Journal of Sound and Vibration (1998) **211**(4), 555–569

AN EXPERIMENTAL INVESTIGATION INTO THE TOPOLOGICAL STABILITY OF A CRACKED CANTILEVER BEAM

J. A. BRANDON

School of Engineering, University of Wales Cardiff, PO Box 917, *Cardiff CF*2 1*XH*, *Wales*

AND

C. SUDRAUD

Ecole Centrale de Lyon, 36 *avenue Guy de Collongue*, *BP* 163, *F*-69131 *Ecully Cedex*, *France*

(*Received* 24 *August* 1995, *and in final form* 7 *April* 1997)

In the study reported here, a vibrating polymeric cantilever beam was found to exhibit two distinct non-linear behaviour patterns. It is demonstrated that switching between these behaviour modes can be initiated by varying modal participation in a number of different ways: by changing the position of the exciter or the crack; by changing the amplitude of excitation; by changing the crack-depth ratio; or by changing the frequency of excitation. The indicator of transition between behaviour modes was the onset of period doubling. Clearly defined regions in the parameter space were identified.

1. INTRODUCTION

Experimental investigations of the non-linear vibration of engineering structures inevitably involve a degree of deviation—often considerably so—from the corresponding idealized analytical model. In order that the experimentalist may gain results that are in any way comparable with theoretical predictions, it is necessary to restrict the complexity of both the analytical model and the experimental analogue. Models may be classified according to three dimensions of complexity (see Figure 1).

First, the excitation has a number of degrees of complexity: it may be deterministic or stochastic; if deterministic it may be sinusoidal, periodic or transient; if stochastic it may be stationary or non-stationary.

Second, the structure itself may be simple or complex in its topology: represented, for instance, by a lumped parameter or distributed parameter topology respectively. For example, the turbine blade model described by Pfeiffer and Hajek [1] is a multi-degree-of-freedom model but the components are representable by lumped parameter components.

Third, the order of the components of the model may be varied to improve correspondence with empirical observations. For example: a first order velocity term will provide a conservative stability bound in aeroelastic galloping; to generate a limit cycle requires a third order velocity term; extension to a seventh order velocity term is necessary to illustrate the existence of multiple limit cycles [2, see, pp. 60–63].

0022–460X/98/140555 + 15 $25.00/0/sv971379

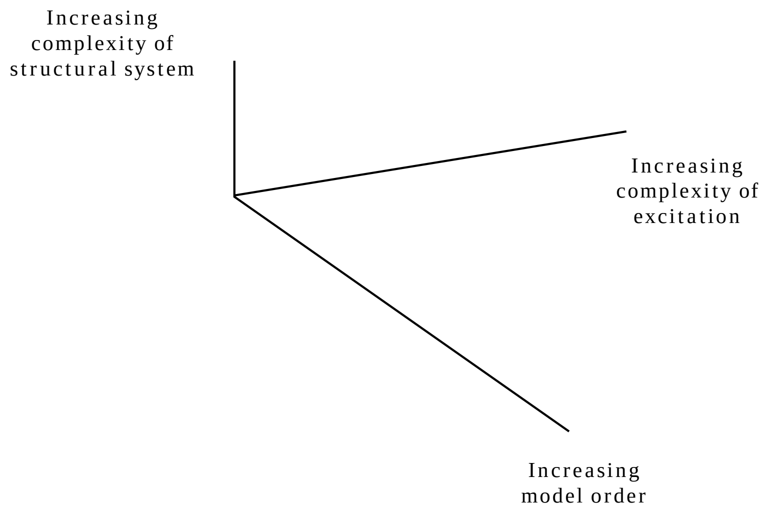


Figure 1. Dimensions of complexity of vibrating systems.

It is well known that even for the simplest analytical systems, response behaviour may be extremely rich for extremely small complexity dimension as assessed by the schema suggested above.

The expression *topological stability* has been chosen here rather than the more common *structural stability*. The latter term may be confused with its use in structural engineering to mean an unbounded physical response; in non-linear dynamics *structural stability* refers to the structure of the phase portrait where small changes in a parameter may cause large variations in response behaviour—although the solution may well remain bounded [2, see p. 110]. Drazin [3] discussed the concept of *structural stability* in some detail, remarking that many of the idealizations used in mathematical modelling ". . *are not found precisely in the* '*real world*' *they are designed to represent*." He emphasised the importance of identifying small irregularities of the idealized model which may cause large perturbations in its behaviour. He offered a formal definition of *structural stability*: "*If the qualitative nature of the set of all solutions of a system is changed by an infinitesimal perturbation of the system . . . then the system is said to be structural unstable*." [3, p. 55].

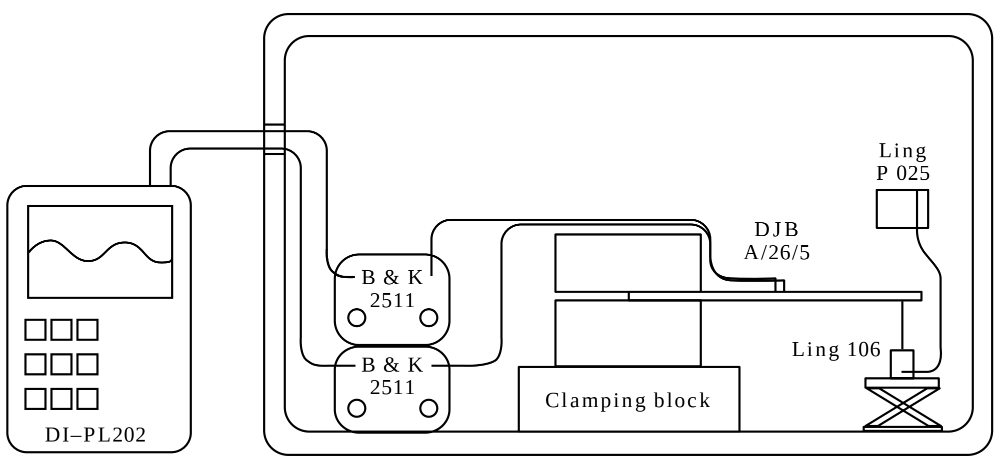


Figure 2. Experimental rig.

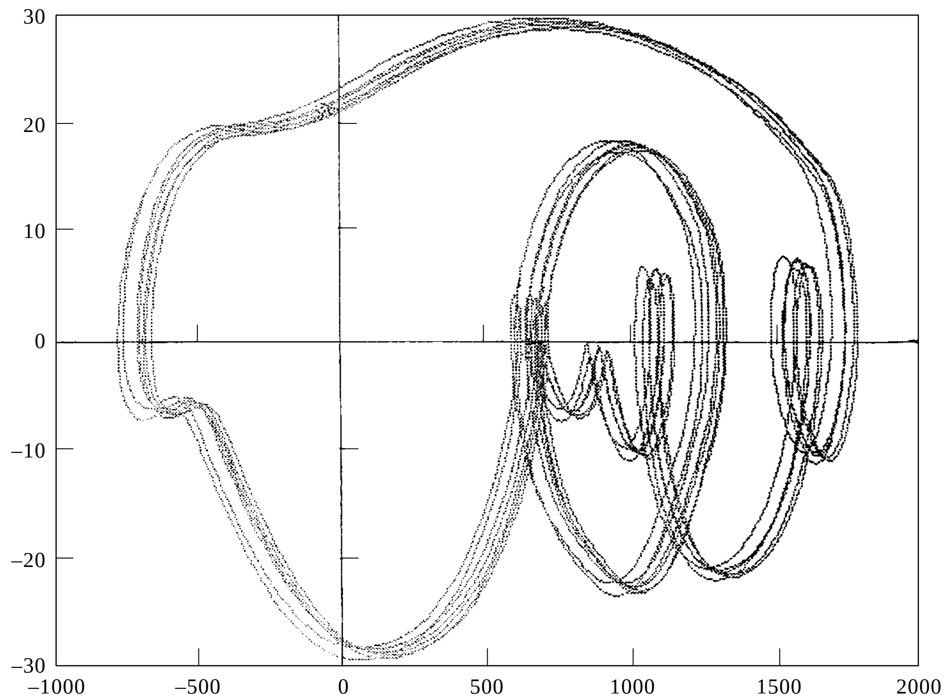


Figure 3. Phase plane trajectory, 10 Hz.

The present paper complements a number of previous theoretical studies [4–6] and extends experimental studies undertaken on the dynamics of cracked beams [7, 8] by the authors and colleagues. Reference [5] contains a substantial bibliography which complements the references cited here.

Because the system under investigation is intended to represent a realistic structure under service conditions, certain aspects of the dimensions of complexity of the model are beyond the control of the analyst. The properties of the structure are inherently distributed. Abraham and Brandon [5] chose to model the cantilever beam as two linear substructures connected by a strongly non-linear interface. Modelled as continua, each substructure had an infinite number of degrees of freedom. From the linear nature of the substructures, however, the degrees of freedom prior to assembly are separable and each of unity model order. In order to attain realistic correspondence with structural properties of the assembled structure, it proved necessary to include a large number of modes from each of the linear substructures. The temporal evolution of vibration was treated as piecewise linear with two linear states: open-crack and closed-crack, with a short interval of transition corresponding to closure or opening of the crack. The modeling of the transition process is far from straightforward and is described in an appendix to the recent paper by Brandon and Abraham [6]. This allowed the incorporation of typical non-linear effects such as impulsive closure of the crack and interface friction. The Lagrange multiplier approach used in this work eliminated anomalous effects at the interface between the two substructures, such as spatial interference, which were inevitable in the modelling approach used by previous workers [5].

Judged according to the criteria of dimensions of model complexity, described above, the assembly of the structural system implicitly converts a complex structural system with components of low order into a simple structural system with high model order. In this case, it was judged that the simplest of excitations—that of sinusoidal form—would generate a rich response behaviour. To date, therefore, the authors and co-workers have restricted the mathematical analysis and experimental programme to the response of cracked beams to two cases, free vibration and sinusoidal excitation.

With growing confidence in the integrity of the analytical model, testable predictions were made. On the basis of these studies it was resolved that focus on the existence and behaviour of subharmonics would give the greatest insight into the physics of the process.

2. EXPERIMENTAL EXPEDIENTS

Although the majority of the analysis [4–6] used a conservative model, it is virtually impossible to induce cracks into materials with negligible damping since crack susceptibility and material damping are inversely related; once a crack has been induced in such materials it is practically impossible to arrest its propagation. Attempts to simulate a crack by saw cuts or laying-up adhesively joined sheets of material failed to provide adequate performance [9].

The material chosen for experimental study in Cardiff in an Ultra-High-Molecular-Weight Polyethylene (UHMW-ethylene). The method of inducing a crack into this material was devised by Brandon and Macleod [9]. It involved alternating cycles of hot and cold soak (boiling water/liquid nitrogen) with loading in a standard three-point bending jig. The cracks induced proved to be sharp yet stable over the lifetime of a testing

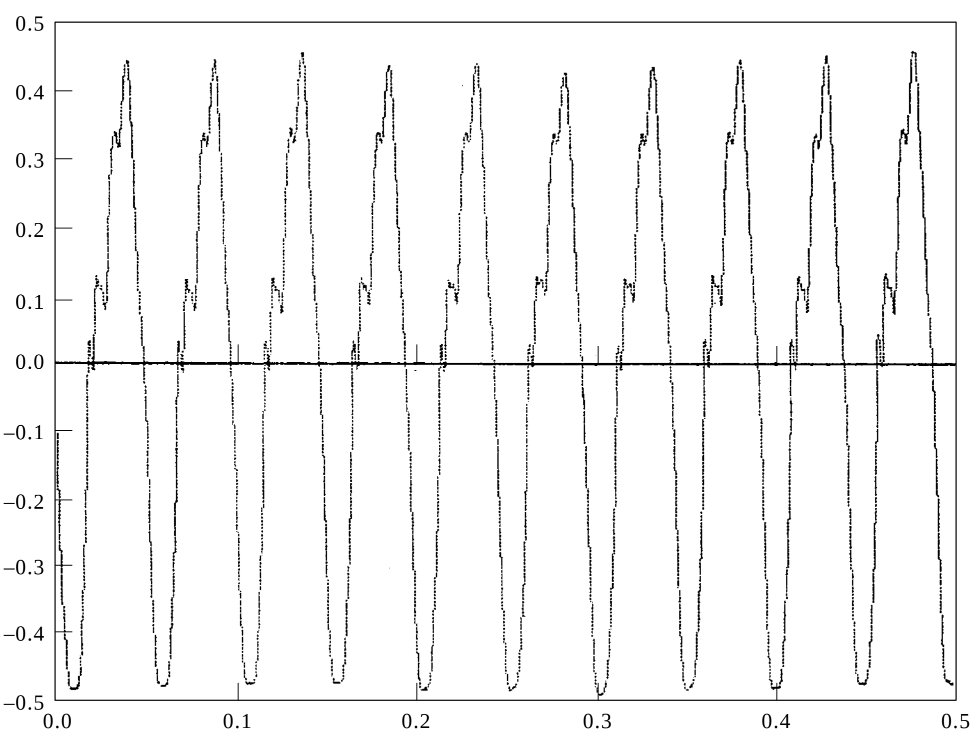


Figure 4. Displacement-time record, 20 Hz.

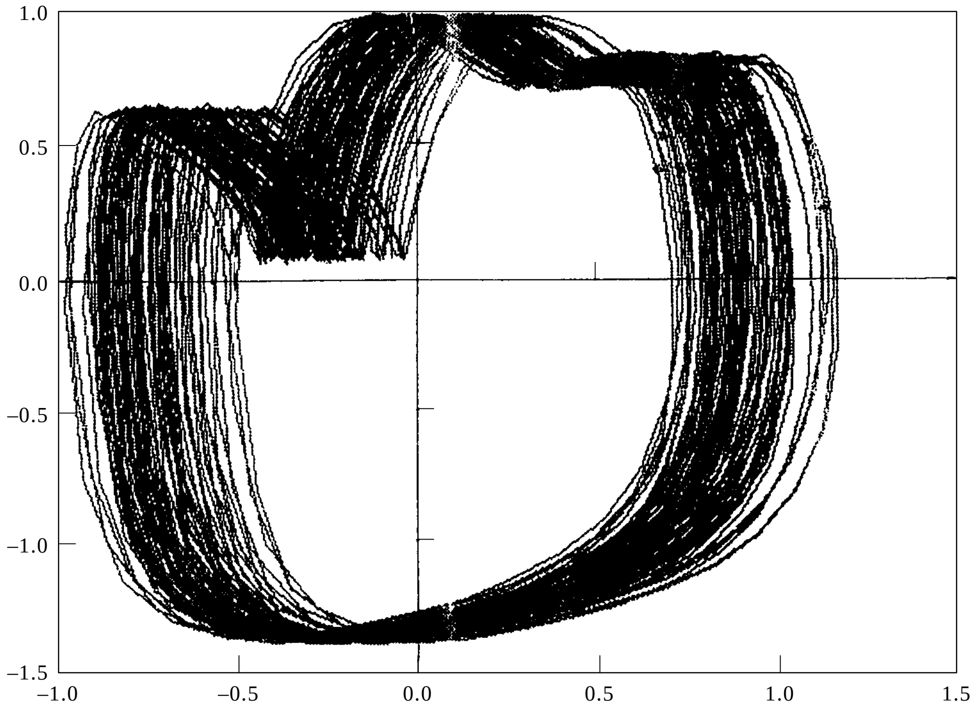


Figure 5. Phase plane trajectory, 20 Hz.

programme. Crack depth was measured post-test by staining the crack followed by destructive examination.

Because of the high amplitude levels commonly used in these tests, the possibility of inducing geometric non-linearities had to be taken into account. For this reason, a control specimen—without a crack—was cut from the same sheet as the experimental specimen and every test repeated on the control specimen under the same excitation conditions [7].

The scale and containment method of the equipment used was the cause of some amusement to visitors to the laboratory. The experimental specimen was extremely flexible (with typical dimensions 0·5 cm depth, 1·3 cm width, 30 cm length) but the exciter used was a Ling Dynamics type 407, most commonly used in Cardiff for excitation of structures of civil engineering scale. This apparent anomaly was to counter the suspicion that the smaller vibrator used in initial tests (LDS 106) was far from being an *ideal energy source* [10, see pp. 7 and 227]. There was also a tendency for the small vibrator to *bottom* at high displacement amplitude levels. The experimental specimen itself was mounted in a (relatively) massive steel clamping block, weighing some 25 kg. The potential for secondary transmission paths was minimized by placing each piece of equipment on expanded polystyrene blocks or foam rubber mouse mats. To eliminate the intrusion of external environmental noise as much as possible the experiment was contained within a commercial acoustic enclosure. To minimize the effects of electro-magnetic interference—particularly at mains frequency [9]—all instrumentation was battery powered. Height adjustment of the vibrator, to ensure that the beam was excited in a *breathing* condition, was achieved by using a precision *lab jack*. The initial experimental set-up is shown in Figure 2.

The transducers used were DJ Birchall A26 miniature charge-mode accelerometers. Signal conditioning and integration (to velocity and displacement) was carried out in hardware (Brüel & Kjaer 2511). A Digital Instruments PL202 instrument was used for: data acquisition and storage, display of spectra and phase plane trajectories. Post processing—largely comprising filtering out low frequency components due to the laboratory environmental conditions—was carried out in MATLAB.

3. PREVIOUS EXPERIMENTAL STUDIES

A conjecture by Brandon and Richards [11], based on experimental data from tests in clinical biomechanics, suggested that two primary effects would be observable in asymmetrically cracked beams. First, there would be a stiffness discontinuity resulting in different stiffnesses for crack compression and tension: second, at high excitation levels, the crack would close impulsively. Although the conjecture was based on intuitive arguments, the predictions were consistent with the subsequent analysis of Chu and Shen [12]. Both of these effects were observed by Macleod [9] together with a third feature which the authors described as the *competence* of the crack: for high amplitude levels and deep cracks the energy transmission capability across the crack section becomes small in comparison with overall energy levels in the two substructures. To all intents and purposes the short term behaviour of the two sections can be analyzed in terms of independent structures.

Each of the effects conjectured and observed is the subject of intense activity in the literature; each is both an analytical and experimental challenge in its own right [13, 14].

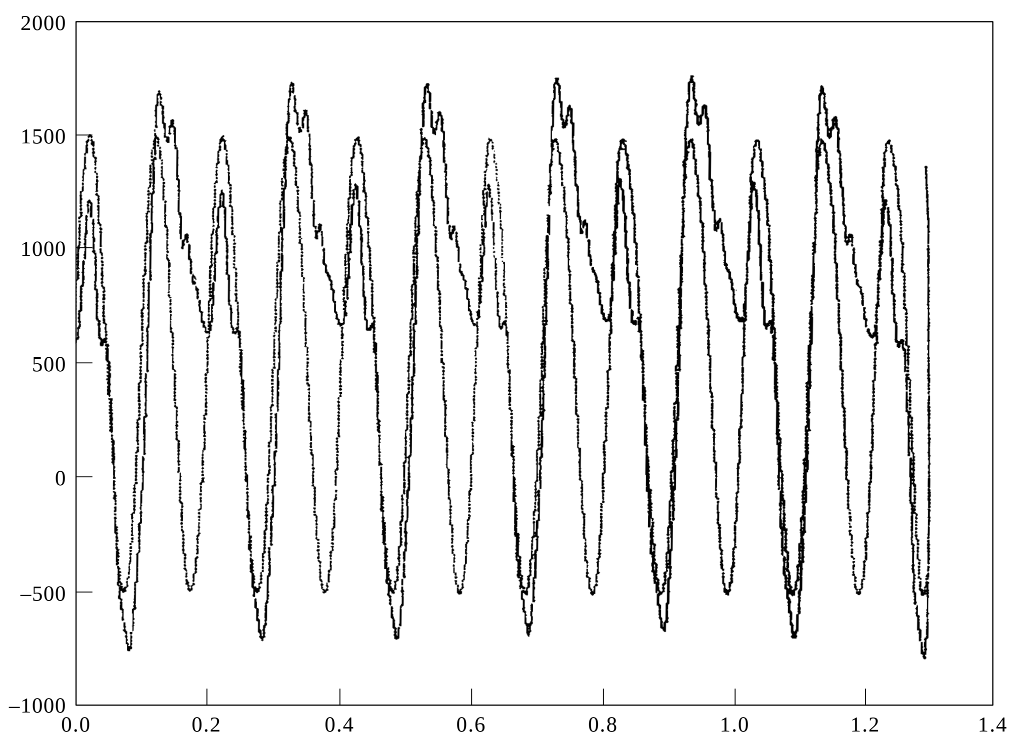


Figure 6. Displacement-time record, 10 Hz.

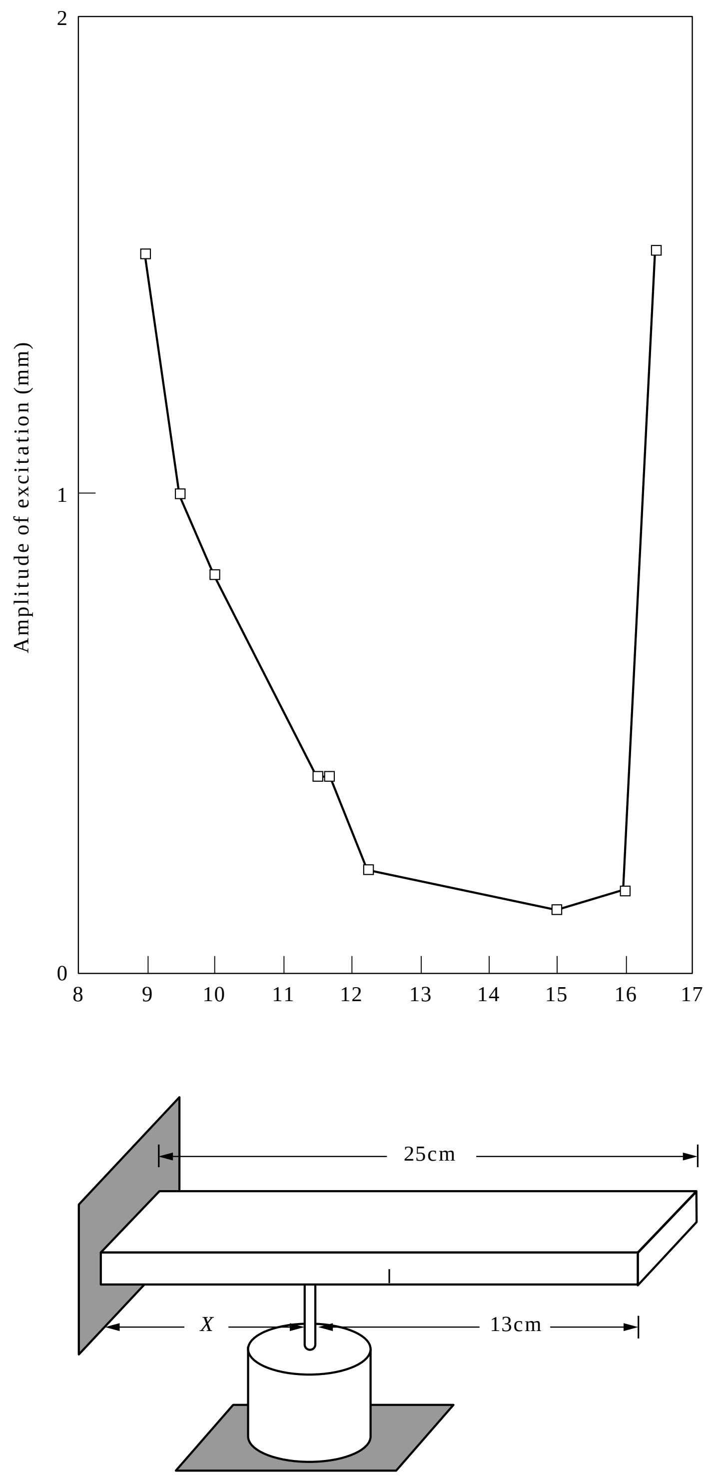


Figure 7. Effect of changing exciter position.

TABLE 1

Boundary of topological stability—changing position of exciter (25 cm beam)

X (cm)	Frequency (Hz)	Amplitude (mm)
9	11·5	0·415
9	12·5	0·664
9	10·25	1·245
9	9·75	1·5
9	15·75	1·5
10	14·5	0·37
10	11·5	0·83
10	12·75	0·5
10	15·75	0·25
10	16	1·5
10	9	1·5
11	15·75	0·25
11	13·25	0·581
11	12	1·245
11	9	1·5
11	16	1·5

When combined, counter-intuitive effects have been observed in both simulations and laboratory studies [6, 8]. In particular, mean-period oscillations observed in low order bilinear systems [2] were not observed in simulations [6], although the periodicities of the component linear systems were unexpectedly perpetuated in the assembled time record. Latterly, an experimental study [8] strongly suggests the existence of a twin-well oscillator. The tentative explanation for this is that the repeated transition between open and closed crack conditions constitutes a higher energy state than either of the alternative conditions.

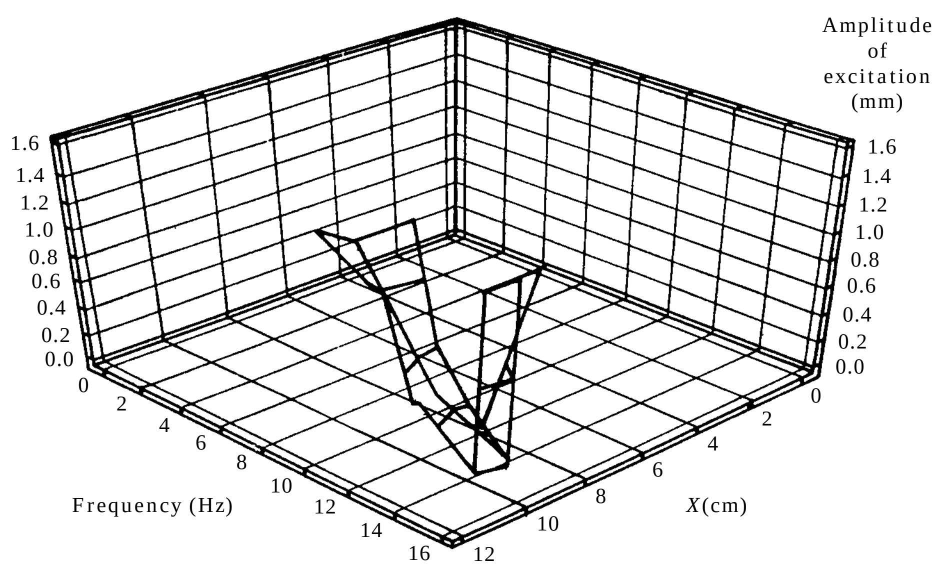


Figure 8. Boundary of period doubling region—changing exciter position.

TABLE 2

Boundary of topological stability—changing position of exciter (28 *cm beam*)

X (cm)	Frequency (Hz)	Excitation (mm)
10	10	0·581
10	8·5	1·24
10	11·75	0·166
10	12	1·5
14	12·25	0·21
14	10	0·83
14	11·5	0·41
14	9·5	1
14	11·75	0·41
14	15	0·13
14	16	0·166
14	16·5	1·5
11	10·75	0·415
11	9·25	0·664
11	8·75	1·24
11	11·5	0·25
11	13·5	1·5
11	12	0·166
12·5	13·75	0·2075
12·5	15·5	1·5
12·5	14	0·2075
12·5	14·25	0·2075
12·5	9·5	0·664
12·5	9	1·08
12·5	11·5	0·23

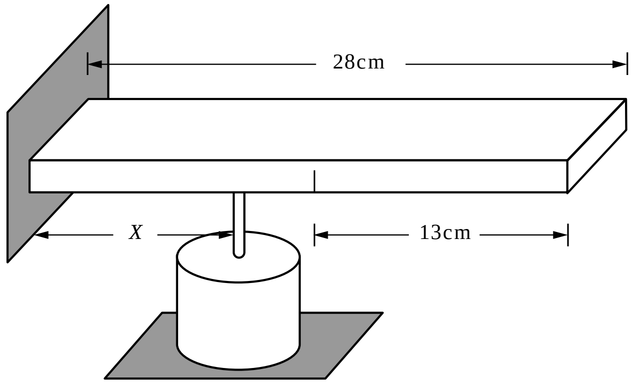
4. SCOPE OF PRESENT STUDY

As has been mentioned, earlier studies have indicated that the study of subharmonic phenomena would probably be the most fruitful area of experimental interest. This was confirmed when regular attractors of significant complexity were observed [7]. Figure 3 shows six cycles of response which correspond to 12 cycles of forcing, i.e. a period doubling (reported more fully elsewhere [7]). Thus, the remainder of the current paper is devoted to investigating the conditions necessary to induce period doubling. There is some evidence for the existence of higher order subharmonics—in particular one of period three—but these have yet to be explored systematically.

The variations in experimental parameters investigated here were determined by the following considerations: (1) changing the position of the exciter would affect the modal participation because of the mode shapes, i.e., the relative excitation of different modes would alter; (2) changing the position of the crack would also affect the modal participation; (3) changing the amplitude of excitation would affect the relative importance of bilinear stiffness and impulsive closure; (4) changing the frequency would also affect modal participation but for a different reason, that the frequency response of the individual modes would vary.

5. EXPERIMENTAL RESULTS

From the work of Collins *et al*. [15], it was expected that significant non-linear effects would be observable marginally below the first natural frequency of the undamaged beam.



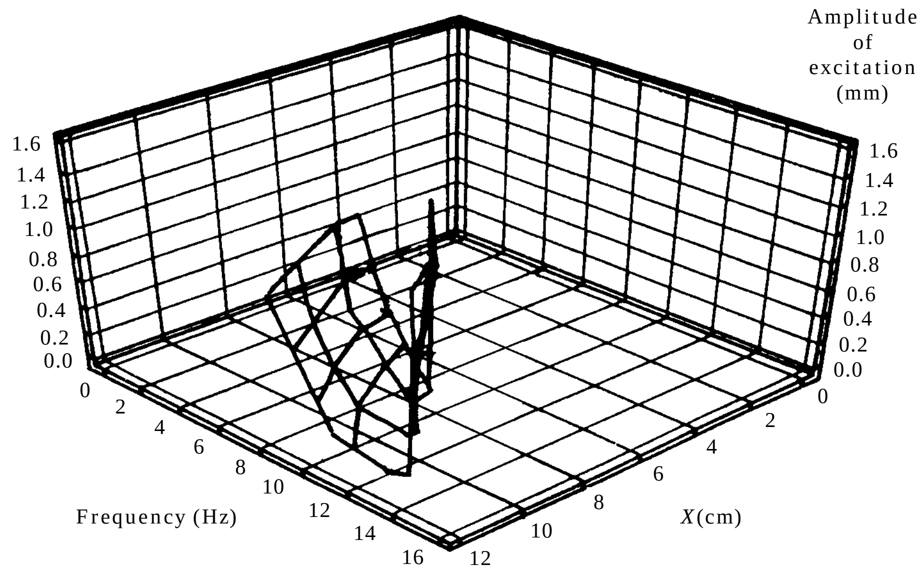


Figure 9. Topological stability boundary—changing exciter position (28 cm beam).

In the work reported here, most significant effects were apparent far below this frequency. In this case the first natural frequency of the undamaged beam was approximately 100 Hz but the most interesting results came from excitations in the 10–30 Hz region.

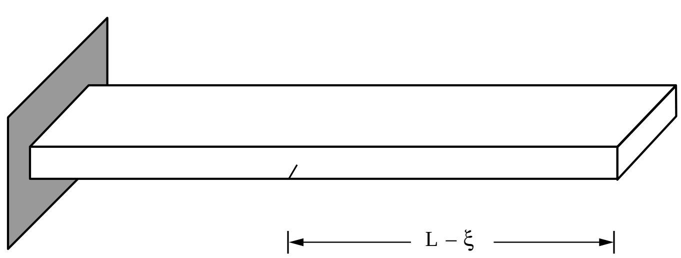


Figure 10. Variation of crack position.

D.R. 846564—JSV 211/4 (Issue)—MS 2290

TABLE 3

Topological stability—varying crack position

$L - \xi$ (cm)	Frequency (Hz)	Amplitude (mm)
20	7	2·5
20	7·75	1
20	8	1·66
20	8·5	0·913
20	9·3	0·587
20	10·75	2·5
19	7·5	2·5
19	8·5	1·24
19	9	0·913
19	9·5	0·5
19	10	0·37
19	10·75	2·5
17	8	2·5
17	8·5	1·66
17	9	1
17	10	0·83
17	11·25	0·41
17	12	2·5
16	8·5	2·5
16	9·25	1·24
16	10·25	1
16	11·5	0·58
16	12·70	2·5
15	8·5	2·5
15	9·25	2
15	10·75	1·24
15	11·5	2·5
13	9·25	2·5
13	10	1·24
13	11	2·5
12	9·75	2·5

In studies based on analytical simulations, the sampling frequency of the Poincaré section is usually fixed and corresponds either to a forcing frequency or some characteristic frequency of the system of interest. In experimental studies, the interference from noise (often itself correlated) can extinguish any evidence of periodicity in the trajectory. The failure of the vibrator to act as an ideal energy source would also lead to variability in the period of the forcing signal. In these circumstances Moon [16] recommends alternative sampling strategies such as a *position triggered* Poincaré map [16, see pp. 135–139]. In this study this concept was extended, with the *event triggered* Poincaré section triggered from the force signal. For this a signal from the peak detector of the B & K 2511 signal conditioning amplifier, derived from a transducer placed directly above the vibrator, was used as an external clock signal to the PL202 analyzer.

Figure 4 shows the time domain behaviour of the displacement response of the beam when driven at a frequency of 20 Hz. Although the driving force signal is sinusoidal, the displacement response is periodic at the same frequency but with clear harmonic components. The transition into the compressive half-cylce is accompanied by a number of impulsive events. The phase plane trajectory for this response is shown in Figure 5.

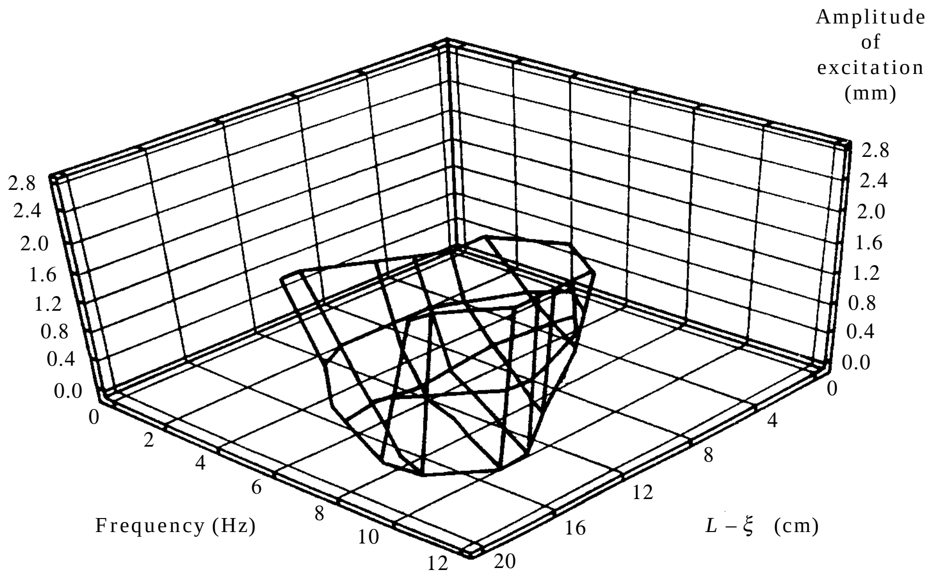


Figure 11. Topological stability—varying crack position.

The interpretation of phase plane response at 10 Hz, shown in Figure 3, is more challenging. Once again, the closure of this response trajectory implies that this is a periodic signal but clearly with a complex set of harmonics. As described elsewhere [7], the event triggered Poincaré section reveals two clusters of points on the trajectory hence period doubling behaviour. This can be seen in the time record, Figure 6, where the sinusoidal force trace is superimposed on the displacement response. It is clear that the displacement response only repeats itself after two periods of the force signal.

Table 4

Topological stability—varying crack depth ratio

Crack depth ratio	Frequency (Hz)	Amplitude (mm)
0·75	7	2·5
0·75	7·75	2
0·75	8	1·66
0·75	8·5	0·913
0·75	9·5	0·587
0·75	10·75	2·5
0·8	7	2·5
0·8	7·75	0·913
0·8	9·25	0·313
0·8	10	0·335
0·8	10·5	0·581
0·8	11	0·415
0·8	12	2·5
0·85	7	2·5
0·85	8	0·45
0·85	8·5	0·332
0·85	10	0·415
0·85	10·25	2·5

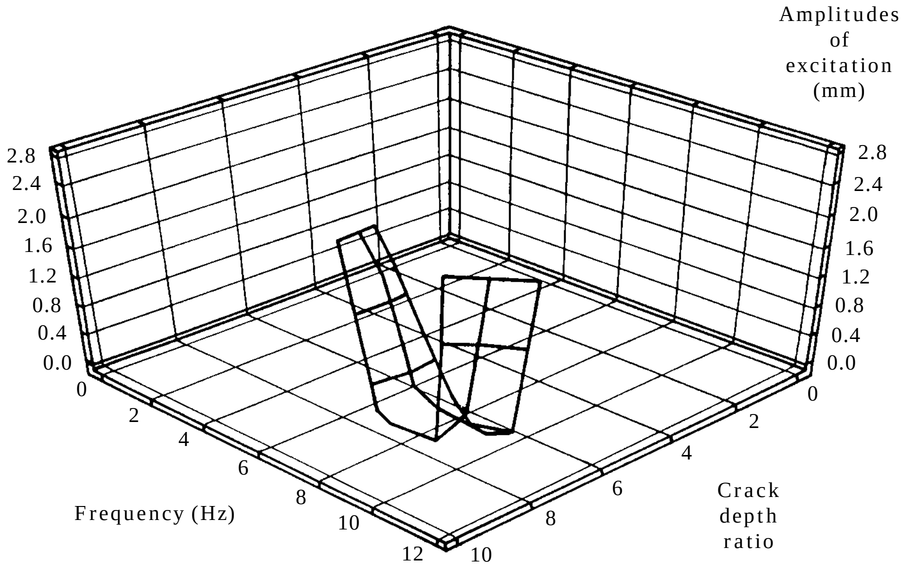


Figure 12. Topological stability boundary—varying crack depth ratio.

The remainder of the paper presents surfaces of onset of period doubling as experimental parameters are varied. In Figure 7 the beam length is set at 25 cm, with the crack 13 cm from the free end. The position of the exciter was varied and the frequency-amplitude pairs for onset of period doubling evaluated. These are presented in Table 1. The response surface is shown in Figure 8. As can be seen, the closer the exciter is placed to the crack the easier it is to induce the period doubling.

The beam was then re-set into the clamping block with an effective length of 28 cm. The results are broadly similar, as shown in Table 2 and Figure 9.

In the next test, the crack position was varied (see Figure 10). The results, shown in Table 3 and Figure 11, indicate that the greater the length of the beam beyond the crack, the easier it is to induce the characteristic period doubling behaviour.

The severity of the crack itself was evaluated by varying the crack-depth ratio progressively. Table 4 and Figure 12 illustrate the response surface for increasing crack-depth ratio.

It should be noted that the scales of the response are somewhat arbitrary. The deformations of the beams were large, typically 100 mm of end displacement for a beam of length 300 mm. Thus, the possibility of geometric non-linearities in response records had to be taken seriously. This was found to be the case and each test on a cracked beam was accompanied by a test on an undamaged specimen under the same experimental conditions. The records for the uncracked specimens contained evidence of harmonics in their spectrum but these were not apparent on visual inspection of the time records—qualitatively different from the low amplitude excitation of the cracked beams.

6. DISCUSSION

As has been remarked, both simulations [6] and experimental studies [8] have revealed response behaviour in systems of high order which are unexpected from low order models [2]. There is reason to believe that the response behaviour of high order systems will be parametric in the excitation frequency [5] and that this response will be sensitive to small changes in excitation conditions or—in the case of the polymeric specimens used here, which have both temperature and frequency dependence in the visco-elastic modulus—to ambient conditions. On the basis of these earlier studies, and the literature, the prospect of identifying regular behaviour at all was far from assured. More likely was the expectation that the coupling between the (multiple incommensurable) modes of the beam and the excitation would lead to irregular response trajectories.

In the examples shown, two distinct regular non-linear responses are identifiable. The response in Figure 4 is periodic and synchronous with the period of the forcing frequency. The response in Figure 6 is again commensurable but now at half the forcing frequency. The transitions between these states could be controlled by varying the parameters under study: the position of the exciter and crack; the amplitude of excitation; frequency of excitation; and the crack depth ratio. Once the transition threshold had been passed, the response behaviour proved to be remarkably robust: e.g., invariant to switching the equipment on and off overnight, leaving the equipment switched on for long periods, dismantling and re-setting.

Perhaps worthy of mention are those phenomena which it proved impossible to induce. From the literature, cascades of period doubling might reasonably have been expected. In the studies reported here, responses of period four, eight, etc., were not observed. What appeared to be a period three trajectory *was* recorded, and would be extremely interesting, but this was not repeatable.

It should be noted that the results reported here were far from typical of the response behaviour. The transition surfaces shown in Figures 8–10 and 12 correspond to small regions of the parameter space and no regions were identified which corresponded to modes above the first.

7. CONCLUSIONS

This study has demonstrated that the response behaviour of a cracked vibrating beam contains regions of topological instability, indicated by period doubling behaviour. It was not possible to induce further bifurcations of the response. Response behaviour below the boundary of topological instability contained characteristic non-linear characteristics such as harmonics of the forcing frequency.

REFERENCES

1. F. PFEIFFER and M. HAJEK 1992 *Philosophical Transactions of the Royal Society of London* **A338**, 503–507. Stick–slip motion of turbine blade dampers.
2. J. M. T. THOMPSON and H. B. STEWART 1986 *Nonlinear Dynamics and Chaos*. Chichester: John Wiley. See pp. 60–63.
3. P. G. DRAZIN 1992 *Nonlinear Systems*. Cambridge: Cambridge University Press.
4. J. A. BRANDON and O. N. L. ABRAHAM 1994 *Proceedings of the Fifth International Conference*: *Recent Advances in Structural Dynamics*, *Institute of Sound and Vibration Research*, *Southampton* **1**, 78–87. A qualitative analysis of the transient behaviour of a cracked Timoshenko beam.
5. O. N. L. ABRAHAM and J. A. BRANDON, 1995 *Transactions of the American Society of Mechanical Engineers*: *Journal of Vibration and Acoustics* **117**, 370–377. The modelling of the opening and closure of a crack.
6. J. A. BRANDON and O. N. L. ABRAHAM 1995 *Journal of Sound and Vibration* **185**, 415–430. Counter-intuitive quasi-periodic motion in the autonomous vibration of cracked beams.
7. J. A. BRANDON, C. SUDRAUD and K. M. HOLFORD 1994 *Proceedings of the Fifth International Conference*: *Recent Advances in Structural Dynamics*, *Institute of Sound and Vibration Research*, *Southampton* **1**, 234–242. An experimental study of the dynamics of a cracked beam.

8. J. A. BRANDON and M. H. MATHIAS 1995 *Journal of Sound and Vibration* **186**, 350–354. Complex oscillatory behaviour in a cracked beam under sinusoidal excitation.
9. J. A. BRANDON and D. MACLEOD 1991 *Proceedings of the 2nd International Conference Interfaces in Medicine and Mechanics*, *Bologna* (K. R. Williams *et al*. editors) 406–412. Amsterdam: Elsevier Applied Science. An investigation into the interface mechanics in fractures, part two: experimental results.
10. A. H. NAYFEH and D. T. MOOK 1979 *Nonlinear Oscillations*. New York: John Wiley.
11. J. A. BRANDON and J. RICHARDS 1989 *Proceedings of the Institution of Mechanical Engineers*, *Series H*: *Journal of Engineering in Medicine* **204**, 203–205. A conjecture on the interface mechanics in fractures based on the interpretation of impulse tests.
12. Y. C. CHU and M.-H. H. SHEN 1992 *AIAA Journal* **30**, 2512–2519. Analysis of forced bilinear oscillators and the application to cracked beam dynamics.
13. M. KLECZKA, E. KREUZER and W. SCHIELEN 1992 *Philosophical Transactions of the Royal Society* **A338**, 533–546. Local and global stability of a piecewise linear oscillator.
14. S. FOALE and S. R. BISHOP 1992 *Philosophical Transactions of the Royal Society* **A338**, 547–557. Dynamical complexities of forced impacting systems.
15. K. R. COLLINS, R. H. PLAUT and J. WAUER 1992 *Transactions of the American Society of Mechanical Engineers*: *Journal of Vibration and Acoustics* **114**, 171–177. Free and forced vibrations of a cantilevered bar with a crack.
16. F. C. MOON 1987 *Chaotic Vibrations*: *An Introduction for Applied Scientists and Engineers*. New York: John Wiley.