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# DISSIPATIVE CONTROL OF CHAOS IN NON-LINEAR VIBRATING SYSTEMS

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## 1. INTRODUCTION

The bifurcation structure and chaotic responses of non-linear systems have been studied extensively. Current interest in this field is to devise control strategies to suppress the instabilities leading to bifurcations and chaos. Extensive literature on control of chaos in various physical and engineering systems has been reviewed by Shinbrot *et al*. [1] and Chen and Dong [2]. These studies include both active and passive control methods [3].

In this paper, the objective is to devise control methods to suppress bifurcations and chaos in non-linear vibrating systems. Towards this end, the canonical example of a non-linear vibrating system, Duffing's oscillator with sinusoidal excitation, is considered [4]. The control scheme to suppress chaos is based on the use of strictly dissipative damping forces. Several types of damping forces, namely, viscous, power-law and even time varying damping forces are considered. This method is termed as dissipative control. The idea of using non-linear dissipation to control chaos is proposed in earlier work [5]. The use of dissipative feedback control to track unstable orbits has also been considered by Rulkov *et al*. [6] for a general dynamical system. In the present case of a vibrating system, the use of variation of damping forces is the obvious choice of achieving dissipative control.

## 2. PASSIVE CONTROL OF CHAOS

Several vibrating systems can be represented by the Duffing's oscillator given in the following form,

$$x' = y,$$
$$y' = -\alpha(-x + x^3) - 2\zeta y + f\cos(\omega t), \tag{1}$$

where $\alpha$ is the stiffness parameter, $\zeta$ ($>0$) is the viscous damping coefficient, $f$ and $\omega$ are respectively the amplitude and frequency of external input. In the present section $\alpha$ is taken to be 0·5.

It is known [4] that equation (1) possesses a chaotic attractor obtained through the period-doubling route to chaos. The first period-doubling bifurcation occurs for the parameter values $\zeta = 0{\cdot}05$, $\omega = 1{\cdot}5$, $f = 0{\cdot}38$. Let $u(x, x', t)$ be the control force applied to achieve the desired objective. Incorporating this control force in equation (1), one can obtain the "controlled" Duffing's equation as

$$x' = y,$$
$$y' = -\alpha(-x + x^3) - 2\zeta y + f\cos(\omega t) + u(x, x', t). \tag{2}$$

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The desired objective could be any of the following: (1) to convert a chaotic response to some periodic orbit, (2) to stabilize the desired unstable periodic orbit embedded in the basin of attraction of the attractor, (3) to convert chaos to a desired periodic orbit which need not be a solution of the system under consideration. The control strategy could be either an open loop (passive) or closed loop (active). If the desired objective is to convert chaotic response to some periodic orbit, one can choose the open loop control of changing the parameters of the system. As pointed out in reference [7], in this case, the problem reduces to locating the periodic and chaotic zones in the parameter space and selecting the relevant set of parameter values. There are several ways of computing such bifurcation diagrams. Some of them are parameter continuation techniques, direct numerical integration, incremental harmonic balance and perturbation techniques. It may be noted that these computations are quite involved and selecting a control strategy based on these may be quite cumbersome. However, it may be noted that as opposed to variation of frequency and forcing amplitude, increase of damping coefficient invariably leads to the suppression of the chaotic attractor and no new instabilities will be encountered in the dynamics. In this case one need not have prior knowledge of the bifurcation diagram. Also practical implementation of this technique is quite easy in the case of vibrating systems as there are several ways of augmenting the damping of a system such as constrained layer damping treatments etc. [8]. Another advantage of this technique is that it is quite robust with respect to noise in the excitation.

In this section, suppression of chaotic orbit by using power law damping is presented. The desired objective is to suppress the chaotic orbit to some periodic orbit whose amplitude and frequency are not specified. In this case, the problem of control of chaos reduces to the determination of critical parameter zones of periodic and chaotic responses. The Melnikov criterion and the period-doubling criterion are used for this purpose.

### 2.1. *Use of power law damping*

Usually damping mechanisms are non-linear in nature and are represented by a general expression which is proportional to some power of velocity. Consider equation (2) with the control force $u(x, x', t) = -Gy|y|^{(p-1)}$. When $p = 1$, this expression reduces to the case of viscous damping. The case when $p = 0$ represents the Coulomb friction. Quadratic damping commonly observed in hydraulic dampers is given by $p = 2$ and pneumatic dampers are also represented by power law damping. This form of power law damping is non-analytic and not amenable for standard stability analysis. The power law form has been shown to be approximated by a combination of cubic ($p = 3$) and viscous damping ($p = 1$) to any degree of accuracy [9]. Hence we consider such a combination in the present work. With this approximation, the controlled Duffing's equation (2) becomes

$$x' = y,$$

$$y' = -\alpha(-x + x^3) - 2\zeta y + f\cos(\omega t) - G_1 y - G_2 y^3. \tag{3}$$

Now the choice of $G_1$ and $G_2$ can be made based on Melnikov criterion or period-doubling criterion.

#### 2.1.2. Melnikov criterion

Extension of Melnikov criterion to take care of the effect of cubic damping force is straightforward and can be obtained [4] as,

$$f = [8(\zeta + G_1/2)/3 + 16G_2/35]\{\cosh(\pi\omega/2)\}/\{\sqrt{2}\pi\omega\}. \tag{4}$$

Now one can choose optimum values of the $G_1$ and $G_2$ based on equation (4) to avoid chaos.

2.1.2. Period-doubling criterion

Assuming the resulting harmonic solution of equation (3) in the form

$$x = a_0 + a\cos(\omega t - \varphi), \tag{5}$$

and using the method of harmonic balance, we get the following expressions for $a_0$ and $a$:

$$a_0 = (2 - 3A^2)/2 \tag{6}$$

and

$$(15^2/8^2)a^6 - (15/4)(1-\omega^2)a^4 + (1-\omega^2)^2a^2 + [(2\zeta + G_1)a\omega + 3G_2a^3\omega^3/4]^2 - f^2 = 0. \tag{7}$$

The stability analysis of the solution given by equation (7), using the standard variational method is carried out. Then using the method of harmonic balance once again for 1/2 subharmonic solution, one can derive that the following condition must be satisfied for a non-trivial solution:

$$(\lambda_0 - \omega^2/4)^2 + [(\zeta + G_1/2) + 3G_2a^2\omega^2/4]^2\omega^2 - \lambda_1^2/4 = 0, \tag{8}$$

where $\lambda_0 = 3a_0^2/2 + 3a^2/4 - 1/2$ and $\lambda_1 = 3a_0a$. Equation (8) in conjunction with (7) determines the boundary of the period-doubling bifurcation in the $f$–$\omega$ plane. Any choice of $G_1$, $G_2$ greater than the values obtained by solving equations (8) and (7) yields a stable harmonic solution. Thus, suppressing the first period-doubling bifurcation automatically ensures suppression of ensuing chaos. Of course, this method yields a conservative estimate.

2.1.3. Application and numerical simulations

In this section a practical application of control of chaos using a special case of power-law damping ($p = 0$) is demonstrated. Vibration isolation is a common method of vibration control. Friction dampers are frequently employed in vibration isolation of space applications. An interesting account of such isolators known as wire-rope isolators, considered in designing an isolation system for the Hubble space telescope can be found in the literature [10]. The governing equation of motion of such a vibration isolation system can be obtained in non-dimensional form as [5],

$$\ddot{\Delta} + 2\zeta\dot{\Delta} + 2\zeta_f\,\mathrm{sgn}(\dot{\Delta}) + \Delta^3 = \Omega^2\cos(\Omega\tau)$$

where $\Delta$ is the relative displacement of the body to be isolated with respect to the base, $\zeta_f$ is the coulomb damping coefficient, $\Omega$ is the frequency of base excitation and $\tau$ is the time. Numerical simulations of the above equation are carried out with $\zeta_f = 0$ and $\zeta = 0{\cdot}01$ for several values of $\Omega$. As the frequency is decreased, the periodic orbit (shown in Figure 1(a) for $\Omega = 0{\cdot}5$) looses its symmetry due to the symmetry-breaking bifurcation and the resulting dual, unsymmetric solutions are shown in Figures 1(b) and (c) with the corresponding initial conditions. Decreasing the frequency further, a period-doubling bifurcation takes place (at $\Omega = 0{\cdot}2$) and the resulting period-two orbit can be seen in Figure 1(d). At a still lower value of the frequency ($\Omega = 0{\cdot}17$), a chaotic attractor (not shown here) resulting from the usual period-doubling route is obtained. Further reduction of $\Omega$ shows the emergence of a period-three orbit followed by the (Type I) intermittency route to chaos.

Numerical simulation of the above equation (for the same value of $\zeta = 0{\cdot}01$) is now carried out after including a small friction damping ($\zeta_f = 0{\cdot}005$). These results are shown in Figures 2(a)–(d). Figure 2(a) shows a symmetric, periodic orbit at $\Omega = 0{\cdot}5$. It can be seen from Figure 2(b) that the symmetry-breaking bifurcation occurs again at $\Omega = 0{\cdot}4$ (the same as that in Figure 1(b)). Thus, it appears that the symmetry-breaking bifurcation is quite insensitive to the value of $\zeta_f$. However, any further reduction of the frequency even as far as 0·15 does not reveal the period-doubling cascade as confirmed by Figures 2(c) and (d). Thus, the addition of friction damping effectively suppresses both period-doubling and intermittency routes to chaos.

## 3. FEEDBACK CONTROL OF CHAOS

In this section two types of controls, namely, an open loop law (time-dependent damping perturbation) and a closed loop law (linear state feedback control law) are considered. This control scheme can be viewed as a combination of the parametric perturbation method and feedback control scheme considered previously in the literature.

### 3.1. *Time varying damping*

A generalized Duffing's oscillator with time varying damping is considered in the following form,

$$x' = y,$$

$$y' = -\alpha x - x^3 - 2\zeta[1 + \psi \cos(2\omega t)]y + f\cos(\omega t), \tag{9}$$

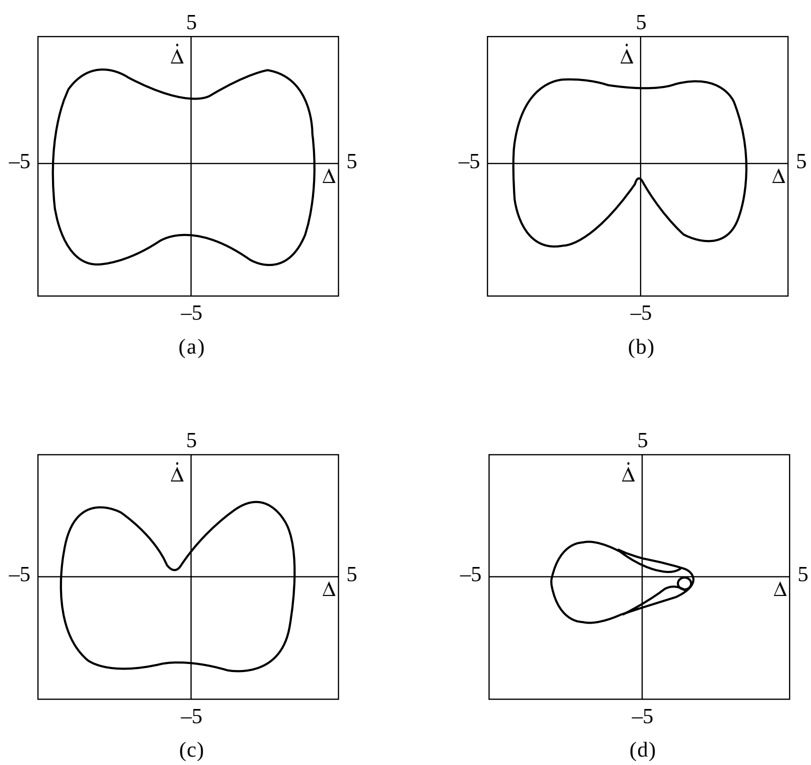


Figure 1. Phase plots of the period-doubling route with $\zeta = 0{\cdot}01$ and $\zeta_f = 0{\cdot}0$. (a) $\Omega = 0{\cdot}5$ and $\Delta_0 = \dot{\Delta}_0 = 0{\cdot}1$. (b) $\Omega = 0{\cdot}4$ and $\Delta_0 = \dot{\Delta}_0 = 0{\cdot}1$. (c) $\Omega = 0{\cdot}4$ and $\Delta_0 = \dot{\Delta}_0 = -0{\cdot}1$. (d) $\Omega = 0{\cdot}2$ and $\Delta_0 = \dot{\Delta}_0 = 0{\cdot}1$.

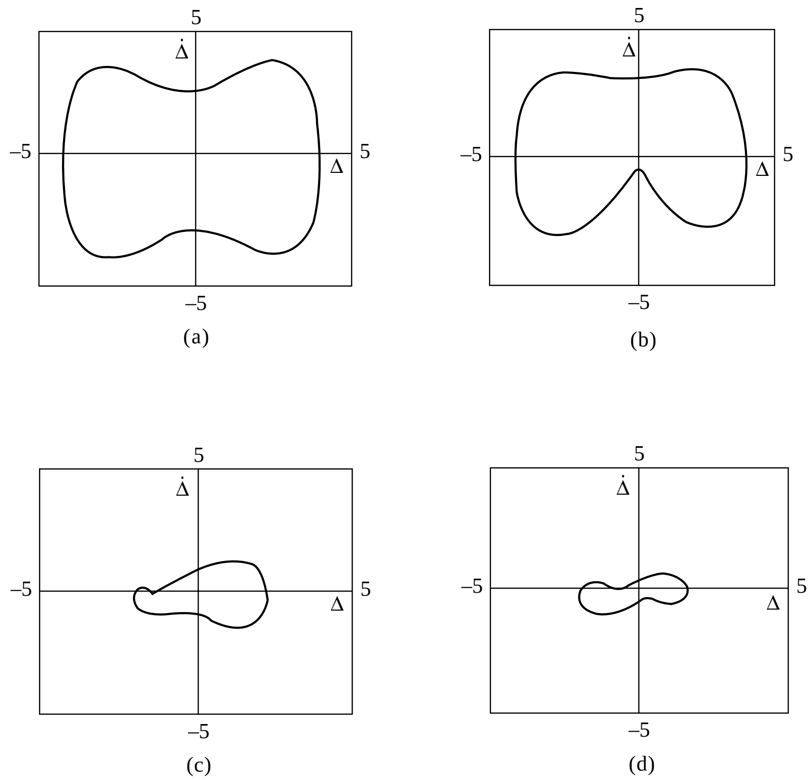


Figure 2. Phase plots with $\zeta = 0{\cdot}01$ and $\zeta_f = 0{\cdot}005$. (a) $\Omega = 0{\cdot}5$ and $\Delta_0 = \dot{\Delta}_0 = 0{\cdot}1$. (b) $\Omega = 0{\cdot}4$ and $\Delta_0 = \dot{\Delta}_0 = 0{\cdot}1$. (c) $\Omega = 0{\cdot}2$ and $\Delta_0 = \dot{\Delta}_0 = 0{\cdot}1$. (d) $\Omega = 0{\cdot}15$ and $\Delta_0 = \dot{\Delta}_0 = 0{\cdot}1$.

where $\alpha$ is the stiffness parameter, $\zeta$ ($>0$) is the viscous damping coefficient, $f$ and $\omega$ are respectively the amplitude and frequency of external input. The time variation of the damping coefficient has an amplitude $\psi$. Linear oscillators with simultaneous time periodic variation in damping and spring constants have been studied in reference [11]. The results obtained in this work indicate that even a little negative damping during some part of the cycle more than offsets the stabilizing effect of the correspondingly higher positive damping during some other part of the same cycle. Therefore, in the present work the time variation in damping is restricted to be positive (or strictly dissipative only). Here, the desired objective is the local stabilization of an unstable known periodic orbit embedded in the chaotic attractor of equation (9). Towards this end, a feedback control scheme based on the conventional proportional derivative strategy is used [12].

The objective is to choose feedback control law such that the response of the resulting system is asymptotically stable and leads to the desired periodic orbit.

Let $(\bar{x}, \bar{y})$ be the desired, known periodic orbit embedded in the chaotic attractor of equation (9). Consider a proportional-derivative feedback controller given by

$$u(x, x', t) = -G_1(x - \bar{x}) - G_2(y - \bar{y}). \tag{10}$$

Adding the control law (10) to (9), gives,

$$x' = y,$$

$$y' = -(G_1 + \alpha)x - x^3 - (G_2 + 2\zeta[1 + \psi \cos(2\omega t)])y - y^3 + (G_1\bar{x} + G_2\bar{y}) + f\cos(\omega t). \tag{11}$$

Now the objective is to choose $G_1$, $G_2$ such that

$$\|x(t) - \bar{x}(t)\| \to 0 \quad \text{and} \quad \|y(t) - \bar{y}(t)\| \to 0 \quad \text{as} \quad t \to \infty. \tag{12}$$

Denoting $e_1 = x - \bar{x}$ and $e_2 = y - \bar{y}$, one can rewrite equations (9) and (11) as

$$e_1' = e_2,$$

$$e_2' = -(G_1 + \alpha + 3\bar{x}^2)e_1 - 3\bar{x}e_1^2 - e_1^3 - (G_2 + 2\zeta[1 + \psi \cos(2\omega t)])e_2 - 3\bar{x}'e_2^2 - e_2^3. \tag{13}$$

It may be noted that $e_1 = 0$, $e_2 = 0$ is an equilibrium point of equation (13). Now choosing $G_1$, $G_2$ to satisfy equation (12) is equivalent to achieving the following objective: choose $G_1$, $G_2$, such that the zero solution of equation (13) is asymptotically stable, i.e.,

$$\|e_1\| \to 0, \qquad \|e_2\| \to 0 \quad \text{as} \quad t \to \infty. \tag{14}$$

Thus, the problem of controlling a chaotic attractor is transformed to the stabilization problem of the origin of (13) (the desired orbit is assumed to be a solution of equation (9)). The design of controller that satisfies equation (14) is discussed in the next section.

### 3.2. *Design of controller*

The local stabilization problem for a general non-linear, non-autonomous system is quite complex. It may be observed that equations (11) and (13) are non-linear and non-autonomous in nature. The linearized version of equation (13) can be written as,

$$e_1' = e_2,$$

$$e_2' = -(\alpha + G_1 + 3\bar{x}^2)e_1 - (G_2 + 2\zeta[1 + \psi \cos(2\omega t)] + 3\bar{x}'^2)e_2. \tag{15}$$

In the present case, since the desired orbit is periodic, the global uniform asymptotic stability of the linearized system (15) can be used to analyse the local uniform asymptotic stability of the original non-linear equation (13).

Assuming further that the given periodic orbit is given by $\bar{x} = b \sin(\omega t)$, and carrying out the usual algebra, one gets equation (15) in the form,

$$e_1'' + [\eta + 2p \cos(2\omega t)]e_1' + [a - 2q \cos(2\omega t)]e_1 = 0 \tag{16}$$

where

$$\eta = 2\zeta + G_2,$$

$$p = \zeta\psi,$$

$$a = \alpha + G_1 + 3b^2/2$$

$$q = 3b^2/4. \tag{17}$$

Imposing the following conditions on parameters in equation (16)

$$\infty > a > 2q > 0,$$

$$\eta > 2p > 0,$$

$$\omega < (\eta - 2p)(a - 2q)/(2q), \tag{18}$$

one can show that the origin of equation (15) is a uniformly asymptotically stable equilibrium by choosing a Lyapunov function

$$V(t, e_1, e_2) = e_1^2 + e_2^2/[a - 2q \cos(2\omega t)].$$

The first and second conditions in equation (18) imply that the restoring and damping forces are always positive and the third one ensures that the nature of the time variation is sufficiently slow. Substituting equation (17) in conditions (18) one obtains the desired control laws.

## 4. CONCLUSIONS

The procedure of dissipative control of chaos is outlined and the use of power-law damping forces for passive control of chaos is demonstrated. Local control of chaos using feedback is shown. Stability analysis is presented based on the Lyapunov second method. A combination of the parametric perturbation method and feedback control strategy is proposed for local control of chaos.

## REFERENCES

1. T. SHINBROT, C. GREBOGI, E. OTT and J. A. YORKE 1993 *Nature* **65**, 3211–3214. Using small perturbations to control chaos.
2. G. CHEN and X. DONG 1993 *International Journal of Bifurcation Theory and Chaos* **3**, 1363–1409. From chaos to order, perspectives and methodologies in controlling chaotic nonlinear dynamical systems.
3. B. BLAZEJCZYK *et al*. 1993 *Applied Mechanics Reviews* **47**, 385–391. Controlling chaos in mechanical systems.
4. B. RAVINDRA and A. K. MALLIK 1994 *Physical Review E* **49**, 4950–4954. Role of nonlinear dissipation in soft Duffing oscillators.
5. B. RAVINDRA and A. K. MALLIK 1995 *Journal of Sound and Vibration* **182**, 345–353. Chaotic response of a mass on an isolator with nonlinear stiffness and damping parameters.
6. N. F. RULKOV *et al*. 1994 *Physical Review E* **50**, 314–324. Tracking unstable orbits in chaos using feedback control.
7. W. SZEMPLINSKA-STUPNICKA 1994 *Journal of Sound and Vibration* **178**, 276–284. A discussion of an analytical method of controlling chaos in oscillator.
8. A. K. MALLIK 1992 *Principles of Vibration Control*. New Delhi: Affiliated East West Press.
9. A. H. NAYFEH and A. A. KHDEIR 1986 *International Ship Building Progress* **33**, 40–49. Nonlinear rolling of ships in regular beam seas.
10. M. L. TINKER and M. A. CUTCHINS 1992 *Journal of Sound and Vibration* **157**, 7–18. Damping phenomena in a wire-rope vibration isolation system.
11. N. CHANDRA SEKHAR and A. K. MALLIK 1994 *Journal of Sound and Vibration* **171**, 411–419. Instability due to simultaneous time variation in damping and spring constants.
12. A. Y. T. LEUNG, C. W. CHAN and B. RAVINDRA 1995 *Proceedings of the International Conference on Structural Dynamics*, *Vibration*, *Noise and Control*, *Hong Kong*. Compensator for nonlinear dynamical systems.