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REANALYSIS OF DAMPED STRUCTURES USING THE SINGLE STEP PERTURBATION METHOD

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The single step perturbation method is a recently developed structural dynamic modification technique. In the present work, this has been applied to complex structures. A typical machine tool structure in form of an F structure is considered and eigenvalue and response reanalysis are carried out for this structure when modified by constrained viscoelastic damping layer treatment. A comparison of results is made for cases using the single step perturbation method, with the erstwhile perturbation method and the re-solved results, i.e., results obtained by repeated solution. Comparison of computational times taken are also indicated. A variation of coverage ratio is considered for the above structure and the variation in natural frequencies and loss factors in case of free vibration, variation in resonant frequencies and amplitudes in case of forced vibration are also studied.

1. INTRODUCTION

Structural dynamic modification or reanalysis methods are methods by which the dynamic characteristics of a modified structure can be obtained from those of the original structure, without repeated analytical solution (re-solution) or experimentation. Brandon [1] discusses strategies for structural dynamic modification.

Perturbation method is a useful and reasonably accurate structural dynamic modification method. Chen *et al*. [2] have given a matrix perturbation method for vibration modal analysis. Shen and Stevens [3] have applied the perturbation method for eigenfrequencies and loss factors of free damping layer treated beams. The eigenvalue reanalysis based on the perturbation method is applied to a sandwich beam with a viscoelastic core by the authors [4]. This modal perturbation method is referred to as multi-step perturbation method, as it considers the structural modification as a number of smaller steps. To and Ewins [5] proposed a non-linear sensitivity analysis for the revised modal properties in structural modification analysis. Recently a new perturbation method named the single step perturbation method has been proposed by the authors [6]. This method has been applied to beams modified with constrained viscoelastic damping layer treatment and found to be more efficient and accurate. In the present work, eigenvalue reanalysis using the single step perturbation method is applied to complex structures, with a typical example of an F structure, for constrained damping layer treatment. A response reanalysis algorithm was developed by the authors [7] and was applied to viscoelastically damped structures.

In the present work, the response reanalysis method has been modified by incorporating single step eigenperturbation, and the same is applied to viscoelastically damped structures. It will be shown that reanalysis based on single step perturbation is more efficient and accurate than reanalysis based on the multi-step perturbation method.

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2. THEORETICAL BACKGROUND

The theory of the reanalysis method and the sandwich beam element are briefly outlined here for the sake of completeness of the present work.

2.1. THE MULTI-STEP PERTURBATION METHOD

As mentioned earlier, the perturbation method that proceeds in small steps or increments of modification will be termed the multi-step perturbation method, to differentiate it from the single step perturbation method, being highlighted in the present work. The equations for the multi-step perturbation method are re-produced below from reference [3]. Changes in the ith eigenvalue, $\Delta\lambda_i$, and the ith eigenvector, $\{\Delta\boldsymbol{\psi}\}_i$, for given changes in stiffness and mass matrices are

$$\Delta\lambda_i \approx [\{\boldsymbol{\psi}\}_i^{\mathrm{T}}[\Delta\mathbf{K} - \lambda_i\Delta\mathbf{M}]\{\boldsymbol{\psi}\}_i]/\{\boldsymbol{\psi}\}_i^{\mathrm{T}}[\mathbf{M}]\{\boldsymbol{\psi}\}_i \tag{1}$$

and

$$\{\Delta\boldsymbol{\psi}\}i \approx \sum_{\substack{j=1 \\ i\neq j}}^{n} p_{ij}\{\boldsymbol{\psi}\}_j, \tag{2}$$

where

$$p_{ij} = [\{\boldsymbol{\psi}\}_j^{\mathrm{T}}[\Delta\mathbf{K} - \lambda_i\Delta\mathbf{M}]\{\boldsymbol{\psi}\}_i]/(\lambda_i - \lambda_j)\{\boldsymbol{\psi}\}_i^{\mathrm{T}}[\mathbf{M}]\{\boldsymbol{\psi}\}_i.$$

2.2. RESPONSE REANALYSIS ALGORITHM

The response reanalysis algorithm [7] proceeds as follows: Perturbation equations are applied to the system for the modification considered. The first m dominant modes only may be considered for the perturbation and further reanalysis. Then, the mode shapes (eigenvectors) thus obtained by perturbation are used for the response calculation by mode summation using the modal analysis equations. The flow diagram for response reanalysis is given in Figure 1.

2.3. SINGLE STEP PERTURBATION

The equations for the single step perturbation method [6] are given in this section. The change in the ith eigenvalue is given by

$$\Delta\lambda_i = [\{\boldsymbol{\psi}\}_i^{\mathrm{T}}[\Delta\mathbf{K} - \lambda_i\Delta\mathbf{M}]\{\dot{\boldsymbol{\psi}}\}_i]/\{\boldsymbol{\psi}\}_i^{\mathrm{T}}[\mathbf{M} + \Delta\mathbf{M}]\{\boldsymbol{\psi}\}_i. \tag{3}$$

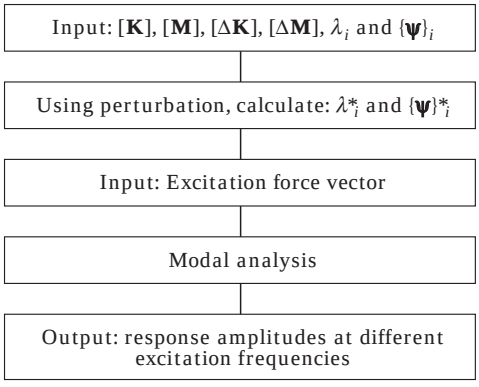

Figure 1. Flow diagram: response reanalysis method.

The change in the ith eigenvector is given by

$$\{\Delta\boldsymbol{\psi}\}_i \approx \sum_{\substack{j=1 \\ i \neq j}}^{n} p_{ij}\{\boldsymbol{\psi}\}_j, \tag{4}$$

where

$$p_{ij} = [\{\boldsymbol{\psi}\}_j^{\mathrm{T}}[\Delta\mathbf{K} - \lambda_i^*\Delta\mathbf{M}]\{\boldsymbol{\psi}\}_i]/(\lambda_i^* - \lambda_j)\{\boldsymbol{\psi}\}_i^{\mathrm{T}}[\mathbf{M} + \Delta\mathbf{M}]\{\boldsymbol{\psi}\}_i.$$

In the single step perturbation method, the entire change in the system is considered in a single step and changes in eigenvalues are obtained and the eigenvalues are updated. These updated eigenvalues are used in calculating the changes in eigenvectors using equation (4). Thus equations (3) and (4) are repeated each time updating eigenvalues and eigenvectors in turn until convergence is obtained. It is observed that in the cases considered one iteration has been found to be accurate enough and in subsequent iterations, the variation is less than 0·1%. The response reanalysis algorithm described above is now modified by incorporating the single step perturbation method at the second step in Figure 1.

The differences between the multi-step perturbation and the single step perturbation are as follows: (1) In the multi-step perturbation, the quantum of modification is divided into a number of smaller steps, and the perturbation proceeds step by step, for each step modifying the eigenvalues and eigenvectors. So, a single large modification is considered as a series of smaller modifications; for each smaller modification, the previous modified system being considered as the base for next perturbation. Whereas, in the single step perturbation, the entire modification is treated as a single step, and the method proceeds in iterations. The original eigenvector is used to update the eigenvalue and these updated eigenvalues are used to update the eigenvector and they in turn are used to update the eigenvalues and so on until convergence. In other words, the multi-step perturbation proceeds step-wise whereas the single step perturbation proceeds in iterations.

(2) The expressions for multi-step and single step perturbations may look similar. However, the derivations are based on different approximations. The approximation used in deriving the expression for multi-step perturbation is that the higher order terms of series expressed for eigenvalue and eigenvector changes are neglected, whereas such approximation is not made for the single step perturbation. The approximation used for single step perturbation is that the mode shape does not change much, and to start with the mode shape of original structure can be used to get changes in eigenvalues due to modification.

(3) The limitation of the multi-step perturbation method is that errors due to approximation are cumulative and hence for large modifications this method becomes inaccurate. Whereas, in the single step perturbation method, the errors due to approximation are not cumulative. But this method may not give accurate results if the modification results in large deviations in mode shapes.

2.4. THE SANDWICH BEAM ELEMENT

The energy expressions for a sandwich beam [9, 10], considering all layers as elastic, and including extensional energies in the outer layers and shear energy in the core

are, for strain energy,

$$U = \frac{1}{2}\left[q \int_0^L w''^2 \, \mathrm{d}x + r \int_0^L u'^2 \, \mathrm{d}x + s \int_0^L \gamma_2^2 \, \mathrm{d}x \right], \tag{5}$$

and for kinetic energy,

$$T = \frac{1}{2}\left[m \int_0^L \dot{w}^2 \, \mathrm{d}x + \sum_{i=1}^{3} \int_0^L b\rho_i (\dot{u}_i^2 t_i + \dot{\bar{u}}^2 t_i^2/12) \right], \tag{6}$$

where $q = (E_1 b t_1^3 + E_3 b t_3^3)/12$, $r = (E_1 b t_1 + E_3 b t_3 e^2)$, $s = G_2 b t_2$ and $m = (\rho_1 t_1 + \rho_2 t_2 + \rho_3 t_3)b$. The following relations relate u_1, u_2, u_3, $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$ and γ_2 in terms of w, w', α:

$$u_1 = (t_2\alpha + t_a w')/(1+e), \quad u_3 = -u_1 e, \qquad \gamma_2 = (\alpha - w'), \qquad \bar{u}_1 = \bar{u}_3 = w',$$

$$u_2 = (u_1 + u_3)/t_2 + w' t_a, \qquad \bar{u}_2 = (u_1 - u_3)/t_2,$$

where,

$$e = (E_1 t_1/E_3 t_3), \qquad t_a = (t_1 + t_3)/2.$$

Substituting the above relations into equations (5) and (6) and simplifying,

$$U = \frac{1}{2}\left[k_1 \int_0^L w^2 \, \mathrm{d}x + k_2 \int_0^L w'^2 \, \mathrm{d}x + k_3 \int_0^L w''^2 \, \mathrm{d}x + k_4 \int_0^L \alpha^2 \, \mathrm{d}x \right.$$

$$\left. + k_5 \int_0^L \alpha'^2 \, \mathrm{d}x + k_6 \int_0^L w'\alpha \, \mathrm{d}x + k_7 \int_0^L w''\alpha' \, \mathrm{d}x \right]. \tag{7}$$

The kinetic energy is obtained by replacing k_1 to k_7 in equation (7) by m_1 to m_7 and differentiating with respect to time. The various parameters are,

$$k_1 = 0, \qquad k_2 = G_2 b t_2, \qquad k_3 = (E_1 b t_1^3 + E_3 b t_3^3)/12 + C_k t_2^2,$$

$$k_4 = G_2 b t_2, \qquad k_5 = C_k t_2^2, \qquad k_6 = -G_2 b t_2, \qquad k_7 = C_k t_2 t_a,$$

$$m_1 = b(\rho_1 t_1 + \rho_2 t_2 + \rho_3 t_3), \qquad m_2 = C_m t_a^2/12, \qquad m_3 = 0,$$

$$m_4 = b\rho_2 t_2^3/12 + C_m t_2^2, \qquad m_5 = 0, \qquad m_6 = C_m t_2 t_a, \qquad m_7 = 0.$$

C_k and C_m are given by $C_k = b(E_1 t_1 + E_3 t_3 e^2)/(1+e)^2$ and $C_m = b(\rho_1 t_1 + \rho_3 t_3 e^2)/(1+e)^2$.

Displacements w and α are expressed in terms of nodal displacements and shape functions. These are substituted into the energy expressions and by the minimization of potential and kinetic energies, stiffness and mass matrices are obtained by the usual variational approach. The shape functions used are beam bending shape functions for w and axial bar shape functions for α which are available in the literature, e.g., Zienkewicz [11]. The matrices derived as above are given in reference [7].

For a viscoelastic core the shear modulus is taken as complex according to the correspondence principle of linear viscoelasticity, $\mathbf{G}_2 = \mathrm{G}^*(1 + \mathrm{i}\beta)$, G* being the in-phase shear modulus and β the material loss factor. Using this complex shear modulus in the above expressions, a complex stiffness matrix is obtained for each element which accounts for the damping in the viscoelastic layer. This results in complex eigenvalues for the system, the real part of which gives the square of natural frequency. The ratio of the imaginary

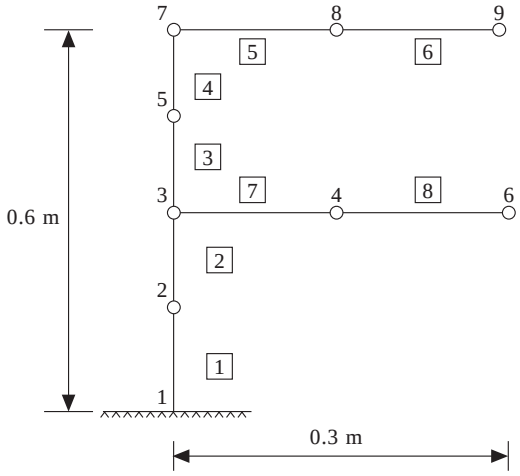

Figure 2. The F structure.

part to the real part of the complex eigenvalue gives the system loss factor η in that mode of vibration.

2.5. TRANSFORMATION MATRICES

The above derivation for sandwich beam elements applies to elements with single orientation. When a structure consists of members with different orientations, the axial terms of the matrix should also be considered. The equivalent section and mass are to be considered in the matrices of the axial bar available in literature [11], and the corresponding rows and columns are introduced in the sandwich beam matrices in the respective quadrants of the matrix. This results in stiffness and mass matrices of size 8×8. In order to take care of different orientations of the elements, a global co-ordinate system has to be defined, into which all the matrices in the local co-ordinates have to be transformed. The 8×8 transformation matrix for a sandwich beam element is

$$\begin{bmatrix} \mathbf{R} & 0 \\ 0 & \mathbf{R} \end{bmatrix}, \qquad [\mathbf{R}] = \begin{bmatrix} \cos\varepsilon & \sin\varepsilon & 0 & 0 \\ -\sin\varepsilon & \cos\varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where ε is the inclination of the axis of the sandwich beam element with the global x-axis. Using this transformation matrix, the sandwich beam matrices in local co-ordinates are transformed into global co-ordinates by postmultiplying the stiffness and mass matrices by the transformation matrix and premultiplying with the transpose of the same.

3. RESULTS AND DISCUSSION

The methods described above are applied to a machine tool structure in a model simplified as an F structure, for the modifications by constrained viscoelastic damping layer treatment. The stiffness and mass matrices for the F structure are obtained by assembling the transformed elemental stiffness and mass matrices of the individual elements. The dimensions of the structure and the finite element discretization are shown in Figure 2. Numbers in boxes show the element numbers and free numbers indicate

node numbers. The cross-sectional details and material properties are: $b = 0{\cdot}05$ m, $t_1 = 0{\cdot}05$ m, $t_3 = t_2 = t_1 \times TR$, where TR = thickness ratio, $E_1 = E_3 = 2{\cdot}07 \times 10^{11}$ N/m^2, $G_2 = 1{\cdot}96 \times 10^{11}$ N/m^2, $\beta = 0{\cdot}6$, and $\rho_1 = 6\rho_2 = \rho_3 = 7860$ kg/m^3.

Because of constrained damping layer treatment additional degrees of freedom, viz., those of α, have to be considered in the matrices. These are later reduced using Guyan's reduction algorithm [8], so as to be compatible with the matrices of the original structures, where α terms do not appear.

3.1. VARIATION OF THICKNESS RATIO

The thickness ratio is varied from 0·2–10 in the order 0·2, 0·4, 0·8, 1, 2, 5, 10 and in each case, the eigenvalue and response reanalysis are performed, using the single step perturbation method to obtain natural frequencies and system loss factors, resonant frequencies and amplitude ratios. Unit base displacement excitation is considered and the amplitude ratio is taken as ratio of displacement amplitude at the top free end of the F structure (Node 9) to the base displacement amplitude. For the sake of comparison, natural frequencies and loss factors, resonant frequencies and amplitude ratios are calculated using re-solution or repeated analytical solution and also by reanalysis based on the multi-step perturbation method. The plots giving the variation of natural frequencies, system loss factors in the first mode, resonant frequencies and response amplitudes for the first resonance, are shown in Figures 3(a) to (d), dashed lines representing re-solved results, dotted lines representing results from reanalysis using the multi-step perturbation method (M.S.P.) and solid lines representing the results from reanalysis based on the single step perturbation (S.S.P.) method. For the multi-step

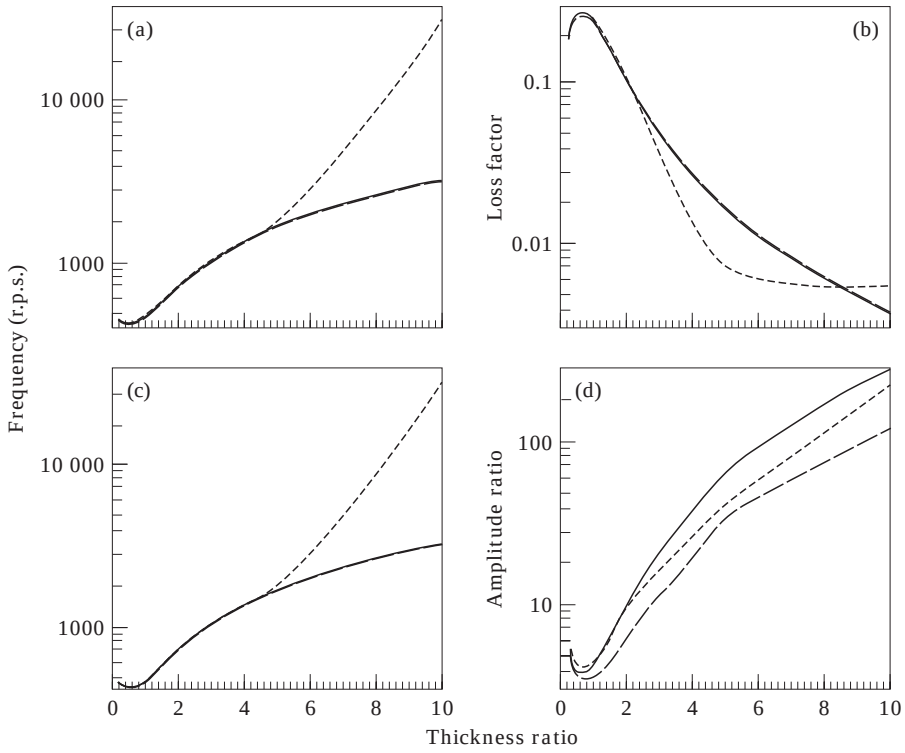

Figure 3. Variation with thickness ratio of (a) natural frequency of first mode. (b) loss factor for first mode, (c) first resonance frequency, (d) amplitude ratio at first resonance. ——, Re-solved; - - - -, M.S.P.; ——, S.S.P.

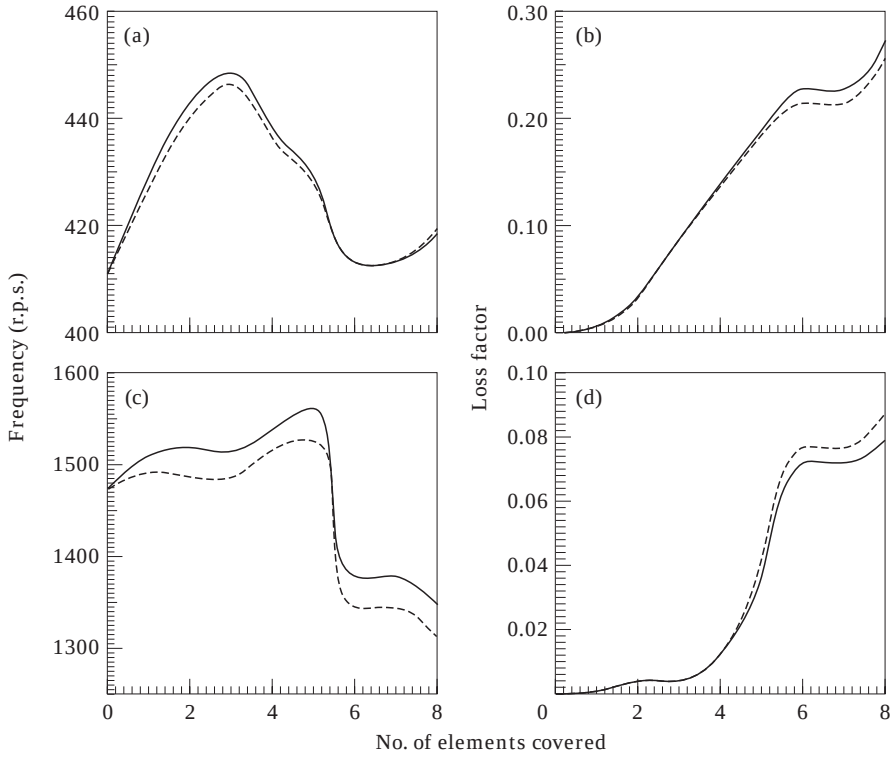

Figure 4. Variation with constrained damping layer coverage of (a) frequency of first mode, (b) loss factor for first mode, (c) frequency of second mode, (d) loss factor for second mode. - - - -, Re-solved; ——, perturbed.

perturbation method, 10 steps are used. For the single step perturbation, only one iteration is used.

It can be observed that the results from reanalysis using single step perturbation are very close and almost coincident with the re-solved results. The results from the multi-step perturbation method, though accurate at low thickness ratios, tend to be inaccurate, as the thickness ratio increases. An increase of the number of steps for multi-step perturbation might have helped to some extent, but any increase in steps is not justified from the consideration of computational time, reduction of which is the main objective of reanalysis.

After crossing an optimum thickness ratio, it can be observed that a further increase in thickness ratio has an adverse effect on damping, i.e., it reduces the loss factor or, equivalently, it increases the response amplitudes.

From the consideration of computational time needed, the single step perturbation method needed one eighth of the time needed for re-solution and one fourth of the time needed for the multi-step perturbation method with five steps, on a PC AT 486, with a numeric processor. The difference would have been a very large amount of CPU seconds for larger structures.

3.2. VARIATION OF CONSTRAINED LAYER COVERAGE

Partial treatment of the structure is considered in this section. Because partial treatment effects the mode shapes of the structures in a different way from a fully treated structure, reanalysis methods have to be verified for partial coverages also as these methods are based

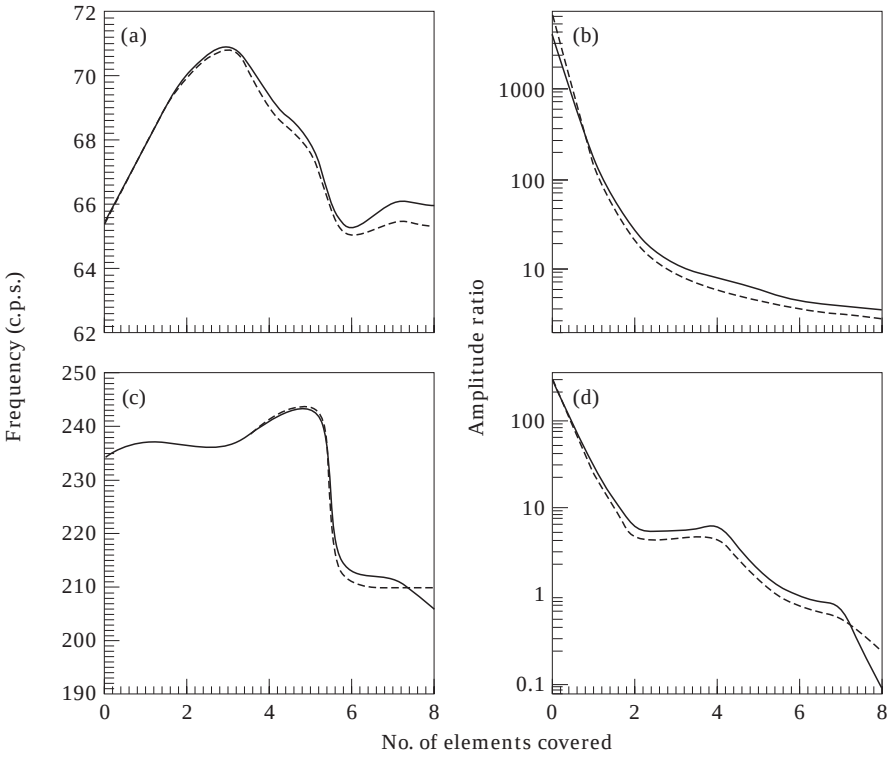

Figure 5. Variation with constrained damping layer coverage of (a) first resonance frequency, (b) amplitude ratio at first resonance, (c) second resonance frequency, (d) amplitude ratio at second resonance. - - - - , Re-solved; ——, re-analyzed.

on updating of eigenvectors. The number of elements covered is increased from one to eight in steps of one in the order of the element numbers shown in Figure 2. And in each case the natural frequencies and system loss factors as well as resonant frequencies and amplitudes for unit base displacement excitation are computed using the reanalysis based on the single step perturbation method. The results of reanalysis as well as those from re-solution for the sake of comparison, are plotted in Figures 4(a) and 4(b), giving the variation of natural frequencies and system loss factors in the first two modes, and in Figure 5(a) to 5(d), giving the variation of resonant frequencies and amplitudes for the first two resonances, with the variation of number of elements covered with constrained damping layer treatment. In these plots, the solid lines show the results of reanalysis (perturbed) and the dotted lines show the re-solved results. It can be once again observed that the reanalysis results are quite accurate in comparison with the re-solved results. It can be seen that, there is no need to cover the entire structure with damping layer treatment, but instead, the same effect can be obtained with less coverage.

4. CONCLUSIONS

Eigenvalue reanalysis and response reanalysis based on single step perturbation method are accurate in comparison with the re-solved results. Also, the accuracy has much improved in comparison with the multi-step perturbation method, whilst the computational time needed has reduced very much. Thus, reanalysis based on the single

step perturbation method is accurate and computationally very efficient and this is illustrated for an F structure modified with viscoelastic damping layer treatment.

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APPENDIX: NOTATION

λ_i, $\{\boldsymbol{\psi}\}_i$	ith eigenvalue and eigenvector of original system
λ_i^*, $\{\boldsymbol{\psi}\}_i^*$	ith eigenvalue and Eigenvector of modified system
$[\mathbf{K}]$, $[\mathbf{M}]$	stiffness and mass matrices of original system
$[\Delta\mathbf{K}]$, $[\Delta\mathbf{M}]$	increments in stiffness and mass matrices
E_1, E_3	Young's modulus of base and constraining layers
$\mathbf{G}_2$	shear modulus of core material
β	material loss factor of core material
ρ_1, ρ_2, ρ_3	mass densities of base, core and constraining layers
b	breadth
t_1, t_2, t_3	thicknesses of base, core and constraining layers
w	transverse deflection
w'	slope
α	rotation angle of transverse plane in the core
γ_2	shear angle in the core
u_1, u_2, u_3	longitudinal displacements in base, core and constraining layers
$\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$	rotation angles in base, core and constraining layers
$\dot{T}_m$	derivative with respect to time