical co-ordinates N_o to the approximate number of effective modal frequencies N_e. Currently, singular value decompositions (SVD) or eigenvalue decompositions (ED) are used to preserve the principal modal information prior to formulating the linear equation solution for unknown matrix coefficients [24–26]. In most cases, even when the spatial information must be condensed, it is necessary to use a matrix dimension greater than the number of effective modal frequencies N_e to compensate for distortion errors or noise in the data and to compensate for the case where the location of the transducers is not sufficient to totally define the structure:

$$[\mathbf{H}'] = [\mathbf{T}][\mathbf{H}], \tag{18}$$

where $[\mathbf{H}']$ is the transformed (condensed) frequency response function matrix, sometimes called the virtual frequency response function matrix; $[\mathbf{T}]$ is the transformation matrix; $[\mathbf{H}]$ is the original FRF matrix.

The difference between the two techniques lies in the method of finding the transformation matrix, $[\mathbf{T}]$. Once $[\mathbf{H}]$ has been condensed, however, the parameter estimation procedure is the same as for the full data set. Because the data eliminated from the parameter estimation process ideally corresponds to the noise in the data, the poles of the condensed data are the same as the poles of the full data set. However, the participation factors calculated from the condensed data may need to be expanded back into the full space:

$$[\boldsymbol{\Psi}] = [\mathbf{T}]^{\mathrm{T}}[\boldsymbol{\Psi}'], \tag{19}$$

where $[\boldsymbol{\Psi}]$ is the full-space participation matrix and $[\boldsymbol{\Psi}']$ is the condensed-space participation matrix.

3.2.1. *Eigenvalue decomposition*

In the eigenvalue decomposition method (sometimes referred to as *Principal Component Analysis* [24]), the $[\mathbf{T}]$ matrix is composed of the eigenvectors corresponding to the $\mathbf{N}_e$ largest eigenvalues of the power spectrum of the FRF matrix as follows:

$$[\mathbf{H}(\omega)]_{N_o \times N_i N_s}[\mathbf{H}(\omega)]^H_{N_i N_s \times N_o} = [\mathbf{V}][\boldsymbol{\Lambda}][\mathbf{V}]^H. \tag{20}$$

The eigenvalues and eigenvectors are then found, and the [**T**] matrix is constructed from the eigenvectors corresponding to the $\mathbf{N}_e$ largest eigenvalues:

$$[\mathbf{T}]_{N_e \times N_o} = [\{\mathbf{v}_1\}\{\mathbf{v}_2\} \cdots \{\mathbf{v}_k\} \cdots \{\mathbf{v}_{N_e}\}]^{\mathrm{T}}, \tag{21}$$

where $\{\mathbf{v}_k\}$ is the $N_o \times 1$ eigenvector corresponding to the kth eigenvalue.

3.2.2. *Singular value decomposition*

The singular value decomposition condensation technique is similar to the egenvalue-based technique, but operates on the FRF matrix directly instead of the power spectrum of the FRF matrix. The basis for this technique is the singular value decomposition [23–27], by which the matrix [**H**] is broken down into three component parts, [**U**], [**Σ**], and [**V**]:

$$[\mathbf{H}]_{N_o \times N_i N_s} = [\mathbf{U}]_{N_o \times N_o}[\mathbf{\Sigma}]_{N_o \times N_o}[\mathbf{V}]^H_{N_o \times N_i N_s}. \tag{22}$$

The left-singular vectors corresponding to the N_e largest singular values are the first N_e columns of [**U**]. These become the transformation matrix [**T**]:

$$[\mathbf{T}]_{N_e \times N_o} = [\{\mathbf{u}_1\}\{\mathbf{u}_2\} \cdots \{\mathbf{u}_k\} \cdots \{\mathbf{u}_{N_e}\}]^{\mathrm{T}}, \tag{23}$$

where $\{\mathbf{u}_k\}$ is the kth column of [**U**], which corresponds to the kth singular value.

3.3. EQUATION CONDENSATION

Equation condensation methods are used to reduce the number of equations generated from measured data to more closely match the number of unknowns in the modal parameter estimation algorithms. There are a large number of condensation algorithms available. Based upon the modal parameter estimation algorithms in use today, the three types of algorithms most often used are:

Least Squares: LS, weighted least squares (WLS), TLS or DLS are used to minimize the squared error between the measured data and the estimation model.

Transformation: The measured data are reduced by approximating the data by the superposition of a limited (reduced) set of independent vectors. The number of significant, independent vectors is chosen equal to the maximum number modes that are expected in the measured data. This set of vectors is used to approximate the measured data and used as input to the parameter estimation procedures. SVD is an example of one of the more popular transformation methods.

Coherent Averaging: Coherent averaging is another popular method for reducing the data. In the coherent averaging method, the data are weighted by performing a dot product between the data and a weighting vector (spatial filter). Information in the data which is not coherent with the weighting vectors is averaged out of the data.

The least squares and the transformation procedures tend to weight those modes of vibration which are well excited. This can be a problem when trying to extract modes which are not well excited. The solution is to use a weighting function for condensation which tends to enhance the mode of interest. This can be accomplished in a number of ways: in the time domain, a spatial filter or a coherent averaging process can be used to filter the response to enhance a particular mode or set of modes; in the frequency domain, the data can be enhanced in the same manner as the time domain plus the data can be additionally enhanced by weighting the data in a frequency band near the natural frequency of the mode of interest. Obviously, the type of equation condensation method that is utilized in a modal identification algorithm has a significant influence on the results.

3.4. MODEL ORDER DETERMINATION

Much of the work concerned with modal parameter estimation since 1975 has involved methodology for determining the correct model order for the modal parameter model. The number of modal frequencies found will be equal to the model order times the size of the matrix coefficients, normally N_o or N_i. For a given algorithm, the size of the matrix coefficients is normally fixed; therefore, determining the model order is directly linked to estimating N, the number of modal frequencies that are of interest in the measured data. As has always been the case, an estimate for the minimum number of modal frequencies can be easily found by counting the number of peaks in the frequency response function in the frequency band of analysis. This is a minimum estimate of N since the frequency response function measurement may be at a node of one or more modes of the system, repeated roots may exist and/or the frequency resolution of the measurement may be too coarse to observe modes that are closely spaced in frequency. Several measurements can be observed and a tabulation of peaks existing in any or all measurements can be used as a more accurate minimum estimate of N. A more automated procedure for including the peaks that are present in several frequency response functions is to observe the summation of frequency response function power. This function represents the auto power or auto moment of the frequency response functions summed over a number of response measurements and is normally formulated as

$$H_{power}(\omega) = \sum_{p=1}^{N_o} \sum_{q=1}^{N_i} H_{pq}(\omega) H_{pq}^*(\omega). \tag{24}$$

These simple techniques are extremely useful but do not provide an accurate estimate of model order when repeated roots exist or when modes are closely spaced in frequency. For these reasons, an appropriate estimate of the order of the model is of prime concern and is the single most important problem in modal parameter estimation.

In order to determine a reasonable estimate of the model order for a set of representative data, a number of techniques have been developed as guides or aids to the user. Much of the user interaction involved in modal parameter estimation involves the use of these tools. Most of the techniques that have been developed allow the user to establish a maximum model order to be evaluated (in many cases, this is set by the memory limits of the computer algorithm). Data are acquired based upon an assumption that the model order is equal to this maximum. In a sequential fashion, the data is evaluated to determine if a model order less than the maximum will describe the data sufficiently. This is the point that the user's judgement and the use of various evaluation aids becomes important. Some of the commonly used techniques are: measurement synthesis and comparison (curve-fit), error chart, stability diagram, mode indication functions, and rank estimation.

3.4.1. *Measurement synthesis*

One of the simplest techniques is to synthesize an impulse response function or a frequency response function and compare it to the measured function to see if modes have obviously been missed. This curve fitting procedure is also used as a measure of the overall success of the modal parameter estimation procedure. Obviously, a poor comparison can be due to many reasons, an incorrect model order simply being one of the possibilities. A good approach is to compare measured and synthesized data where the measured data was used to generate the model used for the synthesized data. A better

approach is to compare measured and synthesized data where the measured data was not used to generate the model used for the synthesized data.

The comparison can be further evaluated by evaluating the normalized error between the measured and synthesized data, computed at each frequency. This error is often plotted on the same plot as the comparison between the measured and synthesized data. The visual match can also be given a numerical value if a correlation coefficient, similar to coherence, is estimated. The basic assumption is that the measured frequency response function and the synthesized frequency response function should be linearly related (unity) at all frequencies. This coefficient is referred to as the synthesis correlation coefficient:

$$COR_{pq} = \Gamma_{pq}^2 = \left| \sum_{\omega=\omega_1}^{\omega_2} H_{pq}(\omega)\hat{H}_{pq}^*(\omega) \right|^2 \Bigg/ \sum_{\omega=\omega_1}^{\omega_2} H_{pq}(\omega)H_{pq}^*(\omega) \sum_{\omega=\omega_1}^{\omega_2} \hat{H}_{pq}(\omega)\hat{H}_{pq}^*(\omega), \tag{25}$$

where $H_{pq}(\omega)$ = measurement, $\hat{H}_{pq}(\omega)$ = synthesis.

3.4.2. *Error chart*

Another method that has been used to indicate more directly the correct model order, for high order model algorithms, is the error chart. Essentially, the error chart is a plot of error in the model as a function of increasing model order. The error in the model is a normalized quantity that represents the ability of the model to predict data that was not involved in the estimate of the model parameters. For example, when using measured data in the form of an impulse response function, only a small percentage of the total number of data values are involved in the estimate of modal parameters. The error in the model can then be estimated by the ability of the model to predict data points not utilized in the estimation of the model. When the model order is insufficient, this model error will be large but when the model error reaches the "correct" value, further increase in the model order will not result in a further decrease in the error. Figure 3 is an example of an error chart.

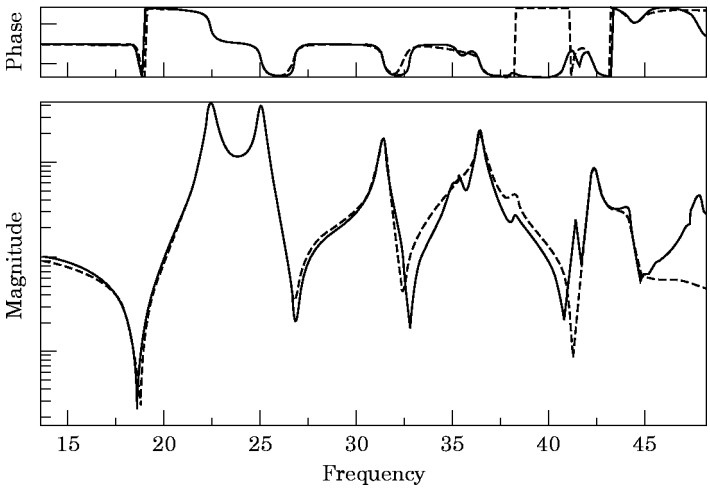

Figure 2. Model order determination: synthesis comparison. ——, Measured data; ---, reconstructed data.

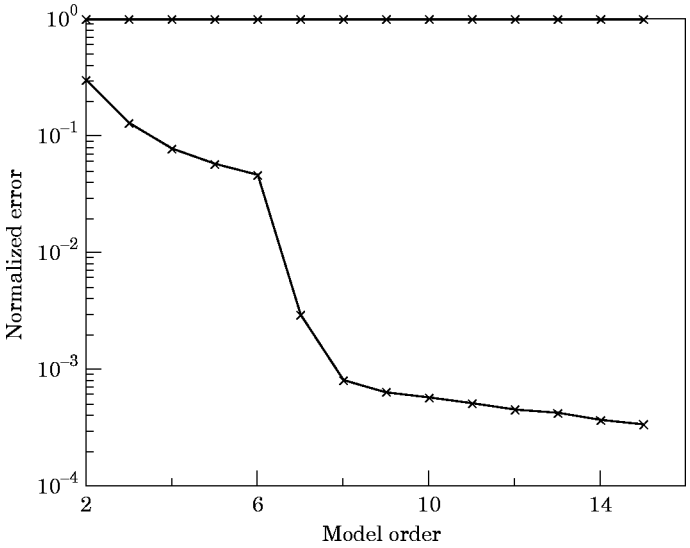

Figure 3. Model order determination: error chart for increasing model order.

3.4.3. *Stability/consistency diagram*

A further enhancement of the error is the stability diagram. The stability diagram has been developed in the same fashion as the error chart and involves tracking the estimates of frequency, damping, and possibly modal participation factors as a function of model order. As the model order is increased, more and more modal frequencies are estimated but, hopefully, the estimates of the physical modal parameters will stabilize as the correct model order is found. For modes that are very active in the measured data, the modal parameters will stabilize at a very low model order. For modes that were poorly excited in the measured data, the modal parameters may not stabilize until a very high model order is chosen. Nevertheless, the non-physical (computational) modes will not stabilize at all during this process and can be sorted out of the modal parameter data set more easily. Note that inconsistencies (frequency shifts, leakage errors, etc.) in the measured data set will obscure the stability and render the stability diagram difficult to use. Normally, a tolerance, in percentage, is given for the stability of each of the modal parameters that are being evaluated. Figure 4 is an example of a stability diagram. In Figure 4, a summation of the frequency response function power is plotted on the stability diagram for reference. Other mode indication functions can be plotted against the stability diagram for reference.

Stability diagrams have traditionally only been used for the high order model cases, such as the LSCE or the PTD. If the concept of stability is broadended to include consistency, a stability/consistency diagram can be constructed for almost any modal parameter estimation method where the vertical axis is a function of a change in model order, a change in the data subset used to estimate the modal frequencies, a change in modal parameter estimation method or, for low order methods, a change in the number of singular values used to represent the data matrix.

3.4.4. *Mode indication functions*

Mode indication functions (MIF) are normally real-valued, frequency domain functions that exhibit local minima or maxima at the natural frequencies of real normal modes. The number of mode indication functions that can be formulated is equal to the

number of references available in the measured data. The primary mode indication function will exhibit a local minimum or maximum at each of the natural frequencies of the system under test. The secondary mode indication function will exhibit a local minimum or maximum at repeated or pseudo-repeated roots of order two or more. Further mode indication functions yield local minima or maxima for successively higher orders of repeated or pseudo-repeated roots of the system under test.

3.4.4.1. Multivariate mode indication function (MvMIF). The MvMIF is based upon finding a force vector $\{\mathbf{F}\}$ that will excite a normal mode at each frequency in the frequency range of interest [28]. If a normal mode can be excited at a particular frequency, the response to such a force vector will exhibit the 90° phase lag characteristic. Therefore, the real part of the response will be as small as possible particularly when compared to the imaginary part or the total response. In order to evaluate this possibility, a minimization problem can be formulated as

$$\min_{\|F\|=1} \frac{\{\mathbf{F}\}^{\mathrm{T}}[\mathbf{H}_{\mathrm{Real}}]^{\mathrm{T}}[\mathbf{H}_{\mathrm{Real}}]\{\mathbf{F}\}}{\{\mathbf{F}\}^{\mathrm{T}}([\mathbf{H}_{\mathrm{Real}}]^{\mathrm{T}}[\mathbf{H}_{\mathrm{Real}}] + [\mathbf{H}_{\mathrm{Imag}}]^{\mathrm{T}}[\mathbf{H}_{\mathrm{Imag}}])\{\mathbf{F}\}} = \lambda. \tag{26}$$

This minimization problem is similar to a Rayleigh quotient and it can be shown that the solution to the problem is found by finding the smallest eigenvalue λ_{min} and the corresponding eigenvector $\{\mathbf{F}\}_{min}$ of the problem

$$[\mathbf{H}_{\mathrm{Real}}]^{\mathrm{T}}[\mathbf{H}_{\mathrm{Real}}]\{\mathbf{F}\} = \lambda([\mathbf{H}_{\mathrm{Real}}]^{\mathrm{T}}[\mathbf{H}_{\mathrm{Real}}] + [\mathbf{H}_{\mathrm{Imag}}]^{\mathrm{T}}[\mathbf{H}_{\mathrm{Imag}}])\{\mathbf{F}\}. \tag{27}$$

The above eigenvalue problem is formulated at each frequency in the frequency range of interest. Note that the result of the matrix product $[\mathbf{H}_{\mathrm{Real}}]^{\mathrm{T}}[\mathbf{H}_{\mathrm{Real}}]$ and $[\mathbf{H}_{\mathrm{Imag}}]^{\mathrm{T}}[\mathbf{H}_{\mathrm{Imag}}]$ in each case is a square, real-valued matrix of a size equal to the number of references in the measured data $N_i \times N_i$. The resulting plot of a mode indication function for a seven reference case can be seen in Figure 5.

3.4.4.2. Complex mode indication function (CMIF). An algorithm based on singular value decomposition (SVD) methods applied to multiple reference FRF measurements, identified as the CMIF, was first developed for traditional FRF data in order to identify the proper number of modal frequencies, particularly when there are closely spaced or repeat modal frequencies [21]. Unlike the MvMIF, which indicates the existence of real

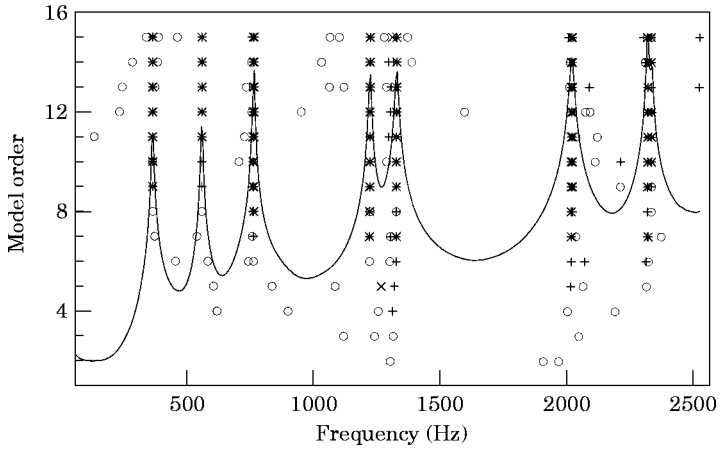

Figure 4. Model order determination: stability diagram. ○, Not stable; +, stable frequency; ×, stable pole; *, stable pole and vector.

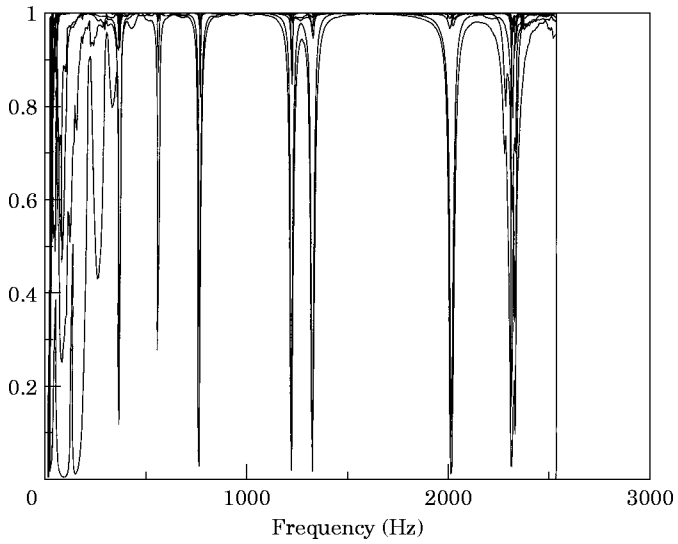

Figure 5. Multivariate mode indication function.

normal modes, CMIF indicates the existence of real normal or complex modes and the relative magnitude of each mode. Furthermore, MvMIF yields a set of force patterns that can best excite the real normal mode, while CMIF yields the corresponding mode shape and/or modal participation vector.

The CMIF is defined as a set of plots, normally in log format, of the singular values of the frequency response function matrix across a frequency range of interest. Each plot represents the nth singular value found for each of the discrete frequencies in the frequency range of interest. The peaks detected in the CMIF indicate the existence of modes, and the corresponding frequencies of these peaks give the damped natural frequencies for each mode. When peaks occur in the first (largest) and successive CMIF plots at the same frequency, this is an indication of a repeat root. In the application of CMIF to traditional modal parameter estimation algorithms, the number of modes detected in CMIF determines the minimum number of degrees of freedom of the system equation for the algorithm. A number of additional degrees of freedom may be needed to take care of residual effects and noise contamination.

$$[\mathbf{H}(\omega)] = [\mathbf{U}(\omega)][\mathbf{\Sigma}(\omega)][\mathbf{V}(\omega)]^H, \tag{28}$$

where N_e is the number of effective modes; the effective modes are the modes that contribute to the response of the structure at this particular frequency ω, $[\mathbf{U}(\omega)]$ is the left singular matrix of size $N_o \times N_e$, which is an unitary matrix; $[\mathbf{\Sigma}(\omega)]$ is the singular value matrix of size $N_e \times N_e$, which is a diagonal matrix; $[\mathbf{V}(\omega)]$ is the right singular matrix of size $N_d \times N_i$, which is also an unitary matrix.

Figure 6 represents a typical CMIF for a multiple reference set of data. It must be noted that not all peaks in CMIF indicate modes. Errors such as noise, leakage, non-linearities and/or a cross-over effects can also make a peak. The cross-over effect is due to the way the CMIF is often plotted.

Since the mode shapes that contribute to each peak do not change much around each peak, several adjacent spectral lines from the FRF matrix can be used simultaneously for a better estimation of mode shapes. By including several spectral lines of data in the

singular value decomposition calculation, the effect of some measurement errors can be minimized.

3.4.5. *Rank estimation*

A more recent model order evaluation technique involves the estimate of the rank of the matrix of measured data. An estimate of the rank of the matrix of measured data gives a good estimate of the model order of the system. Essentially, the rank is an indicator of the number of independent characteristics contributing to the data. While the rank cannot be calculated in an absolute sense, the rank can be estimated from the singular value decomposition (SVD) of the matrix of measured data. For each mode of the system, one singular value should be found by the SVD procedure. The SVD procedure finds the largest singular value first and then successively finds the next largest. The magnitude of the singular values are used in one of two different procedures to estimate the rank. The concept that is used is that the singular values should go to zero when the rank of the matrix is exceeded. For theoretical data, this will happen exactly. For measured data, due to random errors and small inconsistencies in the data, the singular values will not become zero but will be very small. Therefore, the rate of change of the singular values is used as an indicator rather than the absolute values. In one approach, each singular value is divided by the first (largest) to form a normalized ratio. This normalized ratio is treated much like the error chart and the appropriate rank (model order) is chosen when the normalized ratio approaches an asymptote. In another similar approach, each singular value is divided by the previous singular value forming a normalized ratio that will be approximately equal to one if the successive singular values are not changing magnitude. When a rapid decrease in the magnitude of the singular value occurs, the ratio of successive singular values drops (or peaks if the inverse of the ratio is plotted) as an indicator of rank (model order) of the system. Figure 6 shows examples of these rank estimate procedures.

4. CURRENT MODAL IDENTIFICATION METHODS

Using the concepts developed in the previous section, the most commonly used modal identification methods are summarized in Table 2. The high order model is typically used

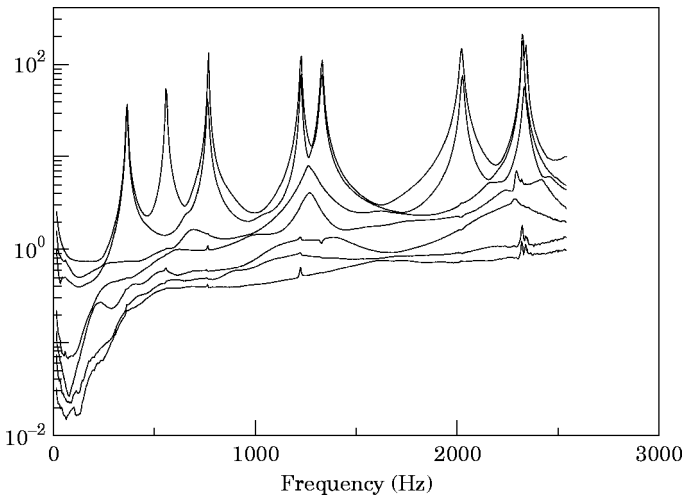

Figure 6. Complex mode indication function.

TABLE 2

Summary of modal parameter estimation algorithms

	Domain		Matrix polynomial order			Coefficients	
Algorithm	Time	Freq	Zero	Low	High	Scalar	Matrix
CEA	●				●	●	
LSCE	●				●	●	
PTD	●				●		$N_i \times N_i$
ITD	●			●			$N_o \times N_o$
MRITD	●			●			$N_o \times N_o$
ERA	●			●			$N_o \times N_o$
PFD		●		●			$N_o \times N_o$
SFD		●		●			$N_o \times N_o$
MRFD		●		●			$N_o \times N_o$
RFP		●			●	●	Both
OP		●			●	●	Both
CMIF		●	●				$N_o \times N_i$

for those cases where the system is undersampled in the spatial domain. For example, the limiting case is when only one measurement is made on the structure. For this case, the left side of the general linear equation corresponds to a scalar polynomial equation with the order equal to or greater than the number of desired modal frequencies. This type of high order model will yield significant numerical problems for the frequency domain case, unless the model order and/or frequency information is severely limited.

The low order model is used for those cases where the spatial information is complete. In other words, the number of physical co-ordinates is greater than the number of desired modal frequencies. For this case, the order of the left side of the general linear equations

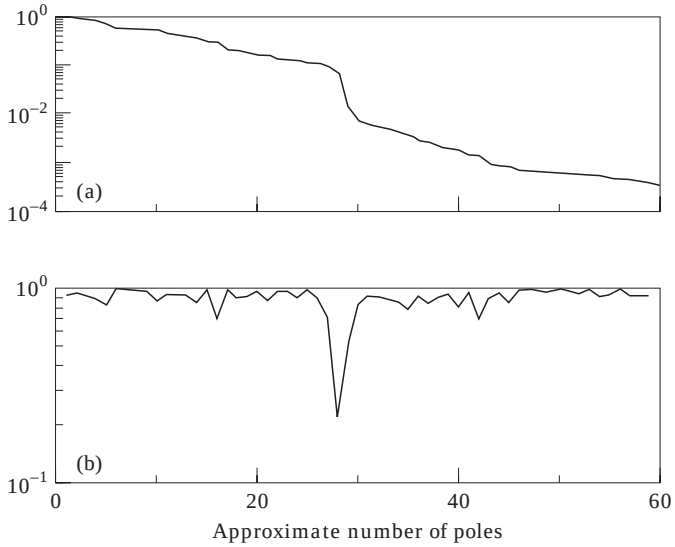

Figure 7. Model order determination: rank estimation. (a) Normalized singular values and (b) ratio of successive singular values for model order 15.

(10) or (13) is equal to one or two. Low order model methods are numerically stable in both time and frequency domain formulations but require more spatial data acquired in a consistent fashion.

The zero order model corresponds to a case where the temporal information is neglected and only the spatial information is used. These methods directly estimate the modal vectors as a first step. In general, these methods are programmed to process data at a single temporal condition (frequency or time). In this case, the method is essentially equivalent to the SDOF methods which have been used with frequency response functions. In other words, the zeroth order matrix polynomial model compared to the higher order matrix polynomial models is similar to the comparison between the SDOF and MDOF methods used historically in modal parameter estimation.

5. SUMMARY—FUTURE TRENDS

Modal parameter estimation is probably one of the most misunderstood aspects of the experimental modal analysis process. Since most modal parameter estimation methods are mathematically intimidating, many users do not fully understand the ramifications of the decisions made during the measurement stages as well as later in the modal parameter estimation process. Ideally, by consolidating the conceptual approach and unifying the theoretical development of modal identification algorithms, increased understanding, with respect to general advantages and disadvantages of different algorithms, can be achieved. This sort of overview of modal parameter estimation can be used simply as a guide toward further study and understanding of the details of the individual modal identification algorithms.

Immediate future trends in modal identification will respond to those situations that cannot be adequately solved today. First of all, specific attention will be given to methodology needed to estimate modal parameters for heavily damped systems, particularly systems with significant modal density. Second, since the measured data used for most modal identification algorithms yields a highly overdetermined solution, increased attention will be given to estimating the statistical information that describes the uncertainty associated with each modal parameter estimate. Finally, in order to address needs of traditional modal identification and needs of control-structure interaction, modal identification algorithms need to be developed that can easily incorporate known or fixed modal parameters into a solution for remaining unknown modal parameters.

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APPENDIX A: MODAL FREQUENCY PARTICIPATION—TIME DOMAIN ALGORITHMS

Typical algorithms:

High order algorithms
- complex exponential (CE)
- least squares complex exponential (LSCE)
- polyreference time domain (PTD)

Low order algorithms
- Ibrahim time domain (ITD)
- multiple-reference time domain (MRITD)
- eigensystem realization algorithm (ERA)

General linear equation formulation:
High order
$m \geqslant 2N/N_i$ or $mN_i \geqslant 2N$
matrix coefficients ($N_i \times N_i$) when $N_o \gg N_i$
measurement matrix $[\mathbf{h}(t_k)]$ needs to be transposed when $N_i \gg N_o$

Low order
$m = 1$ or $m = 2$ with $2N_o = 2N$
matrix coefficients ($2N_o \times 2N_o$ or $N_o \times N_o$) when $N_o \gg N_i$
measurement matrix $[\mathbf{h}(t_k)]$ needs to be transposed when $N_o \gg N_i$

Basic equation:

$$[[\mathbf{h}(t_0)] \quad [\mathbf{h}(t_1)] \quad \cdots \quad [\mathbf{h}(t_{m-1})] \begin{bmatrix} [\alpha_0] \\ [\alpha_1] \\ \vdots \\ [\alpha_{m-1}] \end{bmatrix} = -[\mathbf{h}(t_m)], \tag{A.1}$$

where

$$[\mathbf{h}(t_k)]_{N_o \times N_i} = [\mathbf{h}_{p1}(t_{i+k}) \quad \mathbf{h}_{p2}(t_{i+k}) \quad \mathbf{h}_{p3}(t_{i+k}) \quad \cdots \quad h_{pq}(t_{i+k})].$$

APPENDIX B: MODAL FREQUENCY PARTICIPATION—FREQUENCY DOMAIN ALGORITHMS

Typical algorithms:

High order	Low order
Rational fraction polynomial (RFP)	simultaneous frequency domain (SFD)
Orthogonal polynomial (OP)	multi-reference frequency domain (MRFD)
	frequency domain equivalent to ITD, MRITD, ERA
	polyreference frequency domain (PFD)

General linear equation formulation:
High order
$m \geqslant 2N/N_i$ or $mN_i \geqslant 2N$
matrix coefficients ($N_i \times N_i$) when $N_o \gg N_i$
measurement matrix $[H(\omega_k)]$ needs to be transposed when $N_i \gg N_o$
Low order
$m = 1$ or $m = 2$ with $2N_o \geqslant 2N$
matrix coefficients ($2N_o \times 2N_o$ or $N_o \times N_o$) when $N_o \gg N_i$
measurement matrix $[\mathbf{H}(\omega_k)]$ needs to be transposed when $N_o \gg N_i$

Basic equation:

$$[[\hat{\mathbf{H}}] \quad [\hat{\mathbf{R}}]] \begin{bmatrix} [\alpha_0] \\ [\alpha_1] \\ \vdots \\ [\alpha_{m-1}] \\ [\beta_0] \\ [\beta_1] \\ \vdots \\ [\beta_n] \end{bmatrix} = -(\mathrm{j}\omega_0)^m [\mathbf{H}(\omega_0)], \tag{B.1}$$

where

$$[\hat{\mathbf{H}}] = [[\mathbf{H}(\omega_0)(\mathrm{j}\omega_0)^1[\mathbf{H}(\omega_0)](\mathrm{j}\omega_0)^2[\mathbf{H}(\omega_0)] \cdots (\mathrm{j}\omega_0)^{m-1}[\mathbf{H}(\omega_0)],$$

$$[\hat{\mathbf{R}}] = [-[\mathbf{R}] - (\mathrm{j}\omega_0)^1[\mathbf{R}] - (\mathrm{j}\omega_0)^2[\mathbf{R}] \cdots - (\mathrm{j}\omega_0)^n[\mathbf{R}]],$$

$$[\mathbf{H}(\omega_k)]_{N_o \times N_i} = [[\mathbf{H}_{p1}(\omega_k)] \quad [\mathbf{H}_{p2}(\omega_k)] \quad \cdots \quad [\mathbf{H}_{pp}(\omega_k)] \quad \cdots \quad [\mathbf{H}_{pq}(\omega_k)]],$$

$$[\mathbf{R}]_{N_o \times N_i} = [[\mathbf{0}] \quad [\mathbf{0}] \quad \cdots \quad [\mathbf{I}] \quad \cdots \quad [\mathbf{0}]].$$

APPENDIX C: COMPANION MATRIX—MODEL PARTICIPATION VECTORS

Companion matrix solution:

$$[\mathbf{C}] = \begin{bmatrix} -[\boldsymbol{\alpha}_{m-1}] & -[\boldsymbol{\alpha}_{m-2}] & -[\boldsymbol{\alpha}_{m-3}] & \cdots & \cdots & -[\boldsymbol{\alpha}_2] & -[\boldsymbol{\alpha}_1] & -[\boldsymbol{\alpha}_0] \\ [\mathbf{I}] & [\mathbf{0}] & [\mathbf{0}] & \cdots & \cdots & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{I}] & [\mathbf{0}] & \cdots & \cdots & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{I}] & \cdots & \cdots & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & \cdots & \cdots & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & \cdots & \cdots & [\mathbf{I}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & \cdots & \cdots & [\mathbf{0}] & [\mathbf{I}] & [\mathbf{0}] \end{bmatrix}, \tag{C.1}$$

$$[\mathbf{C}]\{\mathbf{X}\} = \lambda\{\mathbf{X}\}, \quad [\mathbf{C}]\{\mathbf{X}\} = \lambda[\mathbf{I}]\{\mathbf{X}\}. \tag{C.2, 3}$$

The eigenvectors that can be found from the eigenvalue–eigenvector solution utilizing the companion matrix may, or may not, be useful in terms of modal parameters. The eigenvector that is found, associated with each eigenvalue, is of length model order times matrix coefficient size. In fact, the unique (meaningful) portion of the eigenvector is of length equal to the size of the coefficient matrices and is repeated in the eigenvector a model order number of times. Each time the unique portion of the eigenvector is repeated, the unique portion is multiplied by a scalar multiple of the associated modal frequency.

Therefore, the eigenvectors of the companion matrix have the form

$$\{\boldsymbol{\phi}\}_r = \begin{Bmatrix} \lambda_r^m \{\boldsymbol{\psi}\}_r \\ \vdots \\ \lambda_r^2 \{\boldsymbol{\psi}\}_r \\ \lambda_r^1 \{\boldsymbol{\psi}\}_r \\ \lambda_r^0 \{\boldsymbol{\psi}\}_r \end{Bmatrix}_r, \tag{C.4}$$

where $\{\boldsymbol{\phi}\}_r$ is the rth eigenvector of the companion matrix [**C**] associated with eigenvalue λ_r, and $\{\boldsymbol{\psi}\}_r$ is the rth modal vector of the physical system associated with modal frequency λ_r.

APPENDIX D: MODAL RESIDUE—MODAL SCALING ALGORITHMS

Time domain estimation (multiple references):

$$\{\mathbf{h}_{pq}(t_i)\}_{N_i \times 1} = [\mathbf{L}]_{N_i \times 2N} \lceil \mathrm{e}^{\lambda_r t_i} \rfloor_{2N \times 2N} \{\boldsymbol{\psi}\}^{\mathrm{T}}_{2N \times 1}, \tag{D.1}$$

where N_t = number of time points, $N_t \geqslant 2N$; the above equation is repeated N_t times,

$$\{\mathbf{h}_{pq}(t_i)\} = \begin{Bmatrix} \mathbf{h}_{p1}(t_i) \\ \mathbf{h}_{p2}(t_i) \\ \mathbf{h}_{p3}(t_i) \\ \vdots \\ \mathbf{h}_{pq}(t_i) \end{Bmatrix}; \qquad [\mathbf{e}^{\lambda t_i}] = \begin{bmatrix} \mathrm{e}^{\lambda_1 t_i} & 0 & 0 & \cdots & 0 \\ 0 & \mathrm{e}^{\lambda_2 t_i} & 0 & \cdots & 0 \\ 0 & 0 & \mathrm{e}^{\lambda_3 t_i} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathrm{e}^{\lambda_{2N} t_i} \end{bmatrix}. \tag{D.2, 3}$$

The residues are calculated from the modal participation vectors **L** and the modal coefficients ψ ($A_{pqr} = L_{qr}\psi_{pr}$). Note that if one column q of the modal participation matrix **L** is normalized to unity, the modal coefficients that are found will be equal to the residues for that reference A_{pqr}.

$$\{\boldsymbol{\psi}_{pr}\} = \begin{Bmatrix} \psi_{p1} \\ \psi_{p2} \\ \psi_{p3} \\ \vdots \\ \psi_{p2N} \end{Bmatrix}, \qquad [\mathbf{L}_{rq}] = \begin{bmatrix} L_{11} & L_{12} & L_{13} & \cdots & L_{12N} \\ L_{21} & L_{22} & L_{23} & \cdots & L_{22N} \\ L_{31} & L_{32} & L_{33} & \cdots & L_{32N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{N_i 1} & L_{N_i 2} & L_{N_i 3} & \cdots & L_{N_i 2N} \end{bmatrix}. \tag{D.4, 5}$$

APPENDIX E: MODAL RESIDUE—MODAL SCALING ALGORITHMS

Frequency domain estimation (multiple references):

$$\{\mathbf{H}_{pq}(\omega_i)\}_{N_i \times 1} = [\mathbf{L}]_{N_i \times 2N} \left\lceil \frac{1}{\mathrm{j}\omega_i - \lambda_r} \right\rfloor_{2N \times 2N} \{\boldsymbol{\psi}\}^{\mathrm{T}}_{2N \times 1}, \tag{E.1}$$

where N_s = number of spectral lines, $N_s \geqslant 2N$; the above equation is repeated N_s times.

$$\{\mathbf{H}_{pq}(\omega_i)\} = \begin{Bmatrix} H_{p1}(\omega_i) \\ H_{p2}(\omega_i) \\ H_{p3}(\omega_i) \\ \vdots \\ H_{pq}(\omega_i) \end{Bmatrix}, \tag{E.2}$$

$$\left\lceil \frac{1}{j\omega_i - \lambda_r} \right\rfloor = \begin{bmatrix} 1/(j\omega_i - \lambda_1) & 0 & 0 & \cdots & 0 \\ 0 & 1/(j\omega_i - \lambda_2) & 0 & \cdots & 0 \\ 0 & 0 & 1/(j\omega_i - \lambda_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1/(j\omega_i - \lambda_{2N}) \end{bmatrix}. \tag{E.3}$$

The residues are calculated from the modal participation vectors $\mathbf{L}$ and the modal coefficients ψ ($A_{pqr} = L_{qr}\psi_{pr}$). Note that if one column q of the modal participation matrix $\mathbf{L}$ is normalized to unity, the modal coefficients that are found will be equal to the residues for that reference A_{pqr}.

$$\{\boldsymbol{\psi}_{pr}\} = \begin{Bmatrix} \psi_{p1} \\ \psi_{p2} \\ \psi_{p3} \\ \vdots \\ \psi_{p2N} \end{Bmatrix}, \qquad [\mathbf{L}_{rq}] = \begin{bmatrix} L_{11} & L_{12} & L_{13} & \cdots & L_{12N} \\ L_{21} & L_{22} & L_{23} & \cdots & L_{22N} \\ L_{31} & L_{32} & L_{33} & \cdots & L_{32N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{N_i1} & L_{N_i2} & L_{N_i3} & \cdots & L_{N_i2N} \end{bmatrix}. \tag{E4, 5}$$