Journal of Sound and Vibration (1998) **211**(4), 689–701

DYNAMIC STABILITY OF AN AXIALLY TRAVELLING STRING/SLIDER COUPLING SYSTEM WITH MOVING BOUNDARY

R.-F. Fung, J.-S. Huang and J.-J. Chu

Department of Mechanical Engineering, Chung Yuan Christian University, Chung-Li, Taiwan 32023, *Republic of China*

(*Received* 11 *November* 1996, *and in final form* 10 *September* 1997)

1. INTRODUCTION

The string-like systems, including tapes, belts, chains, bands, wires, threads, fibers, and other materials with negligible bending rigidity and a straight, unsagged equilibrium configuration, have broad applications in the areas of chemical, textile, computer and tape recorder industries as well as in many other processes.

A great deal of research about the vibration behavior of the string-like problems has been carried out previously [1–3]. These studies consider the string-like system as a fixed length with no axial motion. Some studies [4–8] have appeared in the literature concerning the vibration and dynamic stability of axially moving materials. Although the string-like systems exhibit movement, the basis of such studies is still as fixed length systems.

For the problem of string vibration with time varying length, Kotera and Kawai [9] analyzed the free vibrations by Laplace transformation. Fung and Cheng [10] studied the free vibration of a non-linearly coupled string/slider system. It should be noted that the methods of solving the problems necessarily differ from the classical methods of treating the fixed length problems. For instance, the concepts of the natural modes and frequencies become meaningless because as the length of the string varies, the natural frequencies become time dependent, and the independence of the natural modes of oscillation is lost. While in theory the motion of a string of variable length can be described to any desired degree of approximation by an infinite system of differential equations, the mathematical difficulty usually becomes prohibitive for all but the first few orders of approximation. As far as string vibration is concerned, little work has appeared on the coupling oscillations in respect of the theoretical formulation of the problem or the analysis of the structural behavior.

The string/slider coupling system developed by Fung and Cheng [10] has been adopted to treat the dynamic stability analysis in this paper. The physical model studied in this paper is shown in Figure 1, the mass–spring–dashpot could be regarded as one subsystem to affect the string system on the steady state amplitudes and stable–unstable regions. The application of such a coupling system may be considered as a vibration reduction mechanism. The energy introduced by the parametric excitation can be transferred between string and slider through the moving boundary. Therefore, the string vibration could be suppressed by adjusting the parameter values of the mass–spring–dashpot system.

The non-linear equations of motion of the string/slider coupling system have been developed using Hamilton's principle. The vibration analysis is based on Galerkin's approximation with time dependent basis functions to discretize the continuous system into a finite-degree-of-freedom system. Finally, the method of multiple time scales is used to solve the uncoupled set of simultaneous differential equations. Combination of the internal and parametric resonances are investigated, and the numerical results are shown.

0022–460X/98/140689 + 13 $25.00/0/sv971415

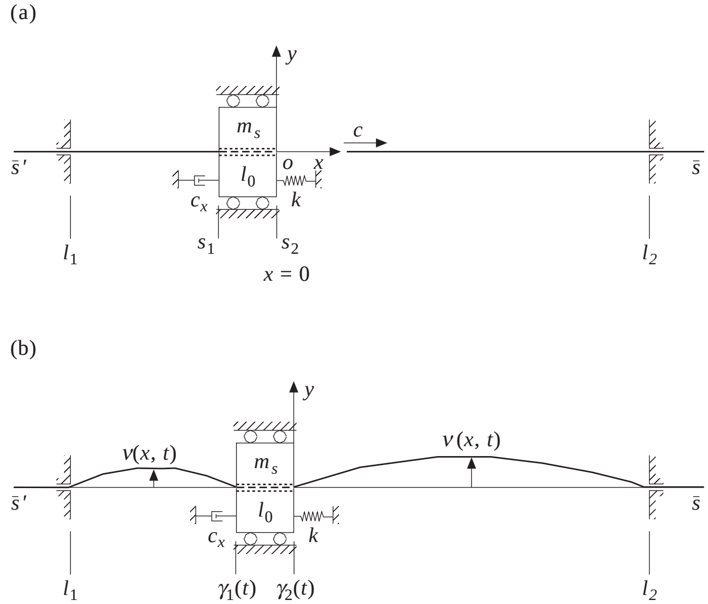

Figure 1. The string/slider coupling system: (a) undeformed configuration; (b) deformed configuration.

2. EQUATION OF MOTION

The string/slider coupling system is shown in Figure 1(a). The string has a constant travelling speed c, and the mass–spring–dashpot is a single-degree-of-freedom system which coupled with the continuous string system $x = 0$ is the static equilibrium position of the slider. In Figure 1(b), $x = \gamma_1(t)$ and $x = \gamma_2(t)$ are the current positions of two sides of the slider and are the moving boundaries for the two-side string in the deformed configuration. The string vibrates transversely in the intervals $l_1 \leqslant x \leqslant \gamma_1(t)$ and $\gamma_2(t) \leqslant x \leqslant l_2$, while the slider has the horizontal oscillations. One assumes that the contact between string and slider is frictionless. Based on Mote [11], the longitudinal elastic motion of the string is neglected for the vibration at lower frequencies and with small amplitudes. To obtain the equation of motion, Hamilton's principle is applied. However, the application of the principle is not straightforward, since there is a moving boundary involved.

When a set of particles tends towards infinity, the aggregate of particles is considered to form the continuum. In the process of variation, the aggregate of particles is considered as fixed between times t_1 and t_2. To this end, one considers the system to include the entire length ($\bar{s}' < x < \bar{s}$) of the string and slider. Hamilton's principle can be written as

$$\int_{t_1}^{t_2} (\delta Ł_t + \delta W)\,\mathrm{d}t = 0, \tag{1}$$

where

$$Ł_t = \int_{\bar{s}'}^{l_1} L_1\,\mathrm{d}x + \int_{l_1}^{\gamma_1} L_2\,\mathrm{d}x + \int_{\gamma_1}^{\gamma_2} L_1\,\mathrm{d}x + \int_{\gamma_2}^{l_2} L_2\,\mathrm{d}x + \int_{l_2}^{\bar{s}} L_1\,\mathrm{d}x + \tfrac{1}{2} m_s \dot{\gamma}^2 - \tfrac{1}{2} k_{\gamma}^{2}, \tag{2}$$

$$\delta W = -C_x\,\dot{\gamma}\delta\gamma + \int_{l_1}^{\gamma_1} [-C_s\,(v_t + cv_x)]\,\delta v\,\mathrm{d}x + \int_{\gamma_2}^{l_2} [-C_s\,(v_t + cv_x)]\,\delta v\,\mathrm{d}x. \qquad (3)$$

L_t is the total Lagrangian function plus the kinetic energy and minus the potential energy of the whole string/slider system; δW is the virtual work by the damping forces [8]; $\gamma(t)$ is the horizontal displacement of mass center of the slider and is measured from the equilibrium position. m_s is the mass of the slider, k is the spring stiffness, C_x is the damping coefficient of the dashpot, C_s is the string damping coefficient; and

$$L_1 = \tfrac{1}{2}\,\rho c^2, \qquad L_2 = \tfrac{1}{2}\,\rho c^2 + \tfrac{1}{2}\,\rho(v_t + cv_x)^2 - \tfrac{1}{2}\,Fv_x^2 - \tfrac{1}{8}\,EAv_x^4, \qquad (4a, b)$$

where L_1 is the Lagrangian density of the string for the ranges $\bar{s}' \leqslant x \leqslant l_1$, $\gamma_1(t) \leqslant x \leqslant \gamma_2(t)$, and $l_2 \leqslant x \leqslant \bar{s}$; L_2 is the Lagrangian density of the string for $l_1 \leqslant x \leqslant \gamma_1(t)$ and $\gamma_2(t) \leqslant x \leqslant l_2$. For simplicity, the same symbol, $v(x, t)$ is used for the two intervals. ρ is the mass density of the string, F is initial tension, E is Young's modulus, A is area of string. The terms $\frac{1}{2}\,Fv_x^2$ and $\frac{1}{8}\,EAv_x^4$ are the strain energies due to the initial tension and the deflection measured from the initially tensioned configuration [8]. Since the transverse deflection and the slope of the string over the other range are zero, consequently the Lagrangian density, L_1, is equal to $\frac{1}{2}\,\rho c^2$.

In Figure 1(b), one considers the boundaries where $v(l_1, t) = v(\gamma_1(t), t) = v(\gamma_2(t), t) = v(l_2, t) = 0$ are specified, but $\gamma_1(t)$ and $\gamma_2(t)$ are free. In the following variation process, it can be found that $\gamma_1(t)$ and $\gamma_2(t)$ are expressed in some relation from the natural boundary condition derived from the process of calculus of variations. Performing the variation in equation (1), using equations (2) and (3) and collecting the like terms, one finds

$$\begin{aligned}
0 = \int_{t_1}^{t_2} \Bigg\{ &\int_{l_1}^{\gamma_1} \left[-\frac{\partial}{\partial t}\frac{\partial L_2}{\partial v_t} - \frac{\partial}{\partial x}\frac{\partial L_2}{\partial v_x} - C_s\,(v_t + cv_x) \right] \delta v\,\mathrm{d}x \\
&+ \int_{\gamma_2}^{l_2} \left[-\frac{\partial}{\partial t}\frac{\partial L_2}{\partial v_t} - \frac{\partial}{\partial x}\frac{\partial L_2}{\partial v_x} - C_s\,(v_t + cv_x) \right] \delta v\,\mathrm{d}x \\
&- \left[\frac{\partial L_2}{\partial v_x}\,\delta v \right]_{x = l_1} + \left[\frac{\partial L_2}{\partial v_x}\,\delta v \right]_{x = l_2} + \Bigg[\left(L_2 - L_1 - v_x \left(\frac{\partial L_2}{\partial v_x} - \dot{\gamma}\,\frac{\partial L_2}{\partial v_t} \right) \right)_{x = \gamma_1} \\
&- \left(L_2 - L_1 - v_x \left(\frac{\partial L_2}{\partial v_x} - \dot{\gamma}\,\frac{\partial L_2}{\partial v_t} \right) \right)_{x = \gamma_2} - m_s\,\ddot{\gamma} - C_x\,\dot{\gamma} - k\gamma \Bigg]_{x = \gamma} \delta\gamma \\
&+ \left[\int_{l_1}^{\gamma_1} \frac{\partial L_2}{\partial v_t}\,\delta v\,\mathrm{d}x + \int_{\gamma_2}^{l_2} \frac{\partial L_2}{\partial v_t}\,\delta v\,\mathrm{d}x + m_s\,\dot{\gamma}\,\delta\gamma \right]_{t_1}^{t_2} \Bigg\} = 0,
\end{aligned} \qquad (5)$$

in which the relations that $\delta v(\gamma_1(t), t) = -v_x(\gamma_1(t), t)\delta\gamma$ and $\delta v(\gamma_2(t), t) = -v_x(\gamma_2(t), t)\delta\gamma$ have been introduced in equation (5) (for details of derivation, see Appendix A). Finally, requiring that δv and $\delta\gamma$ vanish at t_1 and t_2 and noting the arbitrariness of δv in the intervals $l_1 < x < \gamma_1(t)$ and $\gamma_2(t) < x < l_2$, one finds the Euler–Lagrange equation and boundary conditions as follows:

$$-\frac{\partial}{\partial t}\frac{\partial L_2}{\partial v_t} - \frac{\partial}{\partial x}\frac{\partial L_2}{\partial v_x} - C_s\,(v_t + cv_x) = 0, \qquad l_1 < x < \gamma_1 \qquad \text{and} \qquad \gamma_2 < x < l_2, \quad (6a)$$

$$[L_1]_{x=\gamma_2} - [L_1]_{x=\gamma_1} + \left[L_2 - v_x\left(\frac{\partial L_2}{\partial v_x} - \dot{\gamma}\frac{\partial L_2}{\partial v_t}\right)\right]_{x=\gamma_1}$$

$$-\left[L_2 - v_x\left(\frac{\partial L_2}{\partial v_x} - \dot{\gamma}\frac{\partial L_2}{\partial v_t}\right)\right]_{x=\gamma_2} - m_s\ddot{\gamma} - C_x\dot{\gamma} - k\gamma = 0, \qquad x = \gamma(t), \tag{6b}$$

$$v(\gamma_1(t), t) = v(\gamma_2(t), t) = 0, \qquad v(l_1, t) = v(l_2, t) = 0. \tag{6c}$$

Equation (6b) is called the transversality condition [12], which can be treated as the governing equation of the single-degree-of-freedom mass–spring–dashpot system. The expressions for L_1 and L_2 given in equation (4) are substituted into equations (6a, b) and the initial tension F is assumed to be expressed into the form $F = F_0 + F_1\cos\omega_f t$, where F_0 is constant initial tension, F_1 is the perturbed initial tension, and ω_f is its frequency. The equation of motion and the boundary conditions are respectively obtained as:

$$\rho v_{tt} + C_s(v_t + cv_x) + 2\rho c v_{xt} + (\rho c^2 - F_0 - F_1\cos\omega_f t)v_{xx}$$

$$-\tfrac{3}{2}EAv_x^2 v_{xx} = 0, \qquad l_1 < x < \gamma_1 \qquad \text{and} \qquad \gamma_2 < x < l_2, \tag{7a}$$

$$m_s\ddot{\gamma} + C_x\dot{\gamma} + k\gamma + \tfrac{1}{2}[(F_0 + F_1\cos\omega_f t) - \rho(c-\dot{\gamma})^2][v_x^2(\gamma_2(t), t)$$

$$-v_x^2(\gamma_1(t), t)] + \tfrac{3}{8}EA[v_x^4(\gamma_2(t), t) - v_x^4(\gamma_1(t), t)] = 0, \qquad x = \gamma(t), \tag{7b}$$

$$v(\gamma_1(t), t) = v(\gamma_2(t), t) = 0, \qquad v(l_1, t) = v(l_2, t) = 0, \tag{7c}$$

where equation (7a) is the non-linear governing equation of the two-side strings. In the transversality condition (7b), the slider is coupled with the slopes of the strings at the points $x = \gamma_1(t)$ and $x = \gamma_2(t)$. It should be noted that the equation of motion of the slider (7b) coupled with that of string (7a) are of quadratic and quadruple orders. It is seen that the slider vibration is free when the two slopes are equal to each other or equal to, i.e., $v_x(\gamma_1(t), t) = v_x(\gamma_2(t), t) = 0$. Even though the geometric non-linear term $\frac{1}{8}EAv_x^4$ in equation (4b) has vanished, the non-linear coupling arising from the boundary condition at $x = \gamma(t)$ in (7b) still exists.

In order to study the effects of system parameters, let one denote some dimensionless parameters (see Appendix C), then equations (7a), (7b) and (7c) become

$$V_{\tau\tau} + \bar{\eta}_y(V_\tau + \beta V_\xi) + 2\beta V_{\xi\tau} + (\beta^2 - 1 - \bar{v}\cos\Omega_f\tau)V_{\xi\xi}$$

$$-\tfrac{3}{2}\beta_1^2 V_\xi^2 V_{\xi\xi} = 0, \qquad l_{r1} < \xi < \Gamma^{(1)} \qquad \text{and} \qquad \Gamma^{(2)} < \xi < 1, \tag{8a}$$

$$\Gamma_{\tau\tau} + \bar{\eta}_x\Gamma_\tau + \Omega^2\Gamma + \tfrac{1}{2}M[1 + \bar{v}\cos\Omega_f\tau - (\beta - \Gamma_\tau)^2][V_\xi^2(\Gamma^{(2)}, \tau)$$

$$-V_\xi^2(\Gamma^{(1)}, \tau)] + \tfrac{3}{8}M\beta_1^2[V_\xi^4(\Gamma^{(2)}, \tau) - V_\xi^4(\Gamma^{(1)}, \tau)] = 0, \qquad \xi = \Gamma(\tau), \tag{8b}$$

$$V(l_{r1}, \tau) = V(\Gamma^{(1)}(\tau), \tau) = V(\Gamma^{(2)}(\tau), \tau) = V(1, \tau) = 0. \tag{8c}$$

3. APPROXIMATE SOLUTION

In this section, the approximate solution for the transverse motion of string with the moving boundaries at $\xi = \Gamma^{(1)}(\tau)$ and $\xi = \Gamma^{(2)}(\tau)$ will be derived. Consider two boundary value problems

$$\psi_{n,\xi\xi}(\xi, \tau) = -\omega_{ln}^2(\tau)\psi_n(\xi, \tau), \qquad l_{r1} < \xi < \Gamma^{(1)}, \tag{9a}$$

$$\psi_n(l_{r1}, \tau) = \psi_n(\Gamma^{(1)}(\tau), \tau) = 0, \tag{9b}$$

$$\phi_{n,\xi\xi}(\xi, \tau) = -\omega_{rn}^2(\tau)\phi_n(\xi, \tau), \qquad \Gamma^{(2)}(\tau) < \xi < 1, \tag{10a}$$

$$\phi_n\,(\Gamma^{(2)}(\tau), \tau) = \phi_n\,(1, \tau) = 0. \tag{10b}$$

Since the horizontal displacement of slider is time dependent, both $\psi_n\,(\xi, \tau)$ and $\phi_n\,(\xi, \tau)$ and its corresponding eigenvalues $\omega_{ln}\,(\tau)$ and $\omega_{rn}\,(\tau)$ are time dependent, where $\omega_{ln}\,(\tau)$ and $\omega_{rn}\,(\tau)$ are the nth "instantaneous natural frequency" of the system, often called the quasi frequency [13].

It can be verified that $\{\psi_n\,(\xi, \tau), n = 1, 2, \ldots\}$ and $\{\phi_n\,(\xi, \tau), n = 1, 2, \ldots\}$ are the orthonormal basis. Thus, the expansion functions are

$$\psi_n\,(\xi, \tau) = \sqrt{2/\Gamma^{(1)} - l_{r1}}\,\sin \omega_{ln}\,(\tau)\,(\xi - l_{r1}), \qquad n = 1, 2, 3, \ldots, \tag{11a}$$

$$\phi_n\,(\xi, \tau) = \sqrt{2/(1 - \Gamma^{(2)})}\,\sin \omega_{rn}\,(\tau)\,(1 - \xi), \qquad n = 1, 2, 3, \ldots, \tag{11b}$$

where

$$\omega_{ln}\,(\tau) = n\pi/(\Gamma^{(1)} - l_{r1}), \quad n = 1, 2, 3, \ldots, \qquad \omega_{rn}\,(\tau) = n\pi/(1 - \Gamma^{(2)}), \quad n = 1, 2, 3, \ldots.$$

Sequentially, one seeks the solution $V(\xi, \tau)$ of equation (8a) in the form

$$V(\xi, \tau) = \sum_{n=1}^{\infty} f_n\,(\tau)\psi_n\,(\xi, \tau), \qquad l_{r1} < \xi < \Gamma^{(1)}, \tag{12a}$$

$$V(\xi, \tau) = \sum_{n=1}^{\infty} q_n\,(\tau)\phi_n\,(\xi, \tau), \qquad \Gamma^{(2)} < \xi < 1, \tag{12b}$$

where $f_n\,(\tau)$ and $q_n\,(\tau)$ are the generalized co-ordinates. Substituting the expression of equations (12a, b) into equation (8a) and using the Galerkin method, one obtains a system of countably infinite number of non-linear time varying ordinary differential equations

$$\ddot{f}_m + \sum_{n=1}^{\infty} [a^l_{mn}(\Gamma, \dot{\Gamma})\dot{f}_n + b^l_{mn}(\tau, \Gamma, \dot{\Gamma}, \ddot{\Gamma})f_n] - \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c^l_{mnij}(\Gamma)f_n f_i f_j = 0, \tag{13a}$$

$$\ddot{q}_m + \sum_{n=1}^{\infty} [a^r_{mn}(\Gamma, \dot{\Gamma})\dot{q}_n + b^r_{mn}(\tau, \Gamma, \dot{\Gamma}, \ddot{\Gamma})q_n] - \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c^r_{mnij}(\Gamma)q_n q_i q_j = 0, \tag{13b}$$

where

$$a^l_{mn}(\Gamma, \dot{\Gamma}) = 2A^l_{mn}(\Gamma, \dot{\Gamma}) + 2\beta E^l_{mn}(\Gamma) + \bar{\eta}_y\,\delta_{mn},$$

$$a^r_{mn}(\Gamma, \dot{\Gamma}) = 2A^r_{mn}(\Gamma, \dot{\Gamma}) + 2\beta E^r_{mn}(\Gamma) + \bar{\eta}_y\,\delta_{mn},$$

$$b^l_{mn}(\tau, \Gamma, \dot{\Gamma}, \ddot{\Gamma}) = \bar{\eta}_y\,A^l_{mn}(\Gamma, \dot{\Gamma}) + B^l_{mn}(\Gamma, \dot{\Gamma}, \ddot{\Gamma}) + \bar{\eta}_y\,\beta E^l_{mn}(\Gamma)$$
$$+ 2\beta F^l_{mn}(\Gamma, \dot{\Gamma}) + (1 + \bar{v}\cos \Omega_f\tau - \beta^2)\Omega_n^2\delta_{mn},$$

$$b^r_{mn}(\tau, \Gamma, \dot{\Gamma}, \ddot{\Gamma}) = \bar{\eta}_y\,A^r_{mn}(\Gamma, \dot{\Gamma}) + B^r_{mn}(\Gamma, \dot{\Gamma}, \ddot{\Gamma}) + \bar{\eta}_y\,\beta E^r_{mn}(\Gamma)$$
$$+ 2\beta F^r_{mn}(\Gamma, \dot{\Gamma}) + (1 + \bar{v}\cos \Omega_f\tau - \beta^2)\Omega_n^2\delta_{mn},$$

$$c^l_{mnij}(\Gamma) = \tfrac{3}{2}\,\beta_1^2 N^l_{mnij}(\Gamma), \qquad c^r_{mnij}(\Gamma) = \tfrac{3}{2}\,\beta_1^2 N^r_{mnij}(\Gamma),$$

in which the details of $A^l_{mn}(\Gamma, \dot{\Gamma})$, $A^r_{mn}(\Gamma, \dot{\Gamma})$, $B^l_{mn}(\Gamma, \dot{\Gamma}, \ddot{\Gamma})$, $B^r_{mn}(\Gamma, \dot{\Gamma}, \ddot{\Gamma})$, $E^l_{mn}(\Gamma)$, $E^r_{mn}(\dot{\Gamma})$, $F^l_{mn}(\Gamma, \dot{\Gamma})$, $F^r_{mn}(\Gamma, \dot{\Gamma})$, $N^l_{mnij}(\Gamma)$ and $N^r_{mnij}(\Gamma)$ are given in Appendix B.

Substituting $V_\xi\,(\Gamma^{(1)}(\tau),\tau)$ and $V_\xi\,(\Gamma^{(2)}(\tau),\tau)$ into equation (8b), one has

$$\ddot{\Gamma}+\bar{\eta}_x\,\dot{\Gamma}+\Omega^2\Gamma+\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}[d^r_{ni}(\Gamma,\dot{\Gamma})q_n\,q_i-d^l_{ni}(\Gamma,\dot{\Gamma})f_n\,f_i]$$

$$+\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}[e^r_{nijk}(\Gamma)q_n\,q_i\,q_j\,q_k-e^l_{nijk}(\Gamma)f_n\,f_i\,f_j\,f_k]=0, \tag{14}$$

where $\dot{\Gamma}=\mathrm{d}/\mathrm{d}\tau(\Gamma)$, and

$$d^l_{ni}(\Gamma,\dot{\Gamma})=\tfrac{1}{2}\,M(1+\bar{v}\cos\Omega_f\,\tau-(\beta-\dot{\Gamma})^2)\,(-1)^{n+i}(2ni\pi^2/(\Gamma^{(1)}-l_{r1})^3), \tag{15a}$$

$$d^r_{ni}(\Gamma,\dot{\Gamma})=\tfrac{1}{2}\,M(1+\bar{v}\cos\Omega_f\,\tau-(\beta-\dot{\Gamma})^2)\,(-1)^{n+i}(2ni\pi^2/(1-\Gamma^{(2)})^3), \tag{15b}$$

$$e^l_{nijk}(\Gamma)=\tfrac{3}{8}\,M\beta_1^2(-1)^{n+i+j+k}(4nijk\pi^4/(\Gamma^{(1)}-l_{r1})^6), \tag{15c}$$

$$e^r_{nijk}(\Gamma)=\tfrac{3}{8}\,M\beta_1^2(-1)^{n+i+j+k}(4nijk\pi^4/(1-\Gamma^{(2)})^6). \tag{15d}$$

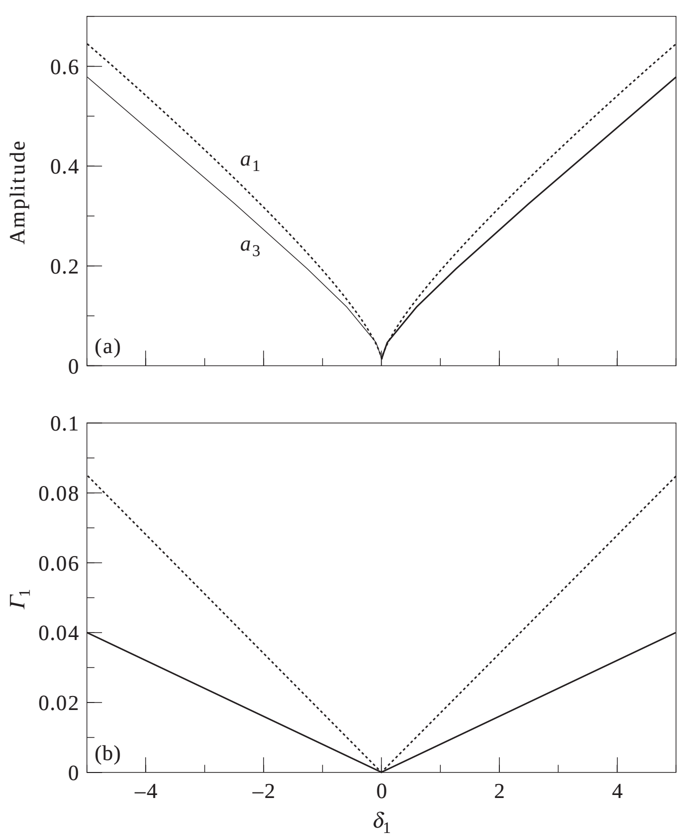

Figure 2. The amplitudes of response of the right-side string with $\beta=0$, $v=1$, $M=1$, $l_{r1}=0{\cdot}7$, $\eta_x=\eta_y=0{\cdot}01$, a_1: ——, steady state amplitude, Ω_f near $2\Omega_1$; a_3: · · · · , steady state amplitude, Ω_f near $2\Omega_3$. (a) The steady state amplitude of the string. (b) The steady state amplitude of the slider.

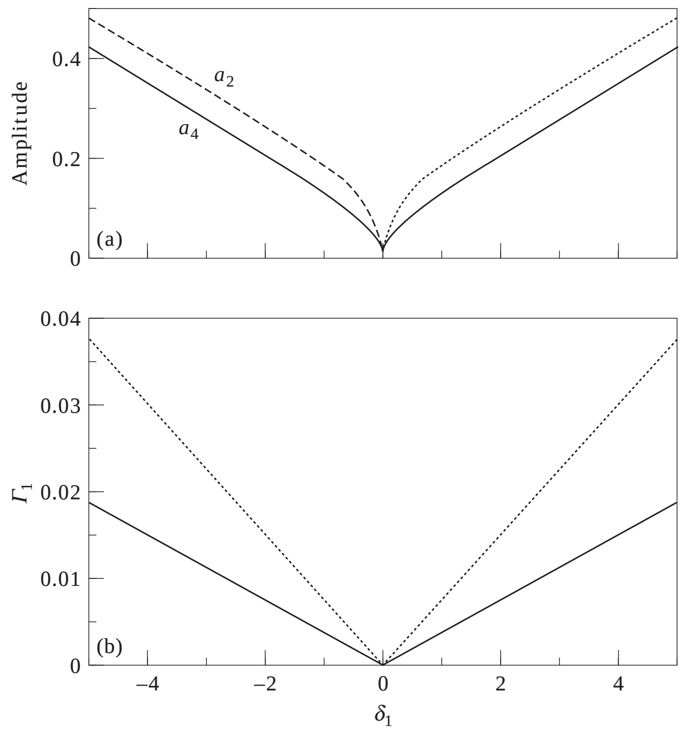

Figure 3. The amplitudes of response of the left-side string with $\beta = 0$, $v = 1$, $M = 1$, $\eta_x = \eta_y = 0{\cdot}01$, a_2: ——, steady state amplitude, Ω_f near $2\Omega_2$; a_4: · · · · , steady state amplitude, Ω_f bear $2\Omega_4$. (a) The steady state amplitude of the string; (b) the steady state amplitude of the slider.

4. PERTURBATION ANALYSIS

Since the perturbation tension $F_1 \cos \omega_f t$ is small compared with the steady state tension F_0, the small parameter ϵ appearing in the perturbation is defined as $\epsilon = F_1/F_0$. In the

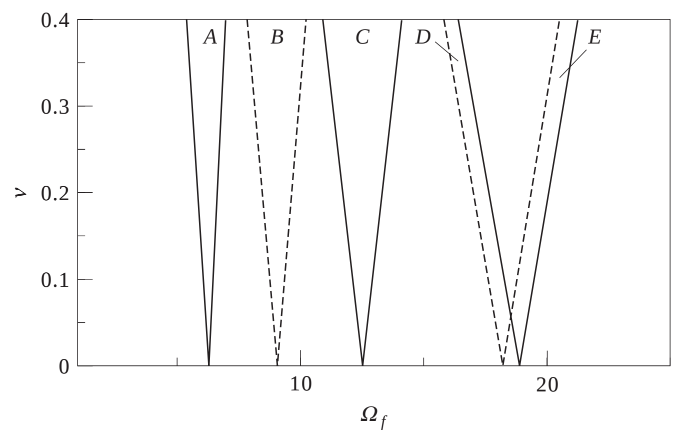

Figure 4. The stable–unstable regions of the first five modes of the string/slider system with $\beta = 0$, $v = 1$, $M = 1$, $l_{r1} = 0{\cdot}7$, $\eta_x = \eta_y = 0{\cdot}01$: A, unstable region, Ω_f near $2\Omega_1$: B, unstable region, Ω_f near $2\Omega_2$: C, unstable region, Ω_f near $2\Omega_3$; D, unstable region, Ω_f near $2\Omega_4$: E, unstable region, Ω_f near $2\Omega_5$. ——, Unstable region of right-side string; ---, unstable region of left-side string.

present work, the method of multiple time scales (Nayfeh [14]) is applied to find an approximate solution of equations (13) and (14). It is assumed that the solution of equations (13) and (14) can be represented by a uniformly valid expansion having the form

$$f_m = \sum_{i=1}^{I} \epsilon^i f_{mi}\,(T_0, T_1, \ldots, T_I) + O(\epsilon^{I+1}), \tag{16a}$$

$$q_m = \sum_{i=1}^{I} \epsilon^i q_{mi}\,(T_0, T_1, \ldots, T_I) + O(\epsilon^{I+1}), \tag{16b}$$

$$\Gamma = \sum_{i=1}^{I} \epsilon^i \Gamma_i\,(T_0, T_1, \ldots, T_I) + O(\epsilon^{I+1}), \tag{16c}$$

where $T_j = \epsilon^j \tau$, $j = 1, 2, \ldots$.

Following the familiar perturbation method, one obtains the resonant solutions. Without the parametric excitation, the free vibration between the string and slider will present the following resonances: (1) $\Omega \neq \Omega_m \pm \Omega_n$, no internal resonance; (2) $\Omega \approx \Omega_p + \Omega_q$, internal resonance of summed-type; (3) $\Omega \approx \Omega_p - \Omega_q$, internal resonance of difference-type. All the resonant cases with parametric excitation will be investigated separately.

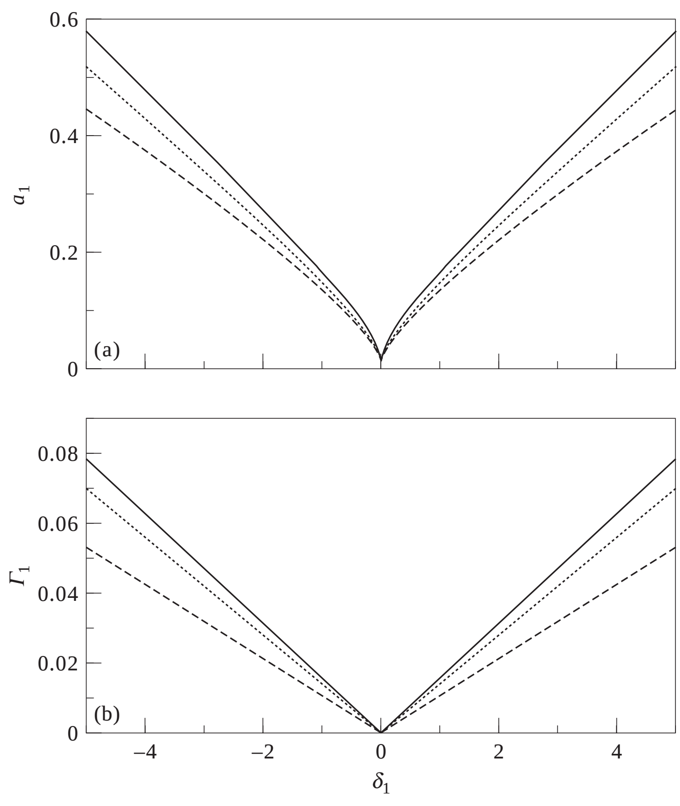

Figure 5. The amplitudes of response of the string/slider coupling system with $v = 1$, $M = 1$, $l_{r1} = 0{\cdot}7$, $\eta_x = \eta_y = 0{\cdot}01$: (a) the steady state amplitude of the string; (b) the steady state amplitude of slider; δ_1; ——, $\beta = 0$; $\cdots\cdot$, $\beta = 0{\cdot}1$; ----, $\beta = 0{\cdot}2$.

5. NUMERICAL RESULTS AND DISCUSSIONS

In this section, the effects of the string transport speed β and parametric excitation v on the steady state amplitudes and stable–unstable regions are analyzed. Figures 2 and 3 present the steady state amplitudes of right- and left-side string and slider versus the detuning parameter δ_1. The first four modes of the string are considered and the small parameter $\epsilon = 0{\cdot}1$ is chosen. The steady state amplitudes a_1 and a_3 of the right-side string are shown in Figure 2(a) for the parametric excitation frequency Ω_f being near $2\Omega_1$ and $2\Omega_2$, respectively. The steady state amplitudes a_2 and a_4 of the right-side string are shown in Figure 3(a) for the parametric excitation frequency Ω_f being near $2\Omega_2$ and $2\Omega_4$, respectively. The steady state amplitude, Γ_1, of the slider is excited through the internal resonance of the summed type. The curves are shown in Figure 2(b) for Ω_f being near $2\Omega_1$ (solid line) and $2\Omega_3$ (dotted line), and shown in Figure 2(b) for Ω_f being near $2\Omega_2$ (solid line) and $2\Omega_4$ (dotted line). In physical terms, it means that energy can be transferred between the string and slider through the boundary at $\xi = \Gamma(\tau)$. Figure 4 presents stable–unstable regions of the parametric resonances for the first five modes of the system. The regions are determined by the eigenvalues of the matrix $[\partial f/\partial \mathbf{Y}]_{\mathbf{Y}_0}$ for $p = q$.

Figures 5 and 6 show the effects of transport speed β and perturbation tension parameter v on the amplitudes of response, respectively. From Figure 6, it is seen that the increase in transport speed value β of the string relates to the decrease in the natural frequency of the string, and the steady state amplitudes are shifted toward the lower frequency direction. In Figure 6, the increase in the perturbation tension v of the string relates to the increase in the steady state amplitudes.

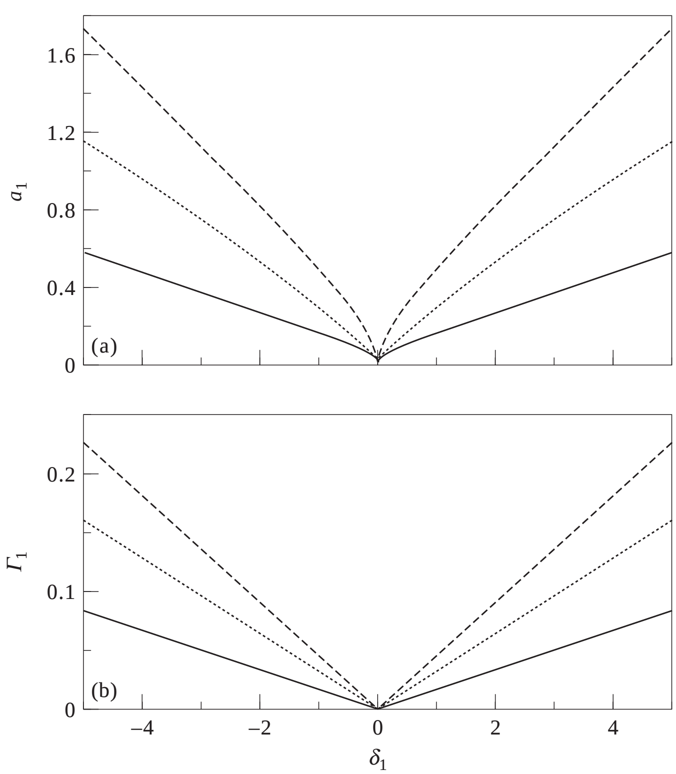

Figure 6. The amplitudes of response of the string/slider coupling system versus the perturbation tension with $\beta = 0$, $M = 1$, $l_{r1} = 0{\cdot}7$, $\eta_x = \eta_y = 0{\cdot}01$: ——, $v = 0{\cdot}1$; · · · · , $v = 0{\cdot}2$; ----, $v = 0{\cdot}3$.

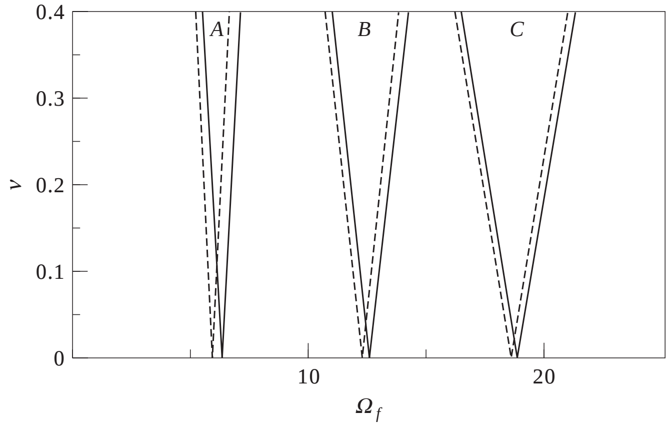

Figure 7. The stable–unstable regions of the first three modes of the right-side string with $\beta = 0$, $v = 1$, $M = 1$, $\eta_x = \eta_y = 0{\cdot}01$: A, unstable region, Ω_f near $2\Omega_1$; B, unstable region, Ω_f near $2\Omega_2$; C, unstable region, Ω_f near $2\Omega_3$; $l_{r1} = 0$; $l_{r1} = 0{\cdot}35$.

Figure 7 shows the effects of string length on the stable–unstable regions for the coupling system. The increase in the length of string on the left side relates to the decrease of the natural frequency of the string, and the unstable regions are shifted to the lower frequency direction.

6. CONCLUSIONS

The method for studying the moving boundary effect on the vibration of a string/slider coupling system has been investigated in this paper. First, the governing equations and boundary conditions were derived by the calculus of variation and Hamilton's principle. Galerkin's procedure with a time-dependent basis function was then used to obtain a discrete system. Next, a technique of modal analysis was applied to decouple the gyroscopic system, and the set of the simultaneous differential equations was solved by the method of multiple-time scales with the small perturbation tension of the string.

From the numerical results, the following conclusions can be drawn:

(1) The frequency of the excitation and the natural frequency of the slider are close to twice of the natural frequencies of the string system, the main resonance and unstable regions exist.

(2) Although combination resonance of the difference-type was predicted to arise in the course of the perturbation analysis, it does not exist in the problem considered.

(3) By increasing the transport speed β of the string, the natural frequency of the string will decrease, and the steady state amplitudes are shifted toward a lower frequency domain.

(4) The natural frequency of the string tends to decrease, while the unstable regions are shifted toward the lower frequency direction with increasing string length.

(5) If non-linear stiffness of spring was introduced, the coupling system will be more complicated and there is a higher likelihood of resonance occurring.

ACKNOWLEDGMENT

Support for this work by the National Science Council of the Republic of China through Contract NSC 85-2212-E033-006 is gratefully acknowledged.

REFERENCES

1. G. V. ANAND 1969 *Journal of the Acoustical Society of America* **45**, 1089–1096. Large-amplitude damped free vibration of a stretched string.
2. J. A. ELLIOTT 1982 *America Journal of Physics* **50**, 1148–1150. Nonlinear resonance in vibrating strings.
3. K. YASUDA and T. TORII 1986 *Bulletin of JSME* **29**, 1250–1260. Nonlinear forced oscillations of a string.
4. C. D. MOTE JR. 1968 *ASME Journal of Applied Mechanics* **35**, 171–172. Parametric excitation of an axially moving string.
5. K. R. KORDE 1985 *ASME Journal of Applied Mechanics* **52**, 493–494. Nonlinear oscillation of moving string.
6. J. A. WICKERT and C. D. MOTE JR. 1988 *Shock and Vibration Digest* **20**, 3–13. Current research on the vibration and stability of axially-moving materials.
7. J. A. WICKERT and C. D. MOTE JR. 1990 *ASME Journal of Applied Mechanics* **57**, 738–744. Classical vibration analysis of axially moving continua.
8. J. S. HUANG, R. F. FUNG and C. H. LIN 1995 *International Journal of Mechanical Science* **37**, 145–160. Dynamic stability of a moving string undergoing three-dimensional vibration.
9. T. KOTERA and R. KAWAI 1988 *Nippon Kikai Gakksi Ronbunshu C Hen* **54**, 9–15. Vibrations of string with time-varying length (second report, free vibrations induced by initial displacements).
10. R. F. FUNG and W. H. CHENG 1993 *Journal of the Chinese Society of Mechanical Engineers* **12**, 229–239. Free vibration of a string/slider nonlinear coupling system.
11. C. D. MOTE JR. 1966 *ASME Journal of Applied Mechanics* **33**, 463–464. On the oscillation of an axially moving string.
12. A. S. LEE and G. H. HANDELMAN 1977 *Mathematics Applied to Continuum Mechanics*. Macmillan; chapter 11.
13. M. S. JANKOVIC 1983 *Journal of Guidance* **7**, 128–130. Dynamics of a Space-craft during Extension of Flexible Appendages.
14. A. H. NAYFEH and D. T. MOOK 1979 *Nonlinear Oscillations*. New York: Wiley; chapter 6.

APPENDIX A

In Figure A1, $v^*(x, t)$ is a comparison curve, while $v(x, t)$ is the true curve. It is apparent that $\delta v(x, t) = v^*(x, t) - v(x, t)$ has meaning only in the interval $[\gamma_2(t) + \delta\gamma(t), l_2]$, since $v^*(x, t)$ is not defined for $x \in (\gamma_2(t), \gamma_2(t) + \delta\gamma(t))$. By inspection of Figure A1 one has

$$\delta\bar{v} = v^*(\gamma_2(t) + \delta\gamma(t), t) - v(\gamma_2(t), t) \doteq v_x(\gamma_2(t), t)\delta\gamma + \delta v(\gamma_2(t), t) = 0,$$

where $\doteq$ means "equal to first order". From the above equation, one gets $\delta v(\gamma_2(t), t) = -v_x(\gamma_2(t), t)\delta\gamma$. Similarly, one has $\delta v(\gamma_1(t), t) = -v_x(\gamma_1(t), t)\delta\gamma$.

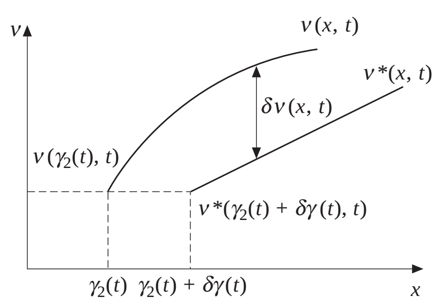

Figure A1. The comparison curve of the moving boundary.

APPENDIX B

The time varying coefficients of equations (13a, b):

$$A^l_{mn}(\Gamma, \dot{\Gamma}) = \int_{l_{r1}}^{\Gamma^{(1)}} \dot{\psi}_n\, \psi_m\, \mathrm{d}\xi, \qquad A^r_{mn}(\Gamma, \dot{\Gamma}) = \int_{\Gamma^{(2)}}^{1} \dot{\phi}_n\, \phi_m\, \mathrm{d}\xi,$$

$$B^l_{mn}(\Gamma, \dot{\Gamma}, \ddot{\Gamma}) = \int_{l_{r1}}^{\Gamma^{(1)}} \ddot{\psi}_n\, \psi_m\, \mathrm{d}\xi, \qquad B^r_{mn}(\Gamma, \dot{\Gamma}, \ddot{\Gamma}) = \int_{\Gamma^{(2)}}^{1} \ddot{\phi}_n\, \phi_m\, \mathrm{d}\xi,$$

$$E^l_{mn}(\Gamma) = \int_{l_{r1}}^{\Gamma^{(1)}} \psi'_n\, \psi_m\, \mathrm{d}\xi, \qquad E^r_{mn}(\Gamma) = \int_{\Gamma^{(2)}}^{1} \phi'_n\, \phi_m\, \mathrm{d}\xi,$$

$$F^l_{mn}(\Gamma, \dot{\Gamma}) = \int_{l_{r1}}^{\Gamma^{(1)}} \dot{\psi}'_n\, \psi_m\, \mathrm{d}\xi, \qquad F^r_{mn}(\Gamma, \dot{\Gamma}) = \int_{\Gamma^{(2)}}^{1} \dot{\phi}'_n\, \phi_m\, \mathrm{d}\xi,$$

$$N^l_{mnij}(\Gamma) = \int_{l_{r1}}^{\Gamma^{(1)}} \psi'_n\, \psi'_i\, \psi''_j\, \psi_m\, \mathrm{d}\xi, \qquad N^r_{mnij}(\Gamma) = \int_{\Gamma^{(2)}}^{1} \phi'_n\, \phi'_i\, \phi''_j\, \phi_m\, \mathrm{d}\xi$$

The coefficients of equations (19a–j):

$$E^{l(0)}_{mn} = \left\{ \begin{array}{ll} -[1-(-1)^{m+n}]\,[2mn/(m^2-n^2)]\,[1/(l_{r0}+l_{r1})], & m \neq n \\ 0 & m = n \end{array} \right\},$$

$$E^{r(0)}_{mn} = \left\{ \begin{array}{ll} -[1-(-1)^{m+n}]\,[2mn/(m^2-n^2)], & m \neq n \\ 0 & m = n \end{array} \right\},$$

$$A^{l(1)}_{mn} = \left\{ \begin{array}{ll} (-1)^{m+n+1}[2mn/(m^2-n^2)]\,[1/(l_{r0}+l_{r1})], & m \neq n \\ 0 & m = n \end{array} \right\},$$

$$A^{r(1)}_{mn} = \left\{ \begin{array}{ll} (-1)^{m+n+1}[2mn/(m^2-n^2)], & m \neq n \\ 0 & m = n \end{array} \right\},$$

$$B^{l(1)}_{mn} = A^{l(1)}_{mn}, \qquad B^{r(1)}_{mn} = A^{r(1)}_{mn}, \qquad E^{l(1)}_{mn} = E^{l(0)}_{mn}[1/(l_{r0}+l_{r1})], \qquad E^{r(1)}_{mn} = E^{r(0)}_{mn},$$

$$F^{l(1)}_{mn} = \left\{ \begin{array}{ll} -[1-(-1)^{m+n}]\,[mn(3m^2+n^2)/(m^2-n^2)^2]\,[1/(l_{r0}+l_{r1})^2], & m \neq n \\ \frac{1}{2}m^2\pi^2[1/(l_{r0}+l_{r1})^2] & m = n \end{array} \right\},$$

$$F^{r(1)}_{mn} = \left\{ \begin{array}{ll} -[1-(-1)^{m+n}]\,[mn(3m^2-7n^2)/(m^2-n^2)^2], & m \neq n \\ \frac{1}{2}m^2\pi^2 & m = n \end{array} \right\}.$$

APPENDIX C: NOMENCLATURE

c_1 $\sqrt{EA/\rho}$, axial wave transport speed of string
c_2 $\sqrt{F_0/\rho}$, wave transport speed of string
c_3 $\sqrt{F_1/\rho}$, variation wave transport speed of string

l_0	length of the slider
l_1, l_2	length of the left- and right-side string, respectively
l_{r1}	l_1/l_2, dimensionless length ratio
l_{r0}	l_0/l_2, dimensionless length ratio
M	$\rho l_2/m_s$, dimensionless mass ratio
V	v/l_2, dimensionless logarithmic displacement
β	c/c_2, dimensionless speed ratio
β_1	c_1/c_2, dimensionless speed ratio
Γ	γ/l_2, dimensionless positions of slider
$\Gamma^{(1)}$	γ_1/l_2, dimensionless positions of slider
$\Gamma^{(2)}$	γ_2/l_2, dimensionless positions of slider
ζ_x	C_x/m_s, damping coefficient of slider
ζ_y	C_s/ρ, damping coefficient of string
$\bar{\eta}_x$	$\zeta_x(l_2/c_2)$, dimensionless damping coefficient of slider system
$\bar{\eta}_y$	$\zeta_y(l_2/c_2)$, dimensionless damping coefficient of string
$\bar{v}$	$F_1/F_0(c_3^2/c_2^2)$, dimensionless initial tension variation
ξ	x/l_2, dimensionless length
τ	$c_2 t/l_2$, dimensionless time
Ω	$\omega_s l_2/c_2$, dimensionless frequency of slider
Ω_f	$\omega_f l_2/c_2$, dimensionless parametric excitation frequency
Ω_p	$\omega_f l_2/c_2$, the pth mode dimensionless frequency of string system
ω_s	$\sqrt{k/m_s}$, frequency of slider