A FREQUENCY DOMAIN BASED NUMERIC–ANALYTICAL METHOD FOR NON-LINEAR DYNAMICAL SYSTEMS

S. Narayanan

Machine Dynamics Laboratory, Department of Applied Mechanics, Indian Institute of Technology, Madras 600 036, India

AND

P. Sekar

National Institute of Ocean Technology, Indian Institute of Technology Campus, Madras 600 036, India

In this paper a multiharmonic balancing technique is used to develop certain algorithms to determine periodic orbits of non-linear dynamical systems with external, parametric and self excitations. Essentially, in this method the non-linear differential equations are transformed into a set of non-linear algebraic equations in terms of the Fourier coefficients of the periodic solutions which are solved by using the Newton–Raphson technique. The method is developed such that both fast Fourier transform and discrete Fourier transform algorithms can be used. It is capable of treating all types of non-linearities and higher dimensional systems. The stability of periodic orbits is investigated by obtaining the monodromy matrix. A path following algorithm based on the predictor–corrector method is also presented to enable the bifurcation analysis. The prediction is done with a cubic extrapolation technique with an arc length incrementation while the correction is done with the use of the least square minimisation technique. The under determined system of equations is solved by singular value decomposition. The suitability of the method is demonstrated by obtaining the bifurcational behaviour of rolling contact vibrations modelled by Hertz contact law.

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1. INTRODUCTION

It is well known that many non-linear dynamical systems exhibit periodic, quasiperiodic and chaotic behaviour depending upon the choice of system parameters and initial conditions. For systems with strong non-linearities, conventional methods such as perturbation techniques or a low order harmonic balancing procedure cannot yield accurate results. A robust procedure to locate and follow periodic and quasiperiodic orbits thus is essential. Numerical integration is computationally expensive especially when the damping level is low or the state of the system is close to bifurcation. Moreover, it is impossible to follow a particular solution branch. Also, in parameter ranges where multiple solutions exist, there is no guarantee that it will follow the same solution branch. This paper presents various developments of efficient numeric–analytical algorithms to obtain periodic solutions of non-linear systems followed by a stability analysis with a view to trace the route to chaos and identify the associated bifurcations. A method for obtaining

periodic solutions in which the advantages of the FFT algorithm and parametric continuation are combined is developed and the efficacy of the method is demonstrated through two example problems pertaining to systems with contact and impact non-linearities. In this method, the non-linear differential equations are transformed into a set of non-linear algebraic equations in terms of Fourier coefficients of the periodic solutions which are solved by using the Newton–Raphson technique. As this method involves Fourier series expansion, the Galerkin process of error minimisation and a Newton–Raphson scheme, it is referred to as the Fourier–Galerkin–Newton (FGN) method. Explicit expressions for the elements of the Jacobian matrix are derived. The local stability of periodic orbits is investigated using the Floquet theory. The continuation of a particular solution branch is effected through a predictor–corrector scheme in which a cubic extrapolation procedure is used for prediction and a least square minimisation with the use of singular value decomposition (SVD) is used for correction.

2. COMPUTATION OF PERIODIC SOLUTIONS

There are essentially two methods of obtaining periodic solutions of non-linear systems, the frequency domain and the time domain methods. The frequency domain method requires the solution to be expressed as a truncated Fourier series. Such methods include the incremental harmonic balance method (IHB), the fast Galerkin method (FG) and the alternate frequency/time domain method (AFT). Cesari [1] obtained periodic solutions of a non-linear system using a truncated Fourier series. Urabe [2] provided mathematical justification for the same. Urabe and Reiter [3] showed that the Fourier coefficients of periodic solutions could be computed using a Newton–Raphson procedure and a discrete Fourier transform. Samoilenko and Ronto [4] developed a matrix method in which matrix relations were formed to transform from frequency to time domain and *vice versa*. Tamura *et al*. [5] developed a matrix method to obtain higher order approximations for steady state periodic solutions using complex Fourier series expansion. The elements of the corresponding Jacobian matrix were obtained by taking advantage of its special structure. The fast Galerkin method was proposed by Ling and Wu [6] in which Urabe's method was implemented using FFT. Cameron and Griffin [7] developed the alternating frequency/time domain method by implementing the FG method where the harmonic components of the non-linear terms were obtained by using FFT while that of the linear terms were obtained directly. Buonomo [8] developed a trigonometric collocation method to obtain periodic solutions of non-linear oscillators applicable to non-autonomous and autonomous systems. The IHB method is ideally suited for parametric studies since it can step from a known state of vibration to a neighbouring state which corresponds to an incremental change in one of the governing parameters of the system [9]. Pierre *et al*. [10] modified the IHB method for the analysis of systems with dry friction. Kim and Noah [11] proposed a modified harmonic balance/Fourier transform approach to determine the steady state periodic solutions and applied the same to an unsymmetrical piecewise linear single-degree-of-freedom system (sdf) subjected to harmonic excitation. Leung and Fung [12] constructed bifurcation curves and the regions of chaos for a Duffing oscillator with zero linear stiffness by using the IHB method. Cheung *et al*. [13] presented the formulation of the IHB method for a multi-degree-of-freedom system (mdf) with cubic non-linearity. An incremental arc length approach combined with a cubic extrapolation technique was adopted to trace the response curves automatically. Leung and Chui [4] further extended the IHB method with a path following algorithm based on arc length prediction method.

The IHB method introduces considerable difficulties because the evaluation of integrals arising out of the Galerkin process is problem dependent. To avoid these difficulties, Narayanan and Sekar [15] developed a fast incremental harmonic balance method in which the integrals were identified as the Fourier coefficients and were calculated using FFT. The corresponding elements of the Jacobian matrix (tangent matrix) obtained during the Newton–Raphson method were also obtained using FFT. Hence, this method will be suitable for problems having both analytical and non-analytical non-linearities. This method was demonstrated for systems containing deviating arguments also. The response of systems with multiple harmonic excitations which could be incommensurate was also dealt with.

In the time domain method a numerical integration procedure is applied to obtain the periodic and aperiodic solutions. They can be classified into the brute force/point mapping method [16] and shooting method [17–19] in which the equations are integrated with the specified initial conditions along with the linearised equations with unit matrix as the initial conditions for one period. The shoots are obtained from the solution of the linearised equations and are added to the current initial conditions to get the new initial conditions. This procedure is continued till the solution settles to a particular attractor. Unstable solutions can also be obtained by this procedure. Kaas-Petersen [20] presented an algorithm to get periodic and biperiodic solutions with use of a Poincaré map. Stability analysis and continuation of solutions with a change in the control parameter were also incorporated. Ling [21] developed a shooting method to obtain the quasiperiodic orbits of quasiperiodically and periodically excited non-linear systems.

3. FOURIER–GALERKIN–NEWTON METHOD FOR PERIODIC SOLUTIONS

Consider the multi-degree-of-freedom (mdf) non-linear dynamical system with L degrees of freedom in the form

$$\omega^2 \mathbf{m}\ddot{\mathbf{x}} + \omega \mathbf{c}\dot{\mathbf{x}} + \mathbf{k}\mathbf{x} + \mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}, \omega, \tau) - \mathbf{f}(p, \omega, \tau) = \mathbf{r}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}, p, \lambda, \omega, \tau) = \mathbf{0}, \qquad (1)$$

where, $\mathbf{m}$, $\mathbf{c}$, $\mathbf{k}$ respectively are the mass, the damping and the stiffness matrices of $L \times L$ dimension, $\mathbf{g}$ is the vector non-linear function, $\mathbf{f}$ is the excitation vector, $\tau = \omega t$ is the non-dimensional time, ω and p are respectively the frequency and amplitude parameter of the periodic excitation and λ is the system parameter. Setting $\mathbf{x} = \mathbf{x}_i + \Delta\mathbf{x}$, $\dot{\mathbf{x}} = \dot{\mathbf{x}}_i + \Delta\dot{\mathbf{x}}$, $\ddot{\mathbf{x}} = \ddot{\mathbf{x}}_i + \Delta\ddot{\mathbf{x}}$, $\lambda = \lambda_i + \Delta\lambda$, $p = p_i + \Delta p$ and $\omega = \omega_i + \Delta\omega$, equation (1) can be expressed as

$$\mathbf{r}_i + \left[\frac{\partial\mathbf{r}}{\partial\mathbf{x}} \quad \frac{\partial\mathbf{r}}{\partial\dot{\mathbf{x}}} \quad \frac{\partial\mathbf{r}}{\partial\ddot{\mathbf{x}}} \quad \frac{\partial\mathbf{r}}{\partial\lambda} \quad \frac{\partial\mathbf{r}}{\partial\omega} \quad \frac{\partial\mathbf{r}}{\partial p}\right]_i [\Delta\mathbf{x} \quad \Delta\dot{\mathbf{x}} \quad \Delta\ddot{\mathbf{x}} \quad \Delta\lambda \quad \Delta\omega \quad \Delta p]^{\mathrm{T}} = \mathbf{0}, \qquad (2)$$

where the left side is the one term Taylor series expansion of $\mathbf{r}(\tau)$ about the known periodic solution $\mathbf{x}_i$. Since $\mathbf{x}$ is periodic, $\Delta\mathbf{x}$ is also periodic and hence these can be represented as Fourier series. Consider a truncated Fourier series expansion for $\mathbf{x}$ and $\Delta\mathbf{x}$ in the form

$$\mathbf{x} = \mathbf{X}_0 + \sum_{j=1}^{M} [\mathbf{X}_{2j-1}\cos(j\tau) + \mathbf{X}_{2j}\sin(j\tau)], \qquad (3)$$

$$\Delta\mathbf{x} = \Delta\mathbf{X}_0 + \sum_{j=1}^{M} [\Delta\mathbf{X}_{2j-1}\cos(j\tau) + \Delta\mathbf{X}_{2j}\sin(j\tau)], \qquad (4)$$

where $\mathbf{X}_0$, $\mathbf{X}_j$'s and $\Delta\mathbf{X}_0$ and $\Delta\mathbf{X}_j$'s are the Fourier coefficients. For the initial trial solution $\mathbf{x}_0$ to converge to the actual solution $\mathbf{x}$ the Galerkin procedure is applied by orthogonalising the error term $\mathbf{r}(\tau)$ with harmonic weighting functions $[1, \cos\tau, \sin\tau, \ldots, \cos j\tau, \sin j\tau, \ldots, \cos M\tau, \sin M\tau]^{\mathrm{T}}$, thereby minimising the error associated with each harmonic coefficient. This procedure yields a set of $(2M + 1)L$ linear algebraic equations for the increments in the Fourier coefficients $\Delta\mathbf{X}$'s and $\Delta\lambda$, $\Delta\omega$ and Δp in the form

$$[\mathbf{J}^L]\{\mathbf{X}\} + \{\mathbf{G}\} - \{\mathbf{F}\} + [\mathbf{J}^L + \mathbf{J}^N]\{\Delta\mathbf{X}\} + \{\mathbf{H}\}\Delta\lambda + \{\mathbf{Q}\}\Delta\omega + \{\mathbf{P}\}\Delta p = \{\mathbf{0}\}, \qquad (5)$$

where $[\mathbf{J}^L]$ and $[\mathbf{J}^N]$ are the Jacobian matrices associated with the linear and non-linear parts of the equation of motion. Here, $\mathbf{X}$ and $\Delta\mathbf{X}$ contain the Fourier coefficients and their increments of $\mathbf{x}$ and $\Delta\mathbf{x}$ respectively. The vectors $\mathbf{G}$, $\mathbf{H}$, $\mathbf{Q}$, $\mathbf{P}$, $\mathbf{F}$ respectively represent the Fourier coefficients of the non-linear function $\mathbf{g}$, derivative of $\mathbf{r}$ with respect to λ, ω and p, and the Fourier coefficients of excitation vector $\mathbf{f}$. The various quantities in equation (5) can be determined either through FFT or DFT as given in Appendix A. For a chosen combination of parameters of the system, that is, when $\Delta\lambda$, $\Delta\omega$ and Δp are identically set to zero, equation (5) contains a set of $(2M + 1)L$ linear algebraic equations with $(2M + 1)L$ unknowns which can be solved for increments in Fourier coefficients [15]. To start the iteration scheme, the initial estimates of the Fourier coefficients are obtained from the linearised solution by neglecting the non-linear part of the equations of motion or by numerical simulation. The number of harmonic coefficients, M is also selected on the basis of the number of significant harmonics expected in the solution. The error vector $\mathbf{R}$ is given by

$$\{\mathbf{R}\} = [\mathbf{J}^L]\{\mathbf{X}\} + \{\mathbf{G}\} - \{\mathbf{F}\} \qquad (6)$$

and the convergence is checked through the error estimates ε_1, ε_2 and ε_3 given by

$$\varepsilon_1 = \sqrt{\mathbf{R}_0^2 + \sum_{j=1}^{M} (\mathbf{R}_{2j-1}^2 + \mathbf{R}_{2j}^2)}, \qquad \varepsilon_2 = \sqrt{\sum_{j=M+1}^{2M} (\mathbf{G}_{2j-1}^2 + \mathbf{G}_{2j}^2)},$$

$$\varepsilon_3 = \sqrt{\Delta\mathbf{X}_0^2 + \sum_{j=1}^{M} (\Delta\mathbf{X}_{2j-1}^2 + \Delta\mathbf{X}_{2j}^2)}. \qquad (7a\text{--}c)$$

The error estimate ε_1 indicates the closeness of the Fourier transform and the actual Fourier coefficients while the error estimate ε_2 indicates the magnitude of higher harmonic coefficients beyond the harmonic number M. The error estimate ε_3 represents the Euclidean norm of the Newton–Raphson increments. To obtain accurate solution the iteration should be continued till acceptable error estimates are obtained. By increasing M it is possible to obtain the periodic solution with the prescribed error tolerance. By making use of the transformation $m\omega t = n\tau$, it is possible to seek periodic solutions of nth order subharmonic and mth order superharmonic.

3.1. ALGORITHM BASED ON THE FAST FOURIER TRANSFORM (FFT)

Initially, a set of approximate Fourier coefficients, $\mathbf{X}$, are assumed and the number of collocation points in the FFT is selected. The collocation nodes should be equal to 2^n, where n is any positive integer greater than or equal to 2. Using the inverse transform $\mathbf{x}(\tau)$, $\dot{\mathbf{x}}(\tau)$, $\ddot{\mathbf{x}}(\tau)$, the functions $\mathbf{g}$, $\partial\mathbf{r}/\partial\lambda$, $\partial\mathbf{r}/\partial\omega$, and $\partial\mathbf{r}/\partial p$ can be evaluated in the time domain. From the time histories of these functions, their Fourier coefficients can be calculated with

the use of the forward transform. The Jacobian matrix corresponding to the linear terms in equation (5) can be directly determined since it is a block diagonal matrix with the corresponding frequency response function in the diagonal. It is worthwhile to note that the elements of the Jacobian matrix corresponding to the non-linear terms are also the Fourier transform of the derivatives of **g** with respect to **x**, $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ and the cosine and sine transforms of the same. Hence, the elements of the Jacobian matrix can be determined with the use of inverse and forward transforms. To reduce aliasing error, the minimum number of collocation nodes, N must be greater than or equal to $4M$, where M is the maximum harmonic number with significant power. Moreover, N should be expressible in 2^n form; where n is any positive integer $\geqslant 2$. By arranging various Fourier coefficients in the same form as arranged for **X** one can solve the linear set of equations to obtain the increments in Fourier coefficients.

3.1.1. *Example* 1: *Impacting non-linear oscillator*

The equation of motion of a sdf system impacting against a non-linear spring modelled by Hertz theory and hysteretic damping in non-dimensional form is given by

$$\ddot{x} + 2\zeta\dot{x} + x + g(x, \dot{x}) = P\cos(\omega t),$$

with

$$g(x, \dot{x}) = \begin{cases} C_H(x - \delta)^{3/2}(1 + \beta\dot{x}), & \text{when } x > \delta, \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

where ζ is the linear damping factor, C_H is the Hertzian constant, β is the hysteretic damping parameter and P and ω are respectively the amplitude and the frequency of harmonic excitation. Equation (8) is solved by the FGN method with a FFT algorithm. The error criteria used to check convergence are as follows: $\varepsilon_1 = \varepsilon_3 = 1\cdot0E - 6$; $\varepsilon_2 = 1\cdot0E - 1$. While ε_1 and ε_3 can be made as small as possible during iteration, ε_2 which is defined only for algorithms which use FFT cannot be reduced beyond a certain limit since the higher harmonic coefficients in general may not be accurate in FFT. This is especially so for systems with non-analytical non-linearities. The parameters used are as follows: $\zeta = 0\cdot05$, $\beta = 0\cdot0$, $C_H = 10\cdot0$, $\delta = 0\cdot0$, $P = 10\cdot0$, and $\omega = 1\cdot2$. For the chosen
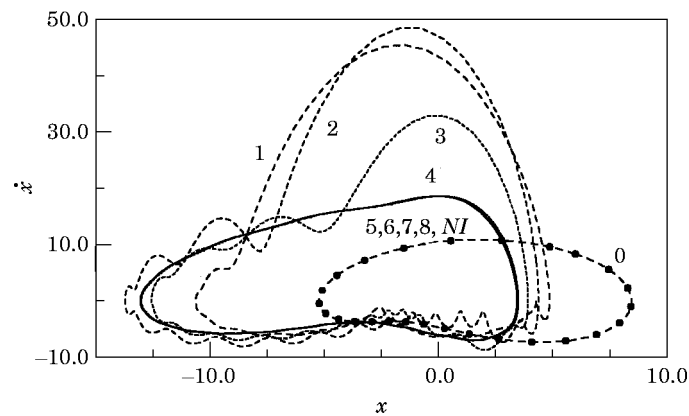

Figure 1. Phase plane plot showing the effect of number of iterations (l) ($\zeta = 0\cdot05$, $\beta = 0\cdot0$, $C_H = 10\cdot0$, $\delta = 0\cdot0$, $P = 10\cdot0$, $\omega = 1\cdot2$); $N = 64$, $M = 16$.

parameters, a period-1 orbit is obtained with 16 harmonics (M) and 64 collocation nodes (N) in the FFT. Shown in Figure 1 is the effect of the number of iterations (I) on the convergence of the period-1 orbit. From the arbitrarily selected initial trial solution ($I = 0$), the algorithm converges to the solution obtained by the numerical integration (NI) in five successive iterations.

3.2. ALGORITHM BASED ON THE DISCRETE FOURIER TRANSFORM (DFT)

In the case of an algorithm based on FFT, the number of collocation nodes should be exactly equal to 2^n. In the DFT based algorithm, this restriction is removed and the number of collocation nodes are calculated as $N = 2M + 1$, where M is the number of harmonics considered in the Fourier series expansion. Samoilenko and Ronto [4] had presented the DFT transforms which are given as (1) time to frequency domain:

$$[\Gamma_{ij}] = \begin{cases} 1/N, & \text{when } i = 1, \\ (2/N)\cos i(j-1)\pi/N], & \text{when } i = 2, 4, \ldots, N-1, \\ (2/N)\sin[(i-1)(j-1)\pi/N], & \text{when } i = 3, 5, \ldots, N, \end{cases}$$

for $j = 1, 2, \ldots, N$, and (2) frequency to time domain: (9)

$$[\Gamma_{ij}^{-1}] = \begin{cases} 1, & \text{when } j = 1, \\ \cos[(i-1)j\pi/N], & \text{when } j = 2, 4, \ldots, N-1, \\ \sin[(i-1)j-1)\pi/N], & \text{when } j = 3, 5, \ldots, N, \end{cases}$$

for $i = 1, 2, \ldots, N.$ (10)

Under these transforms it is possible to determine the various quantities in equation (5). This method is also extended to time delay systems [15].

4. STABILITY AND BIFURCATION ANALYSIS

Stability of the periodic solution is investigated by using the Floquet theory facilitating bifurcation analysis. The state variables of the equation of motion are perturbed about the periodic orbit and the monodromy matrix is constructed [22]. When equation (1) is perturbed and transformed into a state variable form it can be written as

$$\{\dot{\mathbf{z}}\} = [\mathbf{A}]\{\mathbf{z}\},$$ (11)

where

$$[A] = \begin{bmatrix} [\mathbf{0}] & [\mathbf{I}] \\ -[(1/\omega^2)\mathbf{m}^{-1}[\mathbf{k} + \partial\mathbf{g}/\partial\mathbf{x}]] & -[(1/\omega^2)\mathbf{m}^{-1}[\omega\mathbf{c} + \partial\mathbf{g}/\partial\dot{\mathbf{x}}]] \end{bmatrix},$$ (12)

with

$$\{\mathbf{z}\} = [\,\varDelta\mathbf{x}, \varDelta\dot{\mathbf{x}}]^{\mathrm{T}}.$$ (13)
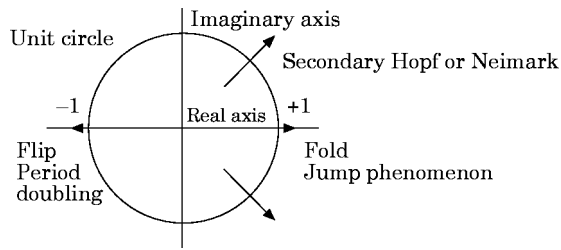
Figure 2. Eigenvalues and the nature of bifurcation in the complex plane.

Let $[\Phi]$ be the matrix solution satisfying

$$[\dot{\Phi}] = [A][\Phi], \tag{14}$$

with

$$[\Phi(0)] = [I], \tag{15}$$

where $[\Phi]$ is the monodromy matrix and $[I]$ is a unit matrix. Bifurcation occurs when one of the eigenvalues of the monodromy matrix crosses the unit circle in the complex plane as illustrated in Figure 2.

5. PARAMETER CONTINUATION/PATH FOLLOWING

Continuing the state of the system as the bifurcation parameter is varied is known as path following or parameter continuation. This is usually performed through a predictor and corrector mechanism. The prediction is done with use of a polynomial (cubic) extrapolation technique. The correction is carried out by using a least square minimisation technique for the resulting system of under determined equations. The least square minimisation is done using singular value decomposition (SVD) which does not require an extra constraint equation in incremental form to be included in the under determined set of equations. Since the prediction is done with a greater number of points, the predicted point will be very close to the actual point on the response curve and hence the correction can be done in few iterations. Path following is based on the assumption that the state of the system varies smoothly as a parameter is changed continuously. During the path following procedure any one of the parameters is varied while all other parameters are kept constant. For example, in equation (5), when λ is varied, $\Delta\omega$, Δp are identically set to zero.
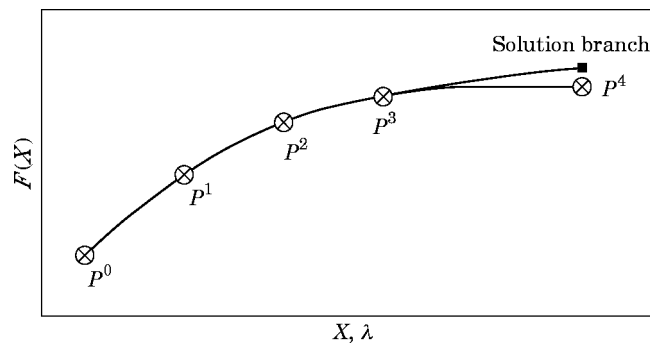

Figure 3. Cubic extrapolation scheme for parametric continuation. $\otimes$, Predicted point.

Assume that a number of points, N_p ($N_p = 4$, cubic polynomial) on the solution branch are obtained *a priori* through some means. Based on these points, the neighbouring point on the solution branch at a specified arc length is predicted using the Lagrangian polynomial extrapolation method [13]. Let P^0, P^1, P^2, P^3, ..., $P^{N_p - 1}$ be the N_p number of points which are known *a priori*. The scheme with four points is shown in Figure 3. Any point on the solution branch is represented at P^i (X^i, λ^i), where X^i and λ^i respectively represent the Fourier coefficients and the bifurcation parameter. Let the arc length between any two consecutive points P^i and P^{i+1} be denoted by s_i. For any system parameter λ, the arc length is given by

$$s_{i+1} = \sqrt{((X^{i+1} - X^i)^{\mathrm{T}}(X^{i+1} - X^i) + (\lambda^{i+1} - \lambda^i)^2)}, \qquad i = 0, 1, \ldots, N_p - 2. \quad (16)$$

The total arc length parameter q_i is given by

$$q_0 = 0, \qquad q_1 = s_1, \qquad q_2 = q_1 + s_2, \qquad q_3 = q_2 + s_3, \ldots, \qquad q_{N_p} = q_{N_p - 1} + \Delta s. \quad (17)$$

Using the Lagrangian extrapolation scheme the neighbouring point at a distance Δs can be predicted and is given by

$$\begin{Bmatrix} \mathbf{X}^{N_p} \\ \lambda^{N_p} \end{Bmatrix} = \sum_{i=0}^{N_p - 1} L_i \begin{Bmatrix} X^i \\ \lambda^i \end{Bmatrix}, \qquad L_i = \prod_{\substack{j=0 \\ i \neq j}}^{N_p - 1} \left(\frac{q_{N_p} - q_j}{q_i - q_j} \right), \qquad i = 0, 1, \ldots, N_p - 1. \quad (18)$$

The predicted point may not lie on the solution branch which is being traced and hence a correction step is required. Correction is performed using SVD. Equation (5) can be written as (note that $\Delta \omega = \Delta p = 0$)

$$[[\mathbf{J}^L + \mathbf{J}^N] \; \vdots \; \{\mathbf{H}\}] \begin{Bmatrix} \Delta \mathbf{X} \\ \Delta \lambda \end{Bmatrix} + \{\mathbf{R}\} = \{\mathbf{0}\}. \quad (19)$$

Since equation (19) contains one more unknown than the number of equations, a least square solution is effected through SVD such that $\sqrt{\{\Delta \mathbf{X}^{\mathrm{T}} \Delta \mathbf{X} + \Delta \lambda^2\}}$ is minimum.

5.1. EXAMPLE 2: NON-LINEAR OSCILLATOR WITH ROLLING HERTZIAN CONTACT

Consider the rigid roller 1 with mass m_1 which is in rolling contact with a flexible roller 2 of mass m_2 as shown in Figure 4. Cylinder 1 is assumed to have a smooth surface, while
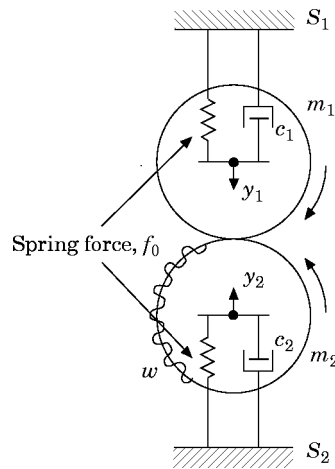

Figure 4. Two elastic bodies in rolling contact one with a smooth surface and the other with a wavy surface.

cylinder 2 has a surface waviness. It is assumed that the system is precompressed by a static load f_0 which brings the two supports S_1 and S_2 together. The suspension springs are assumed to have a very low stiffness so that the spring force always remains at its static value f_0. The damping in the system is assumed to be of viscous type represented by two dampers with damping coefficients c_1 and c_2.

Let y_1 and y_2 be the motion of the cylinders 1 and 2 respectively towards the point of contact, measured with respect to the position when the two masses are just touching in the absence of surface waviness. The contact is assumed to be Hertzian and the contact force between the cylinders is given by [23]

$$f_c = C_H[y_1 + y_2 + w]^{3/2}, \tag{20}$$

where f_c is the contact force, w the surface waviness of the flexible cylinder 2 and C_H is the Hertzian material constant. The equations of motion for the two masses are given by

$$m_1\ddot{y}_1 + c_1\dot{y}_1 + f_c = f_0, \qquad m_2\ddot{y}_2 + c_2\dot{y}_2 + f_c = f_0. \tag{21, 22}$$

Further assuming that $c_1/m_1 = c_2/m_2 = c/m$, and introducing a variable $y = y_1 + y_2$, equations (21) and (22) can be expressed as a single equation as

$$m\ddot{y} + c\dot{y} = f_0 - f_c, \tag{23}$$

where $m = m_1m_2/(m_1 + m_2)$.

Introducing the Hertzian deflection $z = y + w$, the static Hertzian deflection $z_s = (f_0/C_H)^{2/3}$, the non-dimensional Hertzian deflection $x = z/z_s$, the damping factor $\zeta = c/2m\omega_c$ and the contact resonance frequency given by

$$\omega_c^2 = 3f_0/2mz_s, \tag{24}$$

equation (23) becomes

$$\ddot{x} + 2\zeta\dot{x} + \tfrac{2}{3}[H(x)x^{3/2} - 1] = f(\tau), \tag{25}$$

where now the number of overdots represents the order of differentiation with respect to the non-dimensional time $\tau = \omega_c t$, $f(\tau) = \ddot{\xi} + 2\zeta\dot{\xi}$ with $\xi = w/z_s$, and H[x] is the Heaviside unit step function. For $x > 0$ the surfaces of the cylinders are in contact and the restoring force follows the Hertzian law, while for $x < 0$ the surfaces move apart and the contact is lost. Equation (25) represents the equation of motion for contact vibrations. It is now assumed that the surface waviness is harmonic and for a constant rotational speed it can
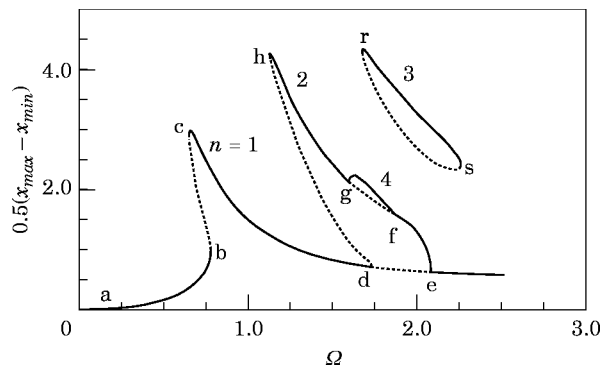

Figure 5. Response diagram-amplitude ($\zeta = 0\cdot05$, $\xi_0 = 0\cdot5$); ——, stable; - - - -, unstable.
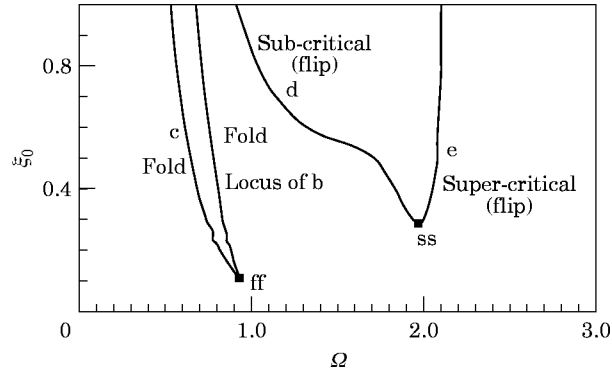
Figure 6. Fold and flip boundaries of period-1 orbits ($\zeta = 0.05$).

be expressed as $\xi = \xi_0 \cos(\Omega\tau)$, where Ω is the non-dimensional frequency dependent on the rotational speed. The equation of contact vibration (25) can now be written as [23]

$$\ddot{x} + 2\zeta\dot{x} + \tfrac{2}{3}[H(x)x^{3/2} - 1] = -[\Omega^2 \cos(\Omega\tau) + 2\zeta\Omega \sin(\Omega\tau)]\xi_0. \qquad (26)$$

Equation (26) is investigated to obtain the bifurcational structure of contact vibrations.

Equation (26) is solved by the FGN method with DFT in conjunction with the path following algorithm and stability analysis. Period-1, 2, 4 and 3 orbits are obtained in the range of the non-dimensional frequency Ω considered. The response diagram for the amplitude of oscillation is shown in Figure 5. Various orders of subharmonics of 2, 3 and 4 are shown in the figure. The stable and the unstable branches are traced by the path following algorithm. These as well as the fold and the flip bifurcation points are also indicated in the figure.

5.2. BIFURCATION BOUNDARIES

The bifurcation boundaries in a two-dimensional parametric space ($\xi_0 - \Omega$) are shown in Figures 5 and 6 for $\zeta = 0.05$. Bifurcation boundaries correspond to the loci of the points b, c, d, e, f, g and h of Figure 5. The bifurcation boundaries are obtained by repeated solution of the problem for various excitation amplitudes.

Shown in Figure 6 is the fold boundary of the period-1 (p1) orbit which corresponds to the upward jump, point b, the downward jump, point c, and also its flip (sub-critical
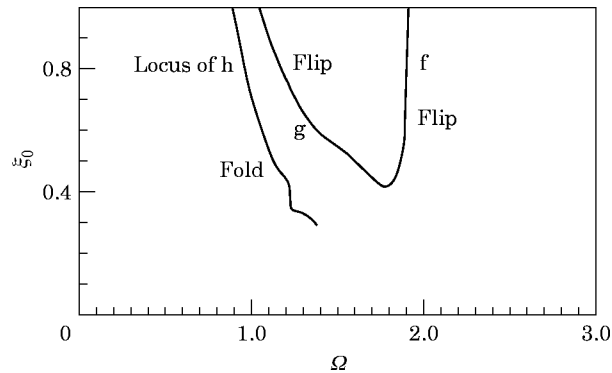

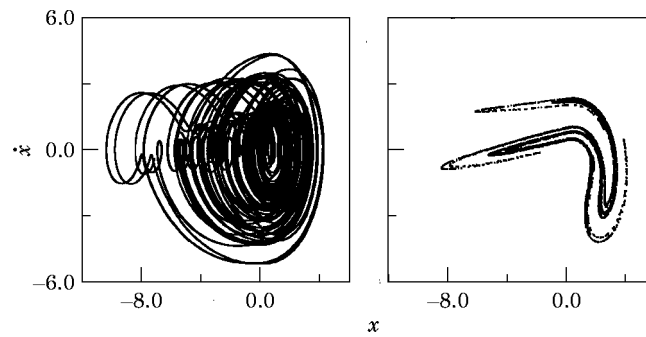Figure 7. Fold and flip boundaries of period-2 orbits ($\zeta = 0.05$).

Figure 8. (a) Phase plane plot and (b) Poincaré section of chaotic orbit ($\zeta = 0.05$, $\xi_0 = 0.90$, $\Omega = 1.5$).

and super-critical) boundaries. The fold bifurcation boundaries (locus of b and c) of the period-1 orbit merge together at the point ff which corresponds to $\Omega = 0.93$ and $\xi_0 = 0.11$, and hence, the system exhibits p1 motion without jump phenomenon for $\xi_0 < 0.11$ in the entire range of Ω considered. Thus, for $\xi_0 > 0.11$, the system exhibits two stable period-1 orbits and one unstable period-1 orbit in the region bracketed by the locus of points b and c. Likewise, the subcritical and supercritical flip bifurcation boundaries merge together at the point ss which corresponds to $\Omega = 1.971$ and $\xi_0 = 0.29$ and hence, for $\xi_0 < 0.29$, the system does not undergo period doubling bifurcations. For $\xi_0 > 0.29$ the system exhibits period-2 motion in the region bracketed by the locus of points d and e. As ξ_0 is increased the width of flip bifurcation boundaries increases sharply as ξ_0 exceeds 0.5. The fold point h, and the flip bifurcation boundaries points f and g, of period-2 (p2) orbit are shown in Figure 7. The flip bifurcation boundaries (loci of f and g) merge together at a point which is close to $\Omega = 1.70$ and $\xi_0 = 0.40$ and hence for $\xi_0 > 0.40$ unstable period-2 motion exists in the region bounded by the lines f and g. For $\xi_0 > 0.55$, it is observed that the system exhibits chaotic behaviour in a certain range of Ω. The phase plane plot and the strange attractor corresponding to chaos for $\xi_0 = 0.90$, $\zeta = 0.05$ and $\Omega = 1.5$ are shown in Figure 8.

6. CONCLUSIONS AND FUTURE RESEARCH

A frequency domain based numeric–analytical method is developed to obtain periodic solutions of non-linear dynamical systems. A stability analysis and a path following scheme is incorporated to carry out the bifurcation analysis. The method is presented in general so that it is applicable for mdf systems with all types of non-linearities since the non-linear functions need to be evaluated in the time domain and transformed into the frequency domain using Fourier transform. However, a large number of linear algebraic equations need to be solved to obtain the higher periodic solutions of mdf systems.

Evaluation of derivatives of non-linear functions with piecewise linear and/or non-linear restoring force characteristics cause considerable difficulties since the derivative function will be piecewise continuous. These functions can be made continuous by using an inverse tangent function [24]. The applicability of the method is demonstrated by solving the non-linear equation of motion of a rolling contact system. The flip and fold bifurcation boundaries of a rolling contact system are presented. For magnitudes of the surface waviness exceeding a certain value, chaotic motion with impacts occurs for certain ranges of frequency parameter (Ω). Such numeric–analytical methods can serve as useful alternatives to numerical integration.

The proposed method in the present form is suitable for bifurcational analysis in which one system parameter (co-dimension 1 bifurcation) is varied. In the case of tracing the bifurcation boundaries in a 2-dimensional plane, two system parameters in addition to the Fourier coefficients should be incremented such that the algorithm follows a bifurcation boundary, along which the modulus of the leading Floquet multiplier is unity. By incorporating the above modification in the proposed method, it is possible to trace the bifurcation boundaries in an effective manner.

The investigation of non-linear dynamical systems can be performed only when the equations of motion together with the values of the parameters are known *a priori*. In most situations, it may not be possible to accurately model and to estimate the values of the parameters of non-linear dynamical systems. However, the actual dynamical behaviour of the systems can be obtained from the measured data of the existing system by using the embedding procedure for the reconstruction of the attractor. Hence, the variation in system parameters can be monitored.

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APPENDIX A: EVALUATION OF FOURIER COEFFICIENTS OF NON-LINEAR FUNCTION AND THEIR DERIVATIVES THROUGH FFT AND DFT

The vector of Fourier coefficients are arranged as follows:

$$\{\mathbf{X}\} = [\mathbf{X}_0^T \quad \mathbf{X}_1^T \quad \mathbf{X}_2^T \quad \cdots \quad \mathbf{X}_{2j-1}^T \quad \mathbf{X}_{2j}^T \quad \cdots \quad \mathbf{X}_{2M-1}^T \quad \mathbf{X}_{2M}^T]^T, \tag{A.1}$$

$[.]^T$ representing the transpose operation. The Fourier coefficients of $\mathbf{g}$, $\mathbf{r}$, $\partial \mathbf{r}/\partial \lambda$, $\partial \mathbf{r}/\partial \omega$, $\partial \mathbf{r}/\partial p$ and $\mathbf{f}$ are given by

$$\begin{Bmatrix} \mathbf{G}_0 \\ \mathbf{H}_0 \\ \mathbf{Q}_0 \\ \mathbf{P}_0 \\ \mathbf{F}_0 \end{Bmatrix} = \frac{1}{2\pi} \int_0^{2\pi} \begin{Bmatrix} \mathbf{g} \\ \partial \mathbf{r}/\partial \lambda \\ \partial \mathbf{r}/\partial \omega \\ \partial \mathbf{r}/\partial p \\ \mathbf{f} \end{Bmatrix} d\tau, \qquad \begin{Bmatrix} \mathbf{G}_{2j-1} \\ \mathbf{H}_{2j-1} \\ \mathbf{Q}_{2j-1} \\ \mathbf{P}_{2j-1} \\ \mathbf{F}_{2j-1} \end{Bmatrix} = \frac{1}{\pi} \int_0^{2\pi} \begin{Bmatrix} \mathbf{g} \\ \partial \mathbf{r}/\partial \lambda \\ \partial \mathbf{r}/\partial \omega \\ \partial \mathbf{r}/\partial p \\ \mathbf{f} \end{Bmatrix} \cos j\tau \, d\tau,$$

$$\begin{Bmatrix} \mathbf{G}_{2j} \\ \mathbf{H}_{2j} \\ \mathbf{Q}_{2j} \\ \mathbf{P}_{2j} \\ \mathbf{F}_{2j} \end{Bmatrix} = \frac{1}{\pi} \int_0^{2\pi} \begin{Bmatrix} \mathbf{g} \\ \partial \mathbf{r}/\partial \lambda \\ \partial \mathbf{r}/\partial \omega \\ \partial \mathbf{r}/\partial p \\ \mathbf{f} \end{Bmatrix} \sin j\tau \, d\tau. \tag{A.2–4}$$

The Jacobian matrices $[\mathbf{J}^L]$ and $[\mathbf{J}^N]$ associated respectively with the linear and non-linear terms are given by

$$[\mathbf{J}^L] = \begin{bmatrix} [\mathbf{k}] & & & & \\ & \begin{bmatrix} -\mathbf{m}\omega^2 + \mathbf{k} & \mathbf{c}\omega \\ -\mathbf{c}\omega & -\mathbf{m}\omega^2 + \mathbf{k} \end{bmatrix} & & & \\ & & \ddots & & \\ & & & \begin{bmatrix} -\mathbf{m}\omega^2 j^2 + \mathbf{k} & \mathbf{c}\omega j \\ -\mathbf{c}\omega j & -\mathbf{m}\omega^2 j^2 + \mathbf{k} \end{bmatrix} & \\ & & & & \ddots \\ & & & & \begin{bmatrix} -\mathbf{m}\omega^2 M^2 + \mathbf{k} & \mathbf{c}\omega M \\ -\mathbf{c}\omega M & -\mathbf{m}\omega^2 M^2 + \mathbf{k} \end{bmatrix} \end{bmatrix}, \quad (A.5)$$

$$[\mathbf{J}^N] = \begin{bmatrix} \mathbf{J}_{0,0} & \mathbf{J}_{0,1} & \mathbf{J}_{0,2} & \cdots & \mathbf{J}_{0,2k-1} & \mathbf{J}_{0,2k} & \cdots & \mathbf{J}_{0,2M-1} & \mathbf{J}_{0,2M} \\ \mathbf{J}_{1,0} & \mathbf{J}_{1,1} & \mathbf{J}_{1,2} & \cdots & \mathbf{J}_{1,2k-1} & \mathbf{J}_{1,2k} & \cdots & \mathbf{J}_{1,2M-1} & \mathbf{J}_{1,2M} \\ \mathbf{J}_{2,0} & \mathbf{J}_{2,1} & \mathbf{J}_{2,2} & \cdots & \mathbf{J}_{2,2k-1} & \mathbf{J}_{2,2k} & \cdots & \mathbf{J}_{2,2M-1} & \mathbf{J}_{2,2M} \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \cdots & \cdots \\ \mathbf{J}_{2j-1,0} & \mathbf{J}_{2j-1,1} & \mathbf{J}_{2j-1,2} & \cdots & \mathbf{J}_{2j-1,2k-1} & \mathbf{J}_{2j-1,2k} & \cdots & \mathbf{J}_{2j-1,2M-1} & \mathbf{J}_{2j-1,2M} \\ \mathbf{J}_{2j,0} & \mathbf{J}_{2j,1} & \mathbf{J}_{2j,2} & \cdots & \mathbf{J}_{2j,2k-1} & \mathbf{J}_{2j,2k} & \cdots & \mathbf{J}_{2j,2M-1} & \mathbf{J}_{2j,2M} \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \cdots & \cdots \\ \mathbf{J}_{2M-1,0} & \mathbf{J}_{2M-1,1} & \mathbf{J}_{2M-1,2} & \cdots & \mathbf{J}_{2M-1,2k-1} & \mathbf{J}_{2M-1,2k} & \cdots & \mathbf{J}_{2M-1,2M-1} & \mathbf{J}_{2M-1,2M} \\ \mathbf{J}_{2M,0} & \mathbf{J}_{2M,1} & \mathbf{J}_{2M,2} & \cdots & \mathbf{J}_{2M,2k-1} & \mathbf{J}_{2M,2k} & \cdots & \mathbf{J}_{2M,2M-1} & \mathbf{J}_{2M,2M} \end{bmatrix}. \quad (A.6)$$

The elements of the non-linear Jacobian matrix are the Fourier transforms of the non-linear function and their cosine and sine transforms of their derivatives with respect to $\Delta\mathbf{x}$, $\Delta\dot{\mathbf{x}}$, and $\Delta\ddot{\mathbf{x}}$ and are given by

$$\begin{Bmatrix} \mathbf{J}_{0,0} \\ \mathbf{J}_{2j-1,0} \\ \mathbf{J}_{2j,0} \end{Bmatrix} = \int_0^{2\pi} \left[\frac{\partial\mathbf{g}}{\partial\mathbf{x}}\right] \begin{Bmatrix} 1/2\pi \\ (1/\pi)\cos j\tau \\ (1/\pi)\sin j\tau \end{Bmatrix} d\tau, \tag{A.7a}$$

$$\begin{Bmatrix} \mathbf{J}_{0,2k-1} \\ \mathbf{J}_{2j-1,2k-1} \\ \mathbf{J}_{2j,2k-1} \end{Bmatrix} = \int_0^{2\pi} \left[\frac{\partial\mathbf{g}}{\partial\mathbf{x}}(\cos k\tau) + \frac{\partial\mathbf{g}}{\partial\dot{\mathbf{x}}}(-k\sin k\tau) + \frac{\partial\mathbf{g}}{\partial\ddot{\mathbf{x}}}(-k^2\cos k\tau)\right] \begin{Bmatrix} 1/2\pi \\ (1/\pi)\cos j\tau \\ (1/\pi)\sin j\tau \end{Bmatrix} d\tau,$$

$$\tag{A.7b}$$

$$\begin{Bmatrix} \mathbf{J}_{0,2k} \\ \mathbf{J}_{2j-1,2k} \\ \mathbf{J}_{2j,2k} \end{Bmatrix} = \int_0^{2\pi} \left[\frac{\partial\mathbf{g}}{\partial\mathbf{x}}(\sin k\tau) + \frac{\partial\mathbf{g}}{\partial\dot{\mathbf{x}}}(+k\cos k\tau) + \frac{\partial\mathbf{g}}{\partial\ddot{\mathbf{x}}}(-k^2\sin k\tau)\right] \begin{Bmatrix} 1/2\pi \\ (1/\pi)\cos j\tau \\ (1/\pi)\sin j\tau \end{Bmatrix} d\tau.$$

$$\tag{A.7c}$$

A.1. ALGORITHM WITH FFT

The elements of the Jacobian matrix given in equation (5) corresponding to the non-linear term are given by (reference to equations A.7a–c)

$$\mathbf{J}_{0,0} + \sum_{j=1}^{M} [\mathbf{J}_{2j-1,0}\cos j\tau + \mathbf{J}_{2j,0}\sin j\tau] = FT\left[\frac{\partial\mathbf{g}}{\partial\mathbf{x}}\right],$$

$$\mathbf{J}_{0,2k-1} + \sum_{j=1}^{M} [\mathbf{J}_{2j-1,2k-1}\cos j\tau + \mathbf{J}_{2j,2k-1}\sin j\tau] = FT\left[\frac{\partial\mathbf{g}}{\partial\mathbf{x}}\{\cos k\theta_i\} + \frac{\partial\mathbf{g}}{\partial\dot{\mathbf{x}}}\{-k\sin k\theta_i\}\right.$$

$$\left. + \frac{\partial\mathbf{g}}{\partial\ddot{\mathbf{x}}}\{-k^2\cos k\theta_i\}\right],$$

$$\mathbf{J}_{0,2k} + \sum_{j=1}^{M} [\mathbf{J}_{2j-1,2k}\cos j\tau + \mathbf{J}_{2j,2k}\sin j\tau] = FT\left[\frac{\partial\mathbf{g}}{\partial\mathbf{x}}\{\sin k\theta_i\} + \frac{\partial\mathbf{g}}{\partial\dot{\mathbf{x}}}\{+k\cos k\theta_i\}\right.$$

$$\left. + \frac{\partial\mathbf{g}}{\partial\ddot{\mathbf{x}}}\{-k^2\sin k\theta_i\}\right],$$

$$\text{for } k = 1, 2, \ldots, M, \ i = 1, 2, \ldots, N, \ \theta_i = 2\pi(i-1)/N, \ N \geqslant 4M. \tag{A.8a–c}$$

or

A.2. ALGORITHM WITH DFT

Denoting the Fourier coefficients of x, $\dot{x}$, $\ddot{x}$ and g respectively by X, $\dot{X}$, $\ddot{X}$ and G, the following relations can be obtained.

$$X = \Gamma x; \qquad \dot{X} = \Lambda X, \qquad \ddot{X} = \Lambda^2 X, \qquad G = \Gamma g, \tag{A.9a–d}$$

where

$$[\varLambda] = \begin{bmatrix} 0 & & & & & \\ & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & & & & \\ & & \ddots & & & \\ & & & \begin{bmatrix} 0 & j \\ -j & 0 \end{bmatrix} & & \\ & & & & \ddots & \\ & & & & & \begin{bmatrix} 0 & M \\ -M & 0 \end{bmatrix} \end{bmatrix}, \qquad (A.10a)$$

$$[\varLambda^2] = \begin{bmatrix} 0 & & & & & \\ & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & & & & \\ & & \ddots & & & \\ & & & \begin{bmatrix} -j^2 & 0 \\ 0 & -j^2 \end{bmatrix} & & \\ & & & & \ddots & \\ & & & & & \begin{bmatrix} -M^2 & 0 \\ 0 & -M^2 \end{bmatrix} \end{bmatrix}. \qquad (A.10b)$$

The Jacobian matrix corresponding to the non-linear terms $[J^N]$ is given by

$$[J^N] = [J_x] + [J_{\dot{x}}] + [J_{\ddot{x}}], \qquad (A.11)$$

$$[J_x] = \varGamma \begin{bmatrix} \ddots & & \\ & \dfrac{\partial g}{\partial x} & \\ & & \ddots \end{bmatrix} \varGamma^{-1}, \qquad (A.12a)$$

$$[J_{\dot{x}}] = \varGamma \begin{bmatrix} \ddots & & \\ & \dfrac{\partial g}{\partial \dot{x}} & \\ & & \ddots \end{bmatrix} \varGamma^{-1}\varLambda, \qquad [J_{\ddot{x}}] = \varGamma \begin{bmatrix} \ddots & & \\ & \dfrac{\partial g}{\partial \ddot{x}} & \\ & & \ddots \end{bmatrix} \varGamma^{-1}\varLambda^2. \quad (A.12b, c)$$