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LETTERS TO THE EDITOR

ON THE DYNAMIC ANALYSIS OF A FLEXIBLE L-SHAPED STRUCTURE

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1. INTRODUCTION

Recently, an interesting study by Bang [1] was published on the natural vibration problem of a flexible L-shaped beam. It was noted that this structure can be used as a space structure model of space antennas and reflectors. Motivated by this publication, the present note deals with the same mechanical system where a tip mass is considered also, through which, in the author's opinion, a more realistic model is obtained. During the efforts of incorporating the tip mass into the formulations, some serious errors were detected in reference [1] which give rise to substantial changes in the frequency equation. The correct form of the frequency equation, which is much longer than noted in reference [1], is given in reference [2].

2. THEORY

The problem to be investigated in the present note is the natural vibration problem of the mechanical system shown in Figure 1. It consists of an L-shaped beam carrying a tip mass M at the free end. The physical properties of the system are as follows. The length, mass per unit length and bending rigidity of the ith beam are L_i, m_i and E_iI_i, respectively ($i = 1, 2$). The beam is considered to vibrate only in the plane of the paper. The two beams are modelled as Bernoulli–Euler beams. The planar bending displacements of the beams in the co-ordinate systems x_1, w_1 and x_2, w_2 are denoted as $w_1(x_1, t)$ and $w_2(x_2, t)$ where both are assumed to be small.

In order to obtain the equations of motion of the system, Hamilton's principle

$$\int_{t_0}^{t_1} \delta(T - V)\,\mathrm{d}t = 0 \tag{1}$$

will be applied, where T and V denote the kinetic and potential energies respectively. The total kinetic energy

$$T = \sum_{i=1}^{5} T_i \tag{2}$$

consists of the following parts:

$$T_1 = \tfrac{1}{2}m_1 \int_0^{L_1} \dot{w}_1^2(x_1, t)\,\mathrm{d}x_1, \qquad T_2 = \tfrac{1}{2}m_2 \int_0^{L_2} (\dot{w}_2 + x_2\dot{\alpha})^2\,\mathrm{d}x_2,$$

$$T_3 = \tfrac{1}{2}m_2 \int_0^{L_2} \dot{w}_1^2(L_1, t)\,\mathrm{d}x_2, \qquad T_4 = \tfrac{1}{2}M\dot{w}_1^2(L_1, t), \qquad T_5 = \tfrac{1}{2}M[(\dot{w}_2 + x_2\dot{\alpha})|_{x_2 = L_2}]^2, \tag{3}$$

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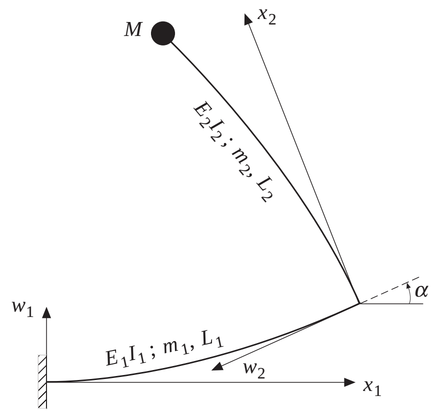

Figure 1. Flexible L-shaped beam.

The potential energy consists of two parts due to bending deformations,

$$V = V_1 + V_2, \tag{4}$$

where

$$V_1 = \tfrac{1}{2} E_1 I_1 \int_0^{L_1} w_1''^2(x_1, t)\,\mathrm{d}x_1, \qquad V_2 = \tfrac{1}{2} E_2 I_2 \int_0^{L_2} w_2''^2(x_2, t)\,\mathrm{d}x_2. \tag{5}$$

In the formulations above $\alpha(t)$ is defined as

$$\alpha \approx \partial w_1(x_1, t)/\partial x_1|_{x_1 = L_1} = w_1'(L_1, t), \tag{6}$$

i.e., the slope of the bent first beam at its end, and dots and primes denote partial derivatives with respect to time t and position co-ordinate x_1 or x_2 respectively. After putting expressions (2) to (5) into expression (1) and carrying out the necessary variations, the following equations of motion of the beams are obtained:

$$E_1 I_1 w_1^{IV}(x_1, t) + m_1 \ddot{w}_1(x_1, t) = 0, \qquad E_2 I_2 w_2^{IV}(x_2, t) + m_2(\ddot{w}_2(x_2, t) + x_2 \ddot{\alpha}(t)) = 0. \tag{7, 8}$$

These are the same differential equations as in reference [1].

The corresponding boundary and matching conditions for the two elastic domains are as follows:

$$w_1(0, t) = 0, \qquad w_1'(0, t) = 0, \qquad w_2(0, t) = 0, \qquad w_2'(0, t) = 0, \tag{9–12}$$

$$\int_0^{L_2} m_2 \ddot{w}_1(L_1, t)\,\mathrm{d}x_2 - E_1 I_1 w_1'''(L_1, t) + M \ddot{w}_1(L_1, t) = 0, \tag{13}$$

$$\int_0^{L_2} m_2(x_2 \ddot{\alpha} + \ddot{w}_2) x_2\,\mathrm{d}x_2 + E_1 I_1 w_1''(L_1, t) + M L_2(L_2 \ddot{\alpha} + \ddot{w}_2(L_2, t)) = 0, \tag{14}$$

$$E_2 I_2 w_2'''(L_2, t) - M(L_2 \ddot{\alpha} + \ddot{w}_2(L_2, t)) = 0, \qquad E_2 I_2 w_2''(L_2, t) = 0. \tag{15, 16}$$

These boundary conditions differ from those given in reference [1] due to the terms containing the tip mass M.

Using the standard method of separation of variables, one assumes

$$w_1(x_1, t) = W_1(x_1)\cos\omega t, \qquad w_2(x_2, t) = W_2(x_2)\cos\omega t, \tag{17, 18}$$

where $W_1(x_1)$, $W_2(x_2)$ are the corresponding amplitude functions of the beams and ω is the unknown eigenfrequency of the combined system. Substituting these expressions into the partial differential equations (7) and (8) results in the following ordinary differential equations:

$$W_1^{IV}(x_1) - \beta^4 W_1(x_1) = 0, \qquad W_2^{IV}(x_2) - \mu^4\beta^4[W_2(x_2) + W_1'(L_1)x_2] = 0, \tag{19, 20}$$

Here, the following abbreviations are introduced:

$$\beta^4 = m_1\omega^2/E_1I_1, \qquad \mu^4 = \alpha_m/\chi, \qquad \chi = E_2I_2/E_1I_1, \qquad \alpha_m = m_2/m_1. \tag{21}$$

Now, the corresponding boundary conditions are

$$W_1(0) = 0, \qquad W_1'(0) = 0, \qquad W_2(0) = 0, \qquad W_2'(0) = 0, \tag{22–25}$$

$$E_1I_1W_1'''(L_1) + \omega^2 W_1(L_1)(M + M_2) = 0, \tag{26}$$

where $M_2 = m_2L_2$,

$$m_2\omega^2\int_0^{L_2}[x_2W_1'(L_1) + W_2(x_2)]x_2\,\mathrm{d}x_2 - E_1I_1W_1''(L_1) + ML_2\omega^2[L_2W_1'(L_1) + W_2(L_2)] = 0, \tag{27}$$

$$E_2I_2W_2'''(L_2) + M\omega^2[L_2W_1'(L_1) + W_2(L_2)] = 0, \qquad E_2I_2W_2''(L_2) = 0. \tag{28, 29}$$

Here primes on $W_1(x_1)$ and $W_2(x_2)$ denote derivatives with respect to position co-ordinates x_1 and x_2, respectively. The general solutions of the ordinary differential equations (19) and (20) are simply

$$W_1(x_1) = C_1\beta x_1 + C_2\cos\beta x_1 + C_3\,\mathrm{sh}\,\beta x_1 + C_4\,\mathrm{ch}\,\beta x_1, \tag{30}$$

$$W_2(x_2) = D_1\sin\mu\beta x_2 + D_2\cos\mu\beta x_2 + D_3\,\mathrm{sh}\,\mu\beta x_2 + D_4\,\mathrm{ch}\,\mu\beta x_2 - W_1'(L_1)x_2, \tag{31}$$

where C_1–C_4 and D_1–D_4 are eight integration constants to be evaluated via conditions (22–29).

In the first step, the application of conditions (22) and (23) to the solution (30) and conditions (24) to the solution (31) yields

$$W_1(x_1) = C_1(\sin\beta x_1 - \mathrm{sh}\,\beta x_1) + C_2(\cos\beta x_1 - \mathrm{ch}\,\beta x_1), \tag{32}$$

$$W_2(x_2) = D_1\sin\mu\beta x_2 + D_2(\cos\mu\beta x_2 - \mathrm{ch}\,\mu\beta x_2) + D_3\,\mathrm{sh}\,\mu\beta x_2 - W_1'(L_1)x_2. \tag{33}$$

Once the shape functions in equations (32) and (33) are substituted into conditions (25–29) a set of five algebraic equations is obtained for the five unknown coefficients C_1, C_2; D_1, D_2 and D_3. This set of equations can be written in matrix notation as

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 \\ 0 & 0 & A_{23} & A_{24} & A_{25} \\ 0 & 0 & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & 0 & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{34}$$

where

$$A_{11} = -(\cos\overline{\beta} + \mathrm{ch}\,\overline{\beta}) + (\alpha_M + \alpha_m\alpha_L)\overline{\beta}(\sin\overline{\beta} - \mathrm{sh}\,\overline{\beta}),$$

$$A_{12} = (\sin\overline{\beta} - \mathrm{sh}\,\overline{\beta}) + (\alpha_M + \alpha_m\alpha_L)\overline{\beta}(\cos\overline{\beta} - \mathrm{ch}\,\overline{\beta}),$$

$$A_{23} = \sin\delta\overline{\beta}, \qquad A_{24} = \cos\delta\overline{\beta} + \mathrm{ch}\,\delta\overline{\beta}, \qquad A_{25} = -\mathrm{sh}\,\delta\overline{\beta},$$

$$A_{33} = -\cos\delta\overline{\beta} + (\alpha_M\mu/\alpha_m)\overline{\beta}\sin\delta\overline{\beta},$$

$$A_{34} = (\sin\delta\overline{\beta} - \mathrm{sh}\,\delta\overline{\beta}) + (\alpha_M\mu/\alpha_m)\overline{\beta}(\cos\delta\overline{\beta} - \mathrm{ch}\,\delta\overline{\beta}),$$

$$A_{35} = \mathrm{ch}\,\delta\overline{\beta} + (\alpha_M\mu/\alpha_m)\overline{\beta}\,\mathrm{sh}\,\delta\overline{\beta}, \qquad A_{41} = -(\cos\overline{\beta} - \mathrm{ch}\,\overline{\beta}), \qquad A_{42} = \sin\overline{\beta} + \mathrm{sh}\,\overline{\beta},$$

$$A_{43} = \mu,$$

$$A_{45} = \mu, \qquad A_{51} = (1/\alpha_m\overline{\beta}^2)(\sin\overline{\beta} + \mathrm{sh}\,\overline{\beta}), \qquad A_{52} = (1/\alpha_m\overline{\beta}^2)(\cos\overline{\beta} + \mathrm{ch}\,\overline{\beta}),$$

$$A_{53} = (\alpha_M\alpha_L/\alpha_m + 1/\mu^2\overline{\beta}^2)\sin\delta\overline{\beta} - (\alpha_L/\mu\beta)\cos\delta\overline{\beta},$$

$$A_{54} = \left(\frac{\alpha_M\alpha_L}{\alpha_m} + \frac{1}{\mu^2\overline{\beta}^2}\right)\cos\delta\overline{\beta} - \left(\frac{\alpha_M\alpha_L}{\alpha_m} - \frac{1}{\mu^2\overline{\beta}^2}\right)\mathrm{ch}\,\delta\overline{\beta} + \frac{\alpha_L}{\mu\overline{\beta}}(\sin\delta\overline{\beta} - \mathrm{sh}\,\delta\overline{\beta}) - \frac{2}{\mu^2\overline{\beta}^2},$$

$$A_{55} = (\alpha_M\alpha_L/\alpha_m - 1/\mu^2\overline{\beta}^2)\,\mathrm{sh}\,\delta\overline{\beta} + (\alpha_L/\mu\overline{\beta})\,\mathrm{ch}\,\delta\overline{\beta}. \tag{35}$$

Here, in addition to those given in equations (21) the following abbreviations are introduced:

$$\overline{\beta} = \beta L_1, \qquad \alpha_L = L_2/L_1, \qquad \alpha_M = M/m_1L_1, \qquad \delta = \mu\alpha_L. \tag{36}$$

For a non-trivial solution to exist, the determinant of the coefficient matrix in equation (34) should be equal to zero. Thus, the frequency equation reads

$$\det\mathbf{A} = 0 \tag{37}$$

where $\mathbf{A}(\overline{\beta})$ denotes the coefficient matrix. The roots of the transcendental frequency equation above give one the dimensionless frequency parameters $\overline{\beta}$ and therefore, by considering the relationship (21), the eigenfrequencies ω of the mechanical system in Figure 1.

The interest here lies not only in obtaining the eigenfrequencies of the system, but also in the mode shapes of the system. To this end, from the first four equations of the homogeneous system (34) the following set of equations is obtained for the determination of the coefficients C_2, D_1, D_2 and D_3:

$$C_2 = -A_{11}/A_{12}, \qquad A_{23}D_1 + A_{24}D_2 + A_{25}D_3 = 0, \tag{38}$$

$$A_{33}D_1 + A_{34}D_2 + A_{35}D_3 = 0, \qquad A_{43}D_1 + A_{45}D_3 = -A_{41} + A_{11}A_{42}/A_{12}. \tag{39}$$

Here $C_1 = 1$ is taken. After solving the set of inhomogeneous equations (39) with respect to D_1, D_2 and D_3 and substituting these values and C_2 from equations (38) into equations (32) and (33), the modes shapes of the system are obtained in the forms

$$W_1(\bar{x}_1) = \sin\overline{\beta}\bar{x}_1 - \mathrm{sh}\,\overline{\beta}\bar{x}_1 + C_2(\cos\overline{\beta}\bar{x}_1 - \mathrm{ch}\,\overline{\beta}\bar{x}_1), \tag{40}$$

$$W_2(\bar{x}_2) = D_1\sin\delta\overline{\beta}\bar{x}_2 + D_2(\cos\delta\overline{\beta}\bar{x}_2 - \mathrm{ch}\,\delta\overline{\beta}\bar{x}_2) + D_3\,\mathrm{sh}\,\delta\overline{\beta}\bar{x}_2 - \alpha_L\overline{\beta}\bar{x}_2[\cos\overline{\beta} - \mathrm{ch}\,\overline{\beta} - C_2(\sin\overline{\beta} + \mathrm{sh}\,\overline{\beta})], \tag{41}$$

where the non-dimensional position co-ordinates $\bar{x}_1 = x_1/L_1$ and $\bar{x}_2 = x_2/L_2$ are introduced.

TABLE 1

Effect of the variation of the tip mass ratio α_M on the dimensionless eigenfrequency parameters

α_M	$\bar{\beta}_1$	$\bar{\beta}_2$	$\bar{\beta}_3$	$\bar{\beta}_4$	$\bar{\beta}_5$
0	1·082488	1·786316	3·969206	4·805330	7·098548
0·1	1·034472	1·703182	3·788805	4·693284	6·850588
0·5	0·907887	1·524326	3·562973	4·592503	6·648926
1	0·817962	1·405996	3·491855	4·555162	6·601915
2	0·717740	1·267142	3·446870	4·523385	6·575042
5	0·587663	1·067160	3·415884	4·493309	6·557635
10	0·499338	0·918684	3·404774	4·479730	6·551591
15	0·452811	0·837192	3·400976	4·474622	6·549547
20	0·422146	0·782512	3·399058	4·471940	6·548520

3. NUMERICAL RESULTS

This section is devoted to the numerical evaluation of formulas established in the preceding section. For the numerical applications, the following values were chosen for the physical data of the mechanical system in Figure 1: $\alpha_L = \alpha_m = \chi = 1$. This means that the two beams of the L-shaped structure are considered to be identical.

The first five dimensionless frequency parameters of the system described above are given in Table 1 for various values of the tip mass ratio $\alpha_M = M/m_1 L_1$. To this end, the transcendental equation (37) was solved with MATLAB. The values in the first row correspond to the case $\alpha_M = 0$: i.e., there is no tip mass. The first two $\bar{\beta}$ values are $\bar{\beta}_1 = 1{\cdot}082488$ and $\bar{\beta}_2 = 1{\cdot}786316$. These yield, via equations (21), the first two eigenfrequencies of the system as $\omega_1 = 1{\cdot}171781\sqrt{EI/mL^4}$ and $\omega_2 = 3{\cdot}190926\sqrt{EI/mL^4}$, where $E_1 I_1 = E_2 I_2 = EI$, $m_1 = m_2 = m$, $L_1 = L_2 = L$ are written for simplicity.

One can compare these results with those of reference [3]. The numerical values for the first two eigenfrequencies of the same system are given as $\omega_1 = 1{\cdot}172\sqrt{EI/mL^4}$ and $\omega_2 = 3{\cdot}198\sqrt{EI/mL^4}$, which were computed with the help of component mode synthesis. Since the present paper is based on the exact analytical formulation of the equations of the motion, the numerical results here are slightly lower than those of reference [3], as is

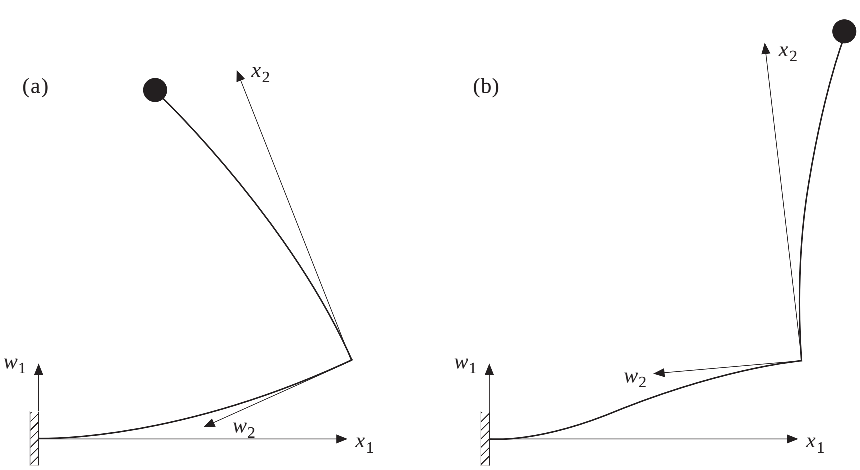

Figure 2. Mode shapes of the L-shaped beam: (a) first mode; (b) second mode.

expected from the theoretical point of view. Further, in accordance with one's experience, all $\bar{\beta}$ values get smaller as the tip mass ratio α_M increases.

Consider the first two mode shapes of the system in Table 1, where the tip mass ratio, i.e., α_M, is taken as 0·5. The corresponding dimensionless eigenfrequency parameters are taken from the table as $\bar{\beta}_1 = 0{\cdot}907887$ and $\bar{\beta}_2 = 1{\cdot}524326$. For $\bar{\beta}_1$, formula (38) gives $C_2 = -1{\cdot}743924$, and the solution of the system of equations in (39) results in $D_1 = 2{\cdot}332021$, $D_2 = -0{\cdot}880287$ and $D_3 = 0{\cdot}026669$.

The corresponding values of the coefficients for the second eigenfrequency parameter $\bar{\beta}_2$ are as follows: $C_2 = -0{\cdot}785347$; $D_1 = -0{\cdot}386409$, $D_2 = 0{\cdot}630439$ and $D_3 = 0{\cdot}530118$. Thus, via expressions (40) and (41), the mode shapes of the system in Figure 1 are determined. They are shown in Figure 2.

4. CONCLUSIONS

The subject of this note is the investigation of the natural vibration problem of a system consisting of an L-shaped beam carrying a tip mass at the free end. On the basis of the Bernoulli-Euler beam theory, the "exact" frequency equation of the system is derived. Then, the frequency equation is solved numerically for some selected values of the system parameters and the corresponding mode shapes are obtained.

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