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STOCHASTIC OPTIMAL ACTIVE CONTROL OF A 2-DOF QUARTER CAR MODEL WITH NON-LINEAR PASSIVE SUSPENSION ELEMENTS

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Active control of the time varying response to a stationary random excitation of a two-degree-of-freedom vehicle model with non-linear passive suspension elements is considered in this paper. The method of equivalent linearization is used to derive the equivalent linear model and the optimal control laws are obtained by using stochastic optimal control theory based on full state information. Velocity squared damping and hysteresis type of stiffness non-linearities are considered. The performance of active suspensions with non-linear passive elements is found to be superior to the corresponding passive suspension systems. As a check on the accuracy of the equivalent linearization technique, a Monte Carlo simulation has also been performed. The results are compared with the results obtained from linearized vehicle model and good agreement between the results has been observed.

1. INTRODUCTION

In recent years much research has been undertaken in the area of active vibration isolation systems with special application to vehicle suspensions. The main functions of a vehicle suspension system are to provide vehicle support, stability and directional control during handling maneuvers and to provide effective isolation from road disturbances. These different tasks result in conflicting design requirements: in conventional vehicle suspensions, the characteristics of suspension elements cannot be altered and a compromise has to be found between ride comfort and road holding. Active suspensions in which externally powered force actuators provide suspension forces according to some prescribed control policy may be used to overcome some of the limitations of passive suspensions. The state of the art reviews by Hedrick and Wormley [1], Goodall and Kortum [2], Sharp and Crolla [3] and Hrovat [4] cover the important developments in active vehicle suspension design.

The suspension system of a large class of road vehicles consists of a combination of coil and leaf springs. Normally in the vibration analysis of vehicle systems, the suspension elements are modelled as linear springs and dampers for simplicity of analysis. In reality, the force deformation characteristics of springs exhibit non-linear behaviour especially of hysteretic type. Similarly the stiffness characteristics of the pneumatic tyre are also non-linear. In many of the suspension systems, the damping characteristics also show non-linear behaviour. Hence for a realistic vehicle dynamic model it is necessary to take into consideration the non-linearities in the passive suspension system.

It is well known fact that the response of a vehicle model with non-linear suspension elements to random road excitation is much more complex than of a linear system. This

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is due to the fact that applying random excitation to a non-linear system does not necessarily yield the same response at a particular excitation frequency that would occur if the system were driven harmonically or periodically at that frequency. Non-linear system responses are dependent on initial conditions and can have multiple solutions for a given excitation amplitude and frequency. Some solutions require a steady state input and take time to build up. While random road profiles usually can be approximated as stochastic inputs, sometimes some elements of periodicity exists. Because of this the tools available for the analysis of random vibration of non-linear system are somewhat limited.

In literature, the response of the non-linear vehicle model has been obtained using either simulation techniques or analytical techniques. Yadav and Nigam [5] and Dailey *et al*. [6] have employed simulation techniques to study the response behaviour of non-linear vehicle models. While simulation techniques are very useful for providing detailed information on the ride behaviour of the vehicle, they tend to be expensive due to the necessity of simulating the time series of the excitation process and applying numerical methods to integrate the equations of motion of the system and due to the time consumed in processing the results obtained into statistical characteristics.

Among the analytical techniques used in the analysis of non-linear systems subjected to random excitations, the equivalent linearization technique has the widest applicability [Iwan and Yang [7]]. In this paper this method has been used to compute the stationary response of a vehicle traversing a rough road with constant velocity. The vehicle is modelled as a two-dof quarter car model with non-linear passive elements such as quadratic dampers and hysteretic stiffness elements [Narayanan and Raju [8]]. The hysteretic nature of the suspension spring is modelled by a modified Bouc's model [9] in which the restoring force of the spring is assumed to be a combination of a pre-yielding component and a hysteretic component. The non-linear equations of motion are approximately linearized by using equivalent linearization techniques. The coefficients of the linear equations are determined in such a way that the mean square value of error, i.e., the difference between the non-linear and linear equations is minimum. By minimizing the mean square value of error, some of the coefficients of the equivalent linear system will in turn depend on the moments of the response and hence the approximate response of the linearized vehicle model is obtained by a cyclic iteration scheme.

The random road profile on which the vehicle is traversing is modelled as the output of a first order shaping filter to white noise excitation. The action of the traverse of the vehicle over the road is imbedded into the dynamics of the entire vehicle road system by inclusion of the road profile as a component of the state vector resulting in an augmented state vector. The control laws are derived in a stochastic framework and the average behaviour of the vehicle response is represented by a zero lag covariance matrix of the state vector and the same is obtained by the solution of the matrix Lyapunov equation. The performance of the vehicle active suspension in terms of ride comfort, road holding and suspension stroke is compared with that of the passive suspension system.

As a check on the accuracy of the equivalent linearization techniques, a Monte Carlo simulation has also been proposed. The road surface excitation has been simulated as a series of cosine functions with weighted amplitudes at evenly spaced frequencies and random phase angles [10]. Both passive as well as active suspension respone time series have been simulated digitally and compared with the vehicle response obtained by the covariance matrix approach. Good agreement between the simulated data and the statistical linearization results for both the suspension systems has been observed.

2. MATHEMATICAL MODELLING

2.1. VEHICLE MODEL

A two-dof quarter car model with quadratic damping and hysteretic stiffness between the sprung and unsprung masses is shown in Figure 1. M is the sprung mass, m is the unsprung mass, c is the coefficient of linear viscous damping, c_1 is the coefficient of non-linear quadratic damper, α and k are the hysteretic suspension spring parameters, and k_t is the tyre stiffness.

Assuming that the wheel does not leave the ground, the equations of motion of the vehicle model can be written as

$$M\ddot{y}_1 + c(\dot{y}_1 - \dot{y}_2) + c_1\,|\dot{y}_1 - \dot{y}_2|(\dot{y}_1 - \dot{y}_2) + \alpha k(y_1 - y_2) + (1-\alpha)kz - u = 0, \tag{1}$$

$$m\ddot{y}_2 + c(\dot{y}_2 - \dot{y}_1) + c|\dot{y}_2 - \dot{y}_1\,|\dot{y}_2 - \dot{y}_1) + \alpha k(y_2 - y_1) + (1-\alpha)kz + k_t\,(y_2 - h) + u = 0, \tag{2}$$

where y_1 and y_2 respectively are absolute displacements of the sprung mass and unsprung mass, u is the control force and the parameter z representing the hysteretic displacement is given by the non-linear differential equation [11]:

$$\dot{z} + \gamma|\dot{y}_1 - \dot{y}_2\,|\,z\,|z|^{n-1} + \eta|\dot{y}_1 - \dot{y}_2\,|\,|z|^n + A|\dot{y}_1 - \dot{y}_2\,|. \tag{3}$$

In the above equation γ, η, A and n are parameters fitting the force–displacement characteristic of the hysteretic system. Parameters γ and η control the shape of the hysteresis loop, A the restoring force amplitude, and n is the parameter representing the smoothness of transition from elastic to plastic response. The values of these parameters have been chosen in such a way that the simulated hysteresis loop matches with the force–deformation characteristics of a road vehicle leaf spring under realistic broad band random loading condition [12]. Figure 2 shows the typical force deformation characteristics of a hysteretic system for $\alpha = 0{\cdot}2$, η, $\gamma = 0{\cdot}5$, $A = 1{\cdot}5$, $n = 1$. In the absence of hysteretic stiffness the equations of motion are obtained by setting $\alpha = 1$ in equations (1) and (2). If the quadratic damping is absent one sets $c_1 = 0$ in equations (1) and (2).

3. EQUIVALENT LINEAR SYSTEM

A multiple-degree-of-freedom non-linear dynamic system can be represented in the form

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} + \mathbf{g}(\mathbf{X}, \dot{\mathbf{X}}) = \mathbf{f}(t), \tag{4}$$

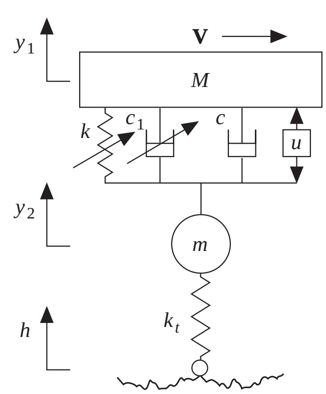

Figure 1. Non-linear vehicle model.

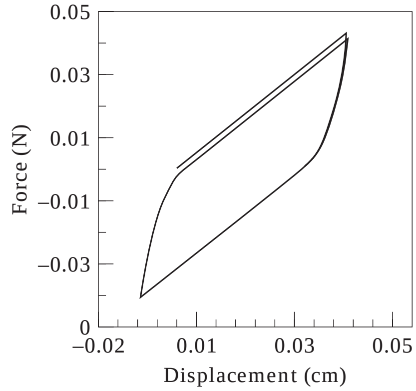

Figure 2. Force deformation characteristics of a hysteretic system, $\alpha = 0{\cdot}2$, $\eta = \gamma = 0{\cdot}5$, $A = 1{\cdot}5$.

where $\mathbf{X}$ is the state vector which includes z, the hysteretic displacement given by equation (3), and $\mathbf{g}(\mathbf{X}, \dot{\mathbf{X}})$ is the vector containing non-linear elements. Let the system of equations (1) and (2) be replaced by the equivalent linear system of equations represented by

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}^{*}\dot{\mathbf{X}} + \mathbf{K}^{*}\mathbf{X} = \mathbf{f}(t), \tag{5}$$

where

$$\mathbf{C}^{*} = \mathbf{C} + \mathbf{C}', \qquad \mathbf{K}^{*} = \mathbf{K} + \mathbf{K}', \tag{6}$$

$\mathbf{C}$ and $\mathbf{K}$ representing, respectively; the linear part of damping and stiffness matrices; $\mathbf{C}'$ and $\mathbf{K}'$ are obtained by minimizing the mean square error $E[\mathbf{e}^{\mathrm{T}}\,\mathbf{e}]$ where $\mathbf{e}$ is defined by

$$\mathbf{e} = \mathbf{g}(\mathbf{X}, \dot{\mathbf{X}}) - \mathbf{C}'\dot{\mathbf{X}} - \mathbf{K}'\mathbf{X}. \tag{7}$$

Assuming the excitation vector $\mathbf{f}(t)$ to be a zero mean Gaussian random vector and since the system of equations (5) is linear, the response of the equivalent linear system can also be considered as Gaussian. Under this condition, using the results of Iwan and Yang [7], and Atalik and Utku [13], it can be shown that the elements of the matrices $\mathbf{C}'$ and $\mathbf{K}'$ are given by

$$\mathbf{C}'_{ij} = E[\partial g_i/\partial \dot{\mathbf{X}}], \qquad \mathbf{K}'_{ij} = E[\partial g_i/\partial \mathbf{X}]. \tag{8, 9}$$

Using the above results, the linearized equations of motion of the 2-dof vehicle model can be obtained as

$$M\ddot{y}_1 + (c + c_{eq})\dot{q} + \alpha k q + (1-\alpha)kz - u = 0, \tag{10}$$

$$m\ddot{y}_2 - (c + c_{eq})\dot{q} - \alpha k q - (1-\alpha)kz + k_t\,(y_2 - h) + u = 0, \tag{11}$$

$$\dot{z} + c_h\,\dot{q} + K_h\,z = 0, \tag{12}$$

where $q = y_1 - y_2$ is the relative displacement between the sprung and unsprung masses and

$$c_{eq} = 2\sqrt{2/\pi}\,c_1\,\sigma_{\dot{q}}, \qquad c_h = \sqrt{2/\pi}\,[\gamma E[\dot{q}Z]/\sigma_{\dot{q}} + \eta\sigma_z] - A, \tag{13, 14}$$

$$k_h = \sqrt{2/\pi}\,[\eta\,E[\dot{q}Z]/\sigma_z + \gamma\sigma_{\dot{q}}]. \tag{15}$$

3.1. ROAD PROFILE MODEL

The power spectral density (*psd*) function of road irregularity is assumed to be of the form

$$S_h(\omega) = (\sigma^2/\pi)a\mathbf{V}/(\omega^2 + (a\mathbf{V})^2), \tag{16}$$

where $\mathbf{V}$ is the vehicle forward velocity, ω is circular frequency, σ^2 denotes the variance of the road irregularity and a is a coefficient depending on the type of road surface. The process $h(t)$ described by equation (16) is the output of a linear first order filter with white noise input given by

$$h(t) + a\mathbf{V}h(t) = \mathbf{W}(t). \tag{17}$$

$\mathbf{W}(t)$ is a white noise process with covariance function $E[\mathbf{W}(t)\,\mathbf{W}^{\mathrm{T}}(t)] = 2\sigma^2 a\mathbf{V}\delta(t_2 - t_1)$ and $E[\cdot]$ denotes the expectation operator.

4. STATE SPACE REPRESENTATION

Defining an augmented state vector as $\mathbf{X}_a(t) = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6]$, where $x_1 = y_1$, $x_2 = \dot{y}_1$, $x_3 = y_2$, $x_4 = \dot{y}_2$, $x_5 = z$ and $x_6 = h$, the linearized equations of motion of the vehicle, and the equation representing the filtered road input can be combined to yield

$$\dot{\mathbf{X}}_a(t) = \mathbf{F}\mathbf{X}_a(t) + \mathbf{G}\mathbf{U}(t) + \mathbf{D}\mathbf{W}(t), \tag{18}$$

where

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_x & \mathbf{D}_x \\ \mathbf{0} & \mathbf{F}_w \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} \mathbf{G}_x \\ \mathbf{0} \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_d \end{bmatrix}, \tag{19}$$

$$\mathbf{F}_\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\alpha k/M & -c_2/M & \alpha k/M & c_2/M & (\alpha - 1)k/M \\ 0 & 0 & 0 & 1 & 0 \\ \alpha k/m & c_2/m & -(\alpha k + k_t)/m & -c_2/m & (1-\alpha)k/m \\ 0 & -c_h & 0 & c_h & -k_h \end{bmatrix},$$

$$\mathbf{D}_x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_t/m \\ 0 \end{bmatrix}, \quad \mathbf{G}_x = \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/m \\ 0 \end{bmatrix}, \tag{20}$$

and where

$$c_2 = c + c_{eq}, \qquad \mathbf{D}_d = 1, \qquad \mathbf{F}_w = -a\mathbf{V}, \qquad \mathbf{Q} = 2\sigma^2 a\mathbf{V}.$$

5. PERFORMANCE INDEX

The objective of optimization of the vehicle suspension is to improve the vehicle performance with respect to ride comfort, suspension stroke and road holding with minimum expenditure of control energy. Ride comfort is measured in terms of the mean square acceleration of the sprung mass, that is,

$$J_1 = E[\ddot{y}_1^2]. \tag{21}$$

The performance with respect to suspension stroke, road holding and control effort are represented, respectively by the mean square values

$$J_2 = E[(y_1 - y_2)^2], \qquad J_3 = E[(y_2 - h_1)^2], \qquad J_4 = E[u^2]. \tag{22}$$

The expected sum of the weighted suspension measures is minimized in the optimization, where ρ_1, ρ_2, ρ_3 and ρ_4 are weighting parameters and $E[\cdot]$ denotes the expectation operator. The overall system performance index can be written as

$$J = \{\rho_1 J_1 + \rho_2 J_2 + \rho_3 J_3 + \rho_4 J_4\}, \tag{23}$$

which can be expressed in the standard form [14] as

$$J = E\left[[\mathbf{X}_a^{\mathrm{T}} \quad \mathbf{U}^{\mathrm{T}}] \begin{bmatrix} \mathbf{A} & \mathbf{N} \\ \mathbf{N}^{\mathrm{T}} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{X}_a \\ \mathbf{U} \end{bmatrix} \right], \tag{24}$$

where matrices **A** and **B** are symmetric positive semi-definite and positive definite, respectively, and are given by

$$\mathbf{A} = \begin{bmatrix} \frac{\rho_1 k^2}{M^2} + \rho_2 & \frac{\rho_1 k c_2}{M^2} & -\frac{\rho_1 k^2}{M^2} - \rho_2 & -\frac{\rho_1 k c_2}{M^2} & 0 \\ & \frac{\rho_1 c_2^2}{M^2} & -\frac{\rho_1 k c_2}{M^2} & -\frac{\rho_1 c_2^2}{M^2} & 0 \\ \text{symmetric} & & \frac{\rho_1 k^2}{M^2} + \rho_2 + \rho_3 & \frac{\rho_1 k c_2}{M^2} & -\rho_3 \\ & & & \frac{\rho_1 c_2^2}{M^2} & 0 \\ & & & & \rho_3 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \frac{\rho_1 k}{M^2} \\ -\frac{\rho_1 c_2}{M^2} \\ \frac{\rho_1 k}{M^2} \\ \frac{\rho_1 c_2}{M^2} \\ 0 \end{bmatrix}$$

$$\mathbf{B} = \left[\frac{1}{M^2} + \rho_4 \right].$$

6. OPTIMAL CONTROL LAW

A control law involving full and perfect state information is considered for the control of the stationary response of the vehicle model. Assuming that the state $\mathbf{X}_a(t)$ of the system is exactly measurable at time t, the admissible control law, minimizing the performance index (24) for the above mentioned system is given by Kwakernak and Sivan [14] as

$$\mathbf{U}(t) = -\mathbf{C}(t)\mathbf{X}_a(t), \tag{25}$$

in which $\mathbf{C}(t)$ is the control gain given by

$$\mathbf{C}(t) = \mathbf{B}^{-1}(\mathbf{N}^{\mathrm{T}} + \mathbf{G}^{\mathrm{T}}\mathbf{S}). \tag{26}$$

S is a symmetric matrix given by the solution of the matrix Riccati equation:

$$\mathbf{S}\mathbf{F}_N + \mathbf{F}_N^{\mathrm{T}}\mathbf{S} - \mathbf{G}\mathbf{B}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{S} + \mathbf{A}_N = 0, \tag{27}$$

where

$$\mathbf{F}_N = \mathbf{F} - \mathbf{G}\mathbf{B}^{-1}\mathbf{N}^{\mathrm{T}}, \qquad \mathbf{A}_N = \mathbf{A} - \mathbf{N}\mathbf{B}^{-1}\mathbf{N}^{\mathrm{T}}. \tag{28}$$

The response of the system can be described by the zero lag covariance matrix of the state vector $\mathbf{P} = E[\mathbf{X}_a(t)\mathbf{X}_a^{\mathrm{T}}(t)]$, which is given by the solution of the matrix differential Lyapunov equation with initial conditions:

$$\dot{\mathbf{P}} = (\mathbf{F} - \mathbf{G}\mathbf{C}(t)) \cdot \mathbf{P} + \mathbf{P} \cdot (\mathbf{F} - \mathbf{G}\mathbf{C}(t))^{\mathrm{T}} + \mathbf{D}\mathbf{Q}\mathbf{D}^{\mathrm{T}}. \tag{29}$$

In the problem considered here the vehicle system matrix **F** is a function of the response statistics of the vehicle and the Riccati equation in turn is a function of system matrix **F**. This means that in order to be able to solve the Riccati equation, the response statistics of the vehicle should be known *a priori*. This problem is overcome by an iterative procedure described in the sequel.

7. ITERATIVE PROCEDURE

In this scheme control gains corresponding to the linear vehicle model (without non-linear elements) are first computed over the entire period of control. These control gains are used in obtaining the response statistics of the non-linear vehicle model. Then, using the current response statistics of the non-linear vehicle model, control gains are computed again over the period of control. This iterative procedure is repeated until the values of control gains and the response statistics at any particular velocity converge to a specified degree of accuracy. In the example considered here, convergence was achieved with respect to the control gains and response statistics within four iterations.

8. NUMERICAL INTEGRATION

Computing the control gain matrix **C** of the controlled system requires the solution of the Riccati equation (27), while the covariance matrix of the state vector of an optimal active control system can be found by solving the matrix Lyapunov equation (29). Algorithms for solving the Riccati equation (27) have been presented in reference [15]. To avoid the divergence and instability of these algorithms, complete controllability of the system is required. As the system considered here is not completely controllable, matrices **A** and **S** of equations (27) are separated into four sub-matrices as mentioned by Hac [16]. Vaughan's algorithm [17] has been used for solving the Riccati equation (27). The Lyapunov equation is evaluated by use of the Smith's algorithm [13].

9. EVALUATION OF PERFORMANCE INDEX TERMS

The mean square values of the sprung mass acceleration J_1, suspension deflection J_2, tyre deflection J_3 and control force J_4 are evaluated from the following relations [16].

Mean square value of ride comfort J_1,

$$J_1 = \sum_{I=1}^{6} \sum_{J=1}^{6} \left[\left(\mathbf{F}(2, I) - \frac{\mathbf{C}(1, I)}{M} \right) \times \left(\mathbf{F}(2, J) - \frac{\mathbf{C}(1, J)}{M} \right) \right] \times \mathbf{P}(I, J). \tag{30}$$

Mean square value of suspension deflection J_2,

$$J_2 = \mathbf{P}(1, 1) - 2\mathbf{P}(1, 3) + \mathbf{P}(3, 3). \tag{31}$$

Mean square value of tyre deflection J_3,

$$J_3 = \mathbf{P}(3, 3) - 2\mathbf{P}(3, 6) + \mathbf{P}(6, 6). \tag{32}$$

Mean square value of control force J_4,

$$J_4 = \sum_{I=1}^{6} \sum_{J=1}^{6} \frac{\mathbf{C}(1, I) \times \mathbf{C}(1, J)}{M^2} \mathbf{P}(I, J). \quad (33)$$

10. RESULTS AND DISCUSSION

The optimal control problem of the stationary response of a non-linear 2-dof vehicle model is solved for the constant velocity run. The numerical data used for the vehicle and road profile model are: $M = 1000$ kg, $m = 100$ kg, $k = 36{\cdot}0$ kN/m, $c = 1398$ Ns/m, $c_1 = 1000$ Ns/m, $k_t = 360$ kN/m, $a = 0{\cdot}15$ and $\sigma^2 = 9{\cdot}0 \times 10^{-6}$ m^2. The values of the parameters for the hysteretic type suspension spring model are chosen as $\alpha = 0{\cdot}2$, η, $\gamma = 0{\cdot}5$, $A = 1{\cdot}5$. The weighting factors used in the performance index are $\rho_1 = 1{\cdot}0$, $\rho_2 = 10^3$, $\rho_3 = 10^4$ and $\rho_4 = 10^{-6}$. This combination of the weighting constants has been used by Hac [16] and gives greater weight to the ride comfort criterion relative to other criteria.

10.1. MONTE CARLO SIMULATION

To check the accuracy of the equivalent linearization technique, Monte Carlo simulation has been performed. The road surface excitation term $h(t)$ in equation (2) is modelled as an output of a first order filter with white noise excitation. The shape of the filter represents the spectral characteristics of the ground excitation with power spectral density

$$S_h(\omega) = \sigma^2 a\mathbf{V}/\pi(\omega^2 + (a\mathbf{V})^2), \quad (16)$$

where $\mathbf{S}_h(\omega)$ is the psd of road surface, which is of insignificant magnitude outside the region defined by $-\infty < \omega_l \leqslant \omega < \omega_u < \infty$. σ^2 denotes the variance of the road, $\mathbf{V}$ is the vehicle velocity and α represents the quality of the road. A first order shaping filter of the form in equation (17) is assumed and a method described by Shinozuka [10] has been used to digitally generate the white noise with psd:

$$\mathbf{S}_w(\omega) = \sigma^2 aV/\pi. \quad (34)$$

The white noise process $\mathbf{W}(t)$ is simulated by the series

$$\mathbf{W}(t) = \sqrt{2} \sum_{k=1}^{N} S_w(\omega_k) \cos(\omega_k' t + \phi_k), \quad (35)$$

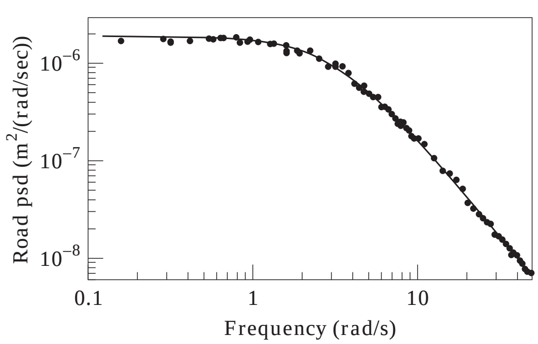

Figure 3. Psd of road profile; ●, Simulated psd; ——, target psd.

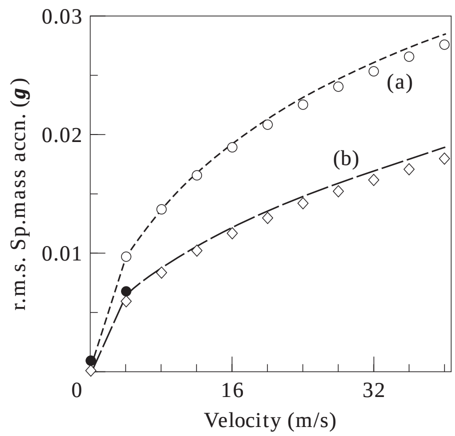

Figure 4. Sprung mass acceleration of vehicle model: (a) passive control; (b) active control. ----, ——, by equivalent linearization; ○, □, by simulation.

where ϕ_k = independent random phase uniformly distributed between 0 and 2π, $\omega_k = \omega_l + (k - \frac{1}{2})\Delta\omega$, $k = 1, 2, \ldots, N$; $\omega'_k = \omega_k + \delta\omega$, $k = 2, \ldots, N$ and $\Delta\omega = (\omega_u - \omega_l)/N$, with N being the number of intervals. $\delta\omega$ is a small random frequency introduced to avoid the periodicity of the simulated process and is uniformly distributed between $-\Delta\omega'/2$ and $\Delta\omega'/2$ with $\Delta\omega' \ll \Delta\omega$.

The white noise is simulated with $N = 1000$, $\omega_l = 0$, $\omega_u = 50$ Hz and $\Delta\omega' = 0{\cdot}05\Delta\omega$. Equation (17) is integrated using the fourth order Runge–Kutta method to obtain $h(t)$. The simulated spectral density is then obtained by taking the Fourier transform of the temporal auto correlation. The resulting simulated spectral density is plotted by symbol • in Figure 3 in which the target spectral density represented by $\mathbf{S}_h$ (equation (16)) is also shown by a solid line. It is seen that the simulated power spectrum is matching very well with the target power spectrum. The sample functions of the responses y_1, $\dot{y}_1$, y_2 and $\dot{y}_2$ are computed from equations (1) and (2) for the vehicle model by the fourth order Runge–Kutta method with a time step of 0·01 and 1000 sample functions.

10.2. ZERO LAG COVARIANCE MATRIX METHOD

The sprung mass acceleration, which is a measure of ride quality, is plotted in Figure 4 for (a) passive suspension and (b) active suspension. From the figure it is immediately clear that with respect to the sprung mass acceleration, the active suspension performs much better than the corresponding passive suspension. The results obtained by Monte Carlo simulation for the full non-linear vehicle model for the two suspension cases are also shown

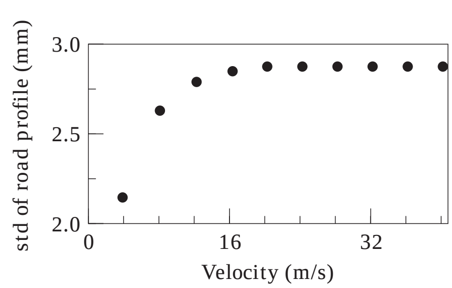

Figure 5. Road profile standard deviation: •••, by simulation.

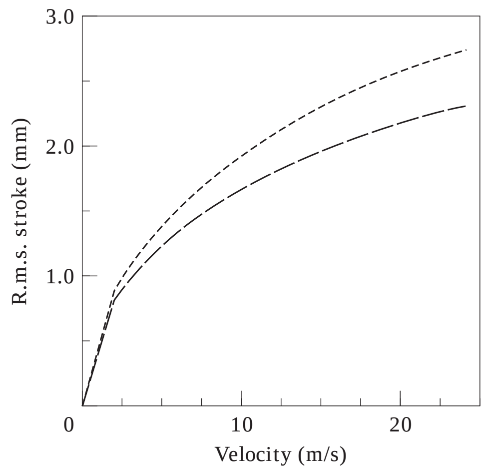

Figure 6. Suspension stroke performance; - - - -, passive; ——, active.

in the Figure 4. It is seen that the results of the equivalent linearization match very well with the simulation results validating the equivalent linearization method for the response prediction of the vehicle with active suspension and with passive suspension along with non-linear passive suspension elements. The small differences between equivalent linearization and simulation which appear to increase linearly with velocity can be attributed to the differences in the simulated road profile standard deviation with the actual value (Figure 5) due to the assumed number of harmonics and numerical inaccuracies in the white noise generation.

In Figure 6 the r.m.s. stroke response is plotted for the passive as well as for active suspension systems. In this case also, the r.m.s. stroke corresponding to the active suspension system is less than that for the passive suspension. Thus, the suspension with active control gives a better performance with respect to stroke response also.

The roadholding characteristics in terms of the r.m.s. relative displacement between the wheel and the road are given in Figure 7 for the two suspension systems. In this case, the active suspension system does not perform well as compared to the passive suspension

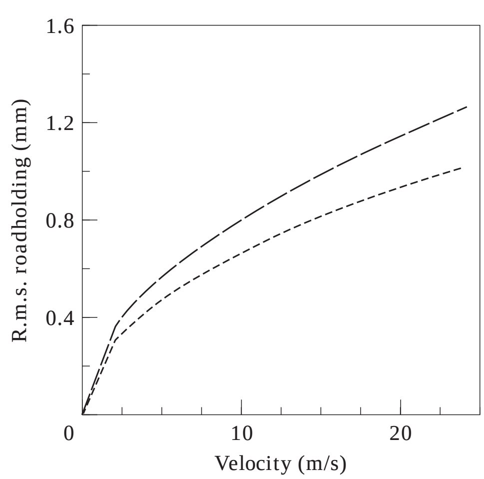

Figure 7. Road holding performance. Key as for figure 6.

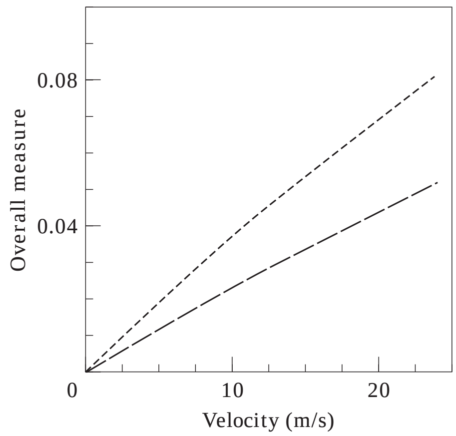

Figure 8. Overall performance. Key as for figure 6.

system. This could be due to the fact that the road holding criterion in the performance index is weighted much less compared to ride comfort criterion. The overall performance measure based on equation (24) is plotted in Figure 8. It is seen that for the weighting parameters considered in the problem, the active suspension gives a much better performance than the passive suspension system.

11. CONCLUSIONS AND FUTURE RESEARCH

Active control of the stationary response of a 2-dof vehicle model with a non-linear suspension system has been considered. The equivalent linearization method has been used in linearizing the system. The control laws have been derived in a stochastic framework and the average behaviour of the linearized vehicle model is represented by a zero lag covariance matrix and the same is obtained by solving the matrix Lyapunov equation. Accuracy of the equivalent linearization technique has been validated using Monte Carlo simulation. For the combination of weighting constants considered, it can be concluded from the computed results that

(1) The vehicle model with active suspension gives a better ride comfort, suspension stroke and overall performance than the vehicle model with passive suspension.

(2) Though the overall performance of the vehicle is improved by the use of an active suspension, the performance of the vehicle with respect to road holding with the active suspension is worse than that of the passive suspension.

(3) By comparing the results with Monte Carlo simulation results, the potential of the zero lag covariance matrix method with equivalent linearization techniques in obtaining the vehicle response statistics is established.

The main factor limiting the performance of the active suspension system is lack of sufficient information about the future road input. The performance could be improved if the future information about the road input is available and if it could be effectively utilized in the control scheme. This type of control problem is referred to as "preview control" and our future research issues include the following: apply proposed techniques to (1) consider optimal preview control of the linear/non-linear stationary and non-stationary response of a quarter car vehicle model, (2) to consider half car and full car linear/non-linear vehicle models to get a better insight about the load transfer among the corners of the vehicle.

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