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INTERACTIVE ASPECTS IN THE NUMERICAL SIMULATION OF THE KINETIC STATE OF A MULTI-BODY SYSTEM WITH FRICTION

H. J. Klepp

Institute of Technical Mechanics, Ruhr-University Bochum, D-44780 Bochum, Germany

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1. INTRODUCTION

Several constructional alternatives are considered in the search for optimal design solutions for machines and mechanisms. The arrangement and properties of connecting members can be demanding in such investigations. If virtual prototyping is used for comparison of design alternatives, the computability of the mathematical models is a strong requirement, for models resulting in multi-body systems that require kinetic determinate structures of the models. With known active forces and torques, the kinetic state parameters of such models, i.e., the accelerations and reaction forces, are the solutions of a system of linear algebraic equations, which can be obtained, for example, from Lagrange's equation of the second kind and/or kinetic equilibrium conditions for parts of the model.

If motion resistances are considered as friction forces and torques acting in the constraints, the kinetic friction laws are the additional equations necessary for the computation of the unknown state parameters [1]. The kinetic friction laws are non-linear equations in the reaction forces. Because kinetic friction forces and torques are active forces and torques, they imply a feedback connection in the block diagram representation of systems with friction, shown in Figure 1. As a mathematical consequence of this interaction, in general, no closed form solutions can be established, and only numerical and iterative computation methods are applicable [2]. The consideration of this feedback connection is essential in the mathematical simulation of typical phenomena caused by friction, such as migration inside equilibrium zones, sticking and friction locking; see, for example, references [3–5].

The aim of this paper is the investigation of a system, which, under the assumption that the connecting members are rigid, has an indeterminate structure. With deformable connecting members, the system has a determinate structure. For this model and motion resistances assumed as friction acting in the constraints, the influences of the stiffnesses and viscosities of the connecting members and of the friction parameter values on the motion and the forces acting in the connecting members are analyzed. In particular interactive aspects due to friction feedback and the sensitivity of the model to small variations of the design parameters and various design variants are considered. The comparison of the results obtained by numerical simulation with different stiffnesses of the connecting members reveals limitations in the possibilities of describing the behaviour of real systems.

2. DESCRIPTION OF THE SYSTEM

The swinging plate shown in Figure 2 is supported by three connecting members having hinge joints with friction at their ends. If the connecting members are considered rigid, this planar system has an indeterminate structure. The twelve independent equations expressing conditions of kinetic equilibrium of the four parts of the system and the six friction laws are not sufficient for the computation of the nineteen unknowns; i.e., the acceleration (one), reaction force components (twelve) and friction torques (six).

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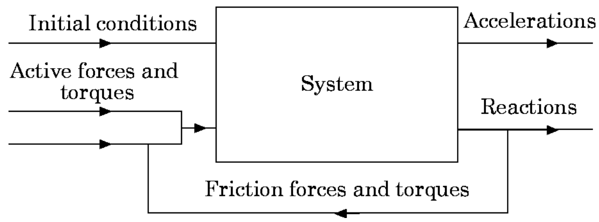

Figure 1. Block diagram representation of a system with friction.

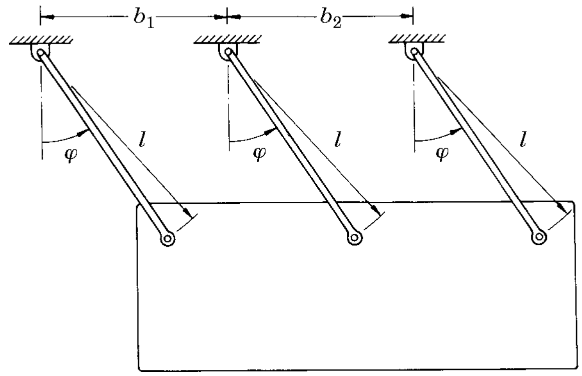

Figure 2. Swinging plate with three connecting members.

Figure 3 presents a three-degree-of-freedom model for the swinging plate, when small longitudinal deformations of the connecting members are admitted. m_i, S_i and J_i, $i = 1, 2, 3, 4$, are the masses, mass centres and centroidal moments of inertia of the bodies. The lengths l_i, d_i $i = 1, 2, 3$, and $O_1O_3 = O_2O_4 = b_1$, $O_3O_5 = O_4O_6 = b_2$, b_i, $i = 3, 4$, are indicated in Figure 3. The elastic and damping properties of the connecting members are represented by spring–dashpot units with stiffnesses c_i and viscosity coefficients β_i, $i = 1, 2, 3$. The longitudinal deformations of the connecting members are denoted by s_i, $i = 1, 2, 3$.

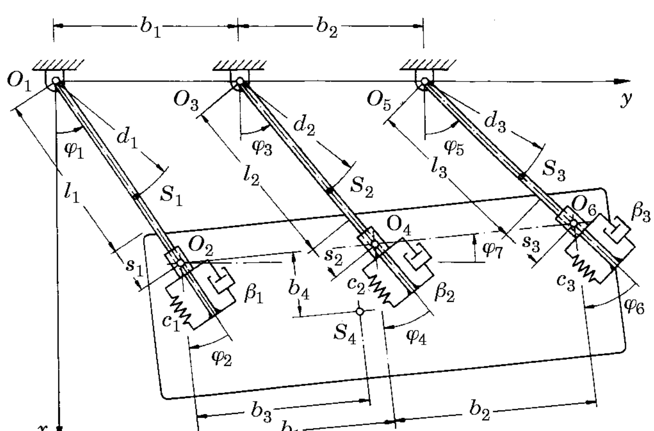

Figure 3. Three-degrees-of-freedom model for the swinging plate with deformable connecting members.

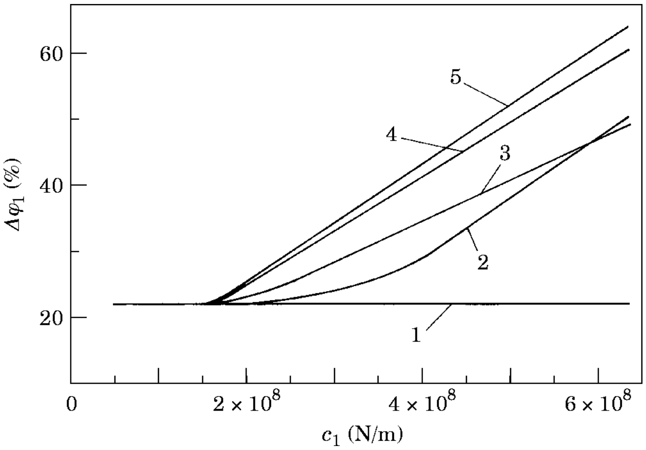

Figure 4. Reductions of the pendular elongations $\Delta\varphi_1$ for viscosity coefficients $\beta_1 = \beta_2 = \beta_3$ leading to the damping factor $D = 0{\cdot}1$ versus stiffnesses $c_1 = c_2 = c_3$ of the connecting members. Curve 1 ($\Delta l = 0$, $\Delta b = 0$ and $\Delta b = 0{\cdot}5$ m); curve 2 ($\Delta l = -0{\cdot}1$ mm, $\Delta b = 0$); curve 3 ($\Delta l = -0{\cdot}1$ mm, $\Delta b = 0{\cdot}5$ m); curve 4 ($\Delta l = 0{\cdot}1$ mm, $\Delta b = 0{\cdot}5$ m); curve 5 ($\Delta l = 0{\cdot}1$ mm, $\Delta b = 0$).

The generalized co-ordinates q_i, $i = 1, 2, 3$, considered are $q_1 = \varphi_1$, $q_2 = l_1 + s_1$, and $q_3 = \varphi_7$. The other position parameters φ_3, φ_5, s_2 and s_3 are related to the generalized co-ordinates by the equations

$$\tan(\varphi_3) = \{(l_1 + s_1)\sin(\varphi_1) - b_1[1 - \cos(\varphi_7)]\}/[(l_1 + s_1)\cos(\varphi_1) - b_1\sin(\varphi_7)], \quad (1)$$

$$(l_2 + s_2)\cos(\varphi_3) = (l_1 + s_1)\cos(\varphi_1) - b_1\sin(\varphi_7), \quad (2)$$

$$\tan(\varphi_5) = \{(l_1 + s_1)\sin(\varphi_1) - (b_1 + b_2)[1 - \cos(\varphi_7)]\}/[(l_1 + s_1)\cos(\varphi_1) - (b_1 + b_2)\sin(\varphi_7)], \quad (3)$$

$$(l_3 + s_3)\cos(\varphi_5) = (l_1 + s_1)\cos(\varphi_1) - (b_1 + b_2)\sin(\varphi_7). \quad (4)$$

The co-ordinates x_i, y_i, of the mass centres S_i are

$$x_i = d_i\cos(\varphi_i), \quad i = 1, 2, 3, \qquad x_4 = (l_1 + s_1)\cos(\varphi_1) - b_3\sin(\varphi_7) + b_4\cos(\varphi_7), \quad (5)$$

$$y_1 = d_1\sin(\varphi_1), \qquad y_2 = b_1 + d_2\sin(\varphi_3), \qquad y_3 = b_1 + b_2 + d_3\sin(\varphi_5), \quad (6)$$

$$y_4 = (l_1 + s_1)\sin(\varphi_1) + b_3\cos(\varphi_7) + b_4\sin(\varphi_7). \quad (7)$$

The scalar parameters determining the relative positions in the friction-affected hinges are φ_1, $\varphi_2 = \varphi_1 - \varphi_7$, φ_3, $\varphi_4 = \varphi_3 - \varphi_7$, φ_5 and $\varphi_6 = \varphi_5 - \varphi_7$.

3. GOVERNING EQUATIONS

The potential and kinetic energies of the system in Figure 3 are

$$V = \sum_{i=1}^{4} m_i g x_i + \tfrac{1}{2}\sum_{i=1}^{3} c_i s_i^2, \qquad T = \tfrac{1}{2}\sum_{i=1}^{4} m_i(\dot{x}_i^2 + \dot{y}_i^2) + \tfrac{1}{2}\sum_{i=1}^{4} J_i\dot{\varphi}_{2i-1}^2. \quad (8)$$

The differential equations of motion established with Lagrange's equations of the second kind are considered in the form

$$\sum_{i=1}^{3} a_{ik}\ddot{q}_k = \Phi_i(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) - Q_i^{(f)}, \quad i = 1, 2, 3. \quad (9)$$

Here, the generalized friction forces $-Q_i^{(f)}$ are given by

$$Q_i^{(f)} = \sum_{j=1}^{6} M_j \frac{\partial \varphi_j}{\partial q_i}, \qquad i = 1, 2, 3. \tag{10}$$

The kinetic friction torques M_j in the hinges O_j are given by the idealized friction laws

$$M_j = [\rho_j \sqrt{N_j^2 + T_j^2} + M_{j0}] \sin(\dot{\varphi}_j), \qquad j = 1, 2, \ldots, 6. \tag{11}$$

Here, the ρ_j are the friction radii, the N_j and T_j are the components of the forces acting in the hinges of the connecting members, in horizontal and vertical directions for the hinges O_j, $j = 1, 3, 5$, and in longitudinal and transversal directions for the hinges O_j, $j = 2, 4, 6$. The longitudinal components N_2, N_4 and N_6 are given by

$$N_{2j} = c_j s_j + \beta_j \dot{s}_j, \qquad j = 1, 2, 3. \tag{12}$$

The friction terms $M_{j0} \geqslant 0$ are due to prestressing or result from supplementary devices used to increase friction effects.

The system of equations (9) can be solved for the accelerations, and, by considering equations (10), the accelerations can be expressed as

$$\ddot{q}_i = \ddot{q}_{i(0)} + \sum_{j=1}^{6} l_{ij} M_j, \qquad i = 1, 2, 3, \tag{13}$$

where $\ddot{q}_{i(0)}$ are the accelerations of the system without friction, and l_{ij} are influence coefficients depending on position parameters.

Because this system has a determinate structure, the reaction force components N_j, $j = 1, 3, 5$, and T_j, $j = 1, 2, \ldots, 6$, can be computed from conditions of kinetic equilibrium for parts of the system and expressed as linear functions of the accelerations and friction torques. By considering equations (13) for the accelerations, the following expressions for the reaction forces components can be obtained:

$$N_j = N_{j(0)} + \sum_{k=1}^{6} \alpha_{jk} M_k; \quad j = 1, 3, 5, \qquad T_j = T_{j(0)} + \sum_{k=1}^{6} \gamma_{jk} M_k; \quad j = 1, 2, \ldots, 6. \tag{14}$$

Here, the $N_{j(0)}$ and $T_{j(0)}$ correspond to the system without friction, and the influence coefficients α_{jk} and γ_{jk} depend on position parameters.

Equations (11)–(14) constitute a non-linear and implicit system of 21 equations for the unknown accelerations (three), force components (twelve) and kinetic friction torques (six). The solutions of this system of equations can be determined only by iterative methods.

The starting values $M_j^{[s]}$, $j = 1, 2, \ldots, 6$, substituted in equations (13) and (14), yield the approximations $\ddot{q}_i^{[s]}$, $i = 1, 2, 3$, $N_j^{[s]}$, $j = 1, 3, 5$ and $T_j^{[s]}$, $j = 1, 2, \ldots, 6$. These values and N_j, $j = 2, 4, 6$, given by equations (12), are substituted into equations (11), yielding the approximations $M_j^{[s+1]}$, $j = 1, 2, \ldots, 6$, of the friction torques. In the case of convergence these computations are repeated for $s = 0, 1, 2, \ldots$, until the desired accuracy is obtained. As starting values $M_j^{[0]}$ one can use the friction terms M_{j0}, $j = 1, 2, \ldots, 6$.

For the computation of the motion and velocity time histories the Runge–Kutta fourth order method is used [6]. Because of the non-linearity and implicitness of the equations for the determination of the accelerations, the computation of the auxiliary values required by the Runge–Kutta method can be carried out only with iterative schemes: i.e., for each step in the Runge–Kutta method four iterative computations of the accelerations with equations (11)–(14) are necessary.

4. NUMERICAL RESULTS

For numerical simulation the following parameter values are used: $m_1 = m_2 = m_3 = 5$ kg, $m_4 = 3000$ kg, $l_1 = 1$ m, $l_2 = 1\text{ m} + \Delta l$, with $\Delta l = +0{\cdot}1$ mm, 0 or $-0{\cdot}1$ mm, $l_3 = 1$ m, $d_1 = d_2 = d_3 = 0{\cdot}5$ m, $b_1 = 1\text{ m} - \Delta b$, $b_2 = 1\text{ m} + \Delta b$, with $\Delta b = 0$ or $0{\cdot}5$ m, $b_3 = 1$ m, $b_4 = 0{\cdot}5$ m, $J_i = 0{\cdot}4167$ kgm^2, $i = 1, 2, 3$, $J_4 = 1250$ kgm^2, and $J_e = 3005 \cdot 892$ kgm^2. Here J_e is the equivalent moment of inertia of the swinging plate (Figure 2).

The stiffness c_j of the springs representing the elastic properties of the deformable connecting members are considered to have values between 5×10^7 N/m and $6{\cdot}5 \times 10^8$ N/m. The value 5×10^7 N/m nearly corresponds to a member of steel composed of a middle part (length 920 mm, cross-section area 400 mm^2) with two rings (outside radius 40 mm, inner radius 30 mm, depth 50 mm) at its ends.

With the viscosity coefficients β_j of the dashpots representing the damping properties of the deformable connecting members, the free vertical oscillations of the plate supported by three identical connecting members, with $\varphi_1 \equiv 0$, and $\varphi_7 \equiv 0$, are characterized by the damping factor $D = 3\beta_j/(2\sqrt{3c_j m_4})$. Here, values of β_j leading to the damping factor D between 0·3 and 0·005 are considered. For the kinetic friction radii ρ_j the values, 0·025 m and 0·050 m are considered and for the friction terms $M_{j0} \leqslant 100$ Nm. For numerical simulations combinations of these parameter values are used which assure unique and finite solutions for the kinetic state parameters and avoid kinetic friction locking [5].

For comparison of time histories of the state parameters during semicycles of pendular motions, the initial conditions are of great importance. It is assumed that the system is in equilibrium at a given position $\varphi_1 = \varphi_1(0) = -30°$ under the action of a horizontal force F acting in the y-direction at the mass centre S_4 and at $t = 0$ this force is removed. The duration of the first pendular semicycle is denoted by τ, the pendular elongation is $\varphi_1(\tau)$ and $\dot{\varphi}_1(\tau) = 0$. As a measure for the reduction of the pendular elongations during this semicycle is considered $\Delta\varphi_1 = [|\varphi_1(0)| - \varphi_1(\tau)]/|\varphi_1(0)|$ expressed in %.

The step for numerical integration is $h = 10^{-3}$ s.

4.1. *Influence of stiffness*

For given values of the friction parameters ($\rho_j = 0{\cdot}025$ m, $M_{j0} = 0, j = 1, 2, \ldots, 6$), design parameters Δl and Δb, stiffnesses $c_1 = c_2 = c_3$ within the domain considered and viscosity coefficients leading to the damping factor $D = 0{\cdot}1$, the time histories of motion and of the force components have been determined. The influence of the stiffness values on the pendular elongation $\varphi_1(\tau)$, i.e., on the values of $\Delta\varphi_1$, are presented in Figure 4. With $\Delta l = 0$ and $\Delta b = 0$ the pendular semicycle starting from $\varphi_1(0) = -30°$ ends after $\tau = 1{\cdot}016$ s in position $\varphi_1(\tau) = 23{\cdot}4249°$, leading to $\Delta\varphi_1 = 21{\cdot}92\%$. For this case, the stiffnesses are without influence on the $\Delta\varphi_1$—values presented in Figure 4 as curve 1. The same value for $\Delta\varphi_1$ is obtained with $\Delta l = 0$ and $\Delta b = 0{\cdot}5$ m. This is an unsatisfactory aspect of the results obtained by numerical simulation with the model having "perfect" lengths: i.e., in this case, with $\Delta l = 0$.

For $\Delta l \neq 0$ increasing stiffnesses cause increasing values of $\Delta\varphi_1$. Curves 2 and 3 are obtained with $\Delta l = -0{\cdot}1$ mm and $\Delta b = 0$ and $\Delta b = 0{\cdot}5$ m, respectively. Curves 4 and 5 are obtained with $\Delta l = 0{\cdot}1$ mm and $\Delta b = 0{\cdot}5$ m and $\Delta b = 0$, respectively.

The results presented in curves 2 and 5 show that the simulation model is sensitive to small variations of the lengths only for relative high values of the stiffnesses of the connecting members. Such small variations of the design parameters can result from manufacturing inaccuracies or are due to temperature variations. In particular for systems which for the assumption that the connecting members are rigid, lead to multi-body models with indeterminate structures, small variations of the lengths have a strong influence on the motion of the systems, and the sensitivity of the simulation model to such variations is essential.

The results presented in curves 3 and 5 show that the simulation model with $\Delta l \neq 0$ is sensitive to modifications in the arrangements of the connecting members only for relative high values of the stiffnesses.

4.2. *Influence of viscosity*

For given values of the stiffnesses and friction parameters, the influence of the viscosity coefficients β_j on the duration τ of the first pendular semicycle, on the pendular elongation $\varphi_1(\tau)$ and on the maximum value of the pendular velocity $\dot{\varphi}_{1max}$ is not significant. For example, with $c_1 = c_2 = c_3 = 5 \times 10^8$ N/m, $\Delta l = 0{\cdot}1$ mm, $\Delta b = 0$, $\rho_j = 0{\cdot}025$ m, $M_{j0} = 0$, $j = 1, 2, \ldots, 6$, and values of $\beta_1 = \beta_2 = \beta_3$ yielding the damping factor D equal to 0·3, 0·2, 0·1, 0·01 and 0·005, the durations τ are 1·016 s, 1·015 s, 1·014 s, 1·015 s and 1·016 s, the pendular elongations $\varphi_1(\tau)$ are 14·5702°, 14·5705°, 14·5712°, 14·5666° and 14·5513°, and the maximum pendular velocities $\dot{\varphi}_{1max}$ are 1·2067 rad/s, 1·2068 rad/s, 1·2068 rad/s, 1·2067 rad/s and 1·2063 rad/s, respectively.

The influence of the viscosity values on the histories of the force components is different. For example, with β_j values yielding the damping factor D equal to 0·1, 0·2 and 0·3, the longitudinal force component N_2 versus angle φ_1, after a short oscillatory interval between initial position $\varphi_1(0) = -30°$ and positions $\varphi_1(0{\cdot}083 \times \tau) = -29{\cdot}2432°$, $\varphi_1(0{\cdot}043 \times \tau) = -29{\cdot}7904°$, and $\varphi_1(0{\cdot}034 \times \tau) = -29{\cdot}8681°$, is a non-oscillatory function increasing from the values 23 749·06 N, 23 602·97 N and 23 585·49 N to the maximum values 30 184·20 N, 30 184·08 N and 30 184·04 N, and after them, decreasing to 29 999·44 N, 29 999·37 N and 28 866·81 N, respectively, at the end of the semicycle.

With lower values of the damping factor the oscillatory interval increases, and for $D \leqslant 0{\cdot}017$ the force component $N_2(\varphi_1)$ is oscillatory during the entire semicycle. The mean values of this oscillatory function obtained with $D \leqslant 0{\cdot}017$ nearly coincide with the non-oscillatory values obtained with higher values of the damping factor D. For example, with β_j values yielding the damping factor $D = 0{\cdot}01$ (0·005), the longitudinal force component N_2 takes values between $-19\,073{\cdot}48$ N ($-20\,744{\cdot}00$ N) and 64 150·38 N (65 191·47 N), the maximum mean value is 30 188·15 N (30 264·28 N), the amplitude at this position is 162·22 N (2167·92 N), and at the end of the semicycle the mean value becomes 29 999·76 N (30 013·11 N) with the amplitude 36·58 N (917·63 N).

These results indicate significant differences in the time histories and maximum values of the force components between the simulations with high or low viscosity coefficients. If limits for the maximum values of the force components are included in the simulation program, then appropriate values for the viscosity coefficients are of great importance.

4.3. *Influence of friction*

Increasing friction parameter values mainly cause reductions of the pendular elongations, i.e., increasing $\Delta\varphi_1$ values, whereas increasing values of the friction radii ρ_j accentuate also the interactive consequences of friction feedback, increasing values of the friction terms M_{j0} produce only an additive growth of the $\Delta\varphi_1$ values.

For the data considered in Figure 4 and friction parameters $\rho_j = 0{\cdot}050$ m, $M_{j0} = 0$, $j = 1, 2, \ldots, 6$, i.e., doubling the friction radii, curve 1 moves from the value 21·92% to the value 42·65%. The gradients of the other curves increase yielding approximately a two times increase of the slopes of the linear intervals.

With the data considered for Figure 4 and friction parameters $\rho_j = 0{\cdot}025$ m, $M_{j0} = 100$ Nm, $j = 1, 2, \ldots, 6$, curves 1–5 translate in the direction of higher $\Delta\varphi_1$ values of $\approx 6{\cdot}50\%$. Similar results are obtained with friction radii $\rho_j = 0{\cdot}050$ m and friction terms $M_{j0} = 0$ and $M_{j0} = 100$ Nm, $j = 1, 2, \ldots, 6$.

The influence of increasing values of the friction parameters on the maximum values of the force components is not uniform. In general, increasing friction creates higher stresses in the connecting members. In addition, increasing friction parameters yields prolongations of the oscillatory intervals of the force components.

5. CONCLUSIONS

A system in the form of a swinging plate is considered, which for the assumption that the connecting members are rigid, yields a multi-body model with indeterminate structure. With deformable connecting members a model with determinate structure is obtained. For this model and motion resistances considered as friction torques acting in the hinges, the importance of the values of the stiffnesses of the connecting members to the possibility to simulate the influence of small system parameter variations and different design alternatives has been investigated. From the numerical simulation of the kinetic state parameters histories, the following facts emerge.

(a) The interactive influence on the pendular elongations due to friction feedback of small variations of the lengths and different arrangements of the connecting members can be simulated only with relative high values of the stiffnesses.

(b) The values of the viscosity coefficients have a significant influence on the maximum values of the force components acting in the connecting members. For a large range of viscosity coefficients, their influence on the pendular elongations is not significant.

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