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FREE AND FORCED RESPONSE OF MISTUNED LINEAR CYCLIC SYSTEMS: A SINGULAR PERTURBATION APPROACH

G. S. Happawana, O. D. I. Nwokah

Mechanical Engineering Department, Southern Methodist University, Dallas, TX 75205, *U.S.A.*

A. K. Bajaj

School of Mechanical Engineering, Purdue University, West Lafayette, IN 47907, *U.S.A.*

and

M. Azene

Department of Civil Engineering, Southern University, Baton Rouge, LA 70813, *U.S.A.*

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The effect of small deterministic parameter perturbations on the forced response of nearly periodic structures with cyclic symmetry is investigated. The general methodology developed herein is applicable to any *n*-degree-of-freedom, strongly coupled cyclic system with two arbitrary and independent variations in system parameters that destroy the cyclic symmetry. The specific system studied may be regarded as a simplified model of a strongly coupled bladed disk assembly. Singular perturbation methods along with modal expansion analysis are applied to gain a physical insight into the effects of parameter perturbations on the eigenvalues, the eigenvectors as well as the forced response amplitudes. The study shows that, under appropriate conditions, the splitting and veering of eigenvalues due to mistunings or small parameter variations increases the amplitude of vibration of some blades significantly compared to what would be predicted by an analysis of the perfectly tuned system. Furthermore, modal bifurcations lead to uneven vibration amplitudes irrespective of the stiffness of the coupling springs. The variations in blade amplitudes are also found to be strongly dependent on damping, and the type of engine order excitation applied to the system for the same set of mistuning parameters.

1. INTRODUCTION

Periodic structures with cyclic symmetry are very often encountered in engineering practice; typical examples of such systems being bladed disk assemblies, large circular space antennas and floppy disks in memory devices. The cyclic symmetry in these systems is destroyed by the small non-uniformities in stiffness and mass properties that invariably arise due to manufacturing and material tolerances. As an example, in bladed disk assemblies the individual blades are really not identical. This variation in system parameters from the ideal cyclic symmetry is known as rotor mistuning. It has received much attention in the vibration literature [1–13] because of the fact that a small amount of mistuning, under appropriate conditions, can cause unexpectedly large amplitude vibrations for some blades, irrespective of the values of the coupling spring constants. Consequently, the fatigue life of a subcomponent of a mistuned cyclic symmetric system

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may be significantly lower than that predicted on the basis of a perfectly tuned cyclic system.

In order to avoid unexpected and premature failures, it is important to pre-identify those few subcomponents in a cyclic symmetric system which could have unusually large amplitudes. Although various research results have consistently predicted that amplitudes of some subcomponents in mistuned cyclic systems could be greatly unequal, there is apparently no consensus on how the subcomponents in a cyclic system with the largest amplitudes may be identified. For instance, El-Bayuomy and Srinivasan [3], and later, Griffin and Hoosac [11] concluded from their studies with bladed disk assemblies that the blades with cantilever frequencies close to the coupled blade–disk resonance usually respond with greatest amplitude. Afolabi's investigations [4] concluded that blades with the largest amplitude are likely to be the blades with the most mistuning, while El-Bayuomy and Srinivasan [3], and Griffin and Hoosac [11], could not reach this conclusion. Also, conflicting results were reached by Sogliero and Srinivasan [12], and Griffin and Hoosac [11], on the effect of mistuning standard deviation on the rotor's largest amplitude. These discrepancies may have originated from the different models and parameter values used in the studies. Wei and Pierre [13] proposed a physical explanation for most of these discrepancies, and utilized for it the occurrence (or non-occurrence) of mode localization depending on modal parameter values. In general, all the studies show that mistuning may have an undesirable effect on the forced response through an increase in the maximum amplitude experienced by some blades [1–4, 11–16]. Also, the works in references [2, 5, 10, 13, 15, 17, 18] explored the modal properties of tuned and mistuned systems, and showed that multiple eigenvalues of the tuned system are split by the introduction of small perturbations in parameters, thereby resulting in additional peaks in the frequency response.

In this work, the singular perturbation methodology and modal analysis are used to investigate the effects of mistuning on the free and forced response of "nearly" periodic structures with cyclic symmetry. The methodology adopted here is general and systematic, and is valid for any cyclic system with two independent perturbations. Although independent, in the neighborhood of parameter values where the usual power series expansions break down and become non-uniform, the two perturbations have to be related in a specified manner. The singular perturbation technique allows us to determine this specific relationship in a systematic way such that the resulting expansion is rendered uniform. In this respect, the present work can be considered a generalization of the singular perturbation approach introduced in reference [10] for the analysis of eigenvalue veering phenomenon. In reference [10], a three-degree-of-freedom mistuned cyclic system was analyzed for the variation of its eigenvalues as a function of the mistuning parameters. The generalization undertaken here for the analysis of modal properties of mistuned systems builds on the modal properties of perfectly tuned cyclic systems [19] and the one-parameter perturbation theory for multiple eigenvalues in non-defective eigenvalue problems [16]. Furthermore, the problems of forced vibratory response with engine-order excitation are solved using modal expansion and perturbation analysis [20].

The analysis presented mainly concerns strongly coupled cyclic systems because of the fact that for strongly coupled cyclic systems, all degenerate double eigenvalues appear in well isolated pairs [19], thus leading to an analysis of the splitting of doublets. This implies that two independent and arbitrary mistunings are sufficient to unfold the double degenerate eigenvalues [21]. In weakly coupled cyclic systems, however, the eigenvalues occur in closely spaced doublets which become coincident as the coupling strength goes to zero, thus leading to an nth order degenerate eigenvalue problem. A completely general characterization of the eigenproperties of the mistuned system will then require the introduction of n independent parameters, and an analytical study is presently not possible for this situation. We should note here that the splitting and perturbation of eigenvalue doublets in strongly coupled cyclic systems leads to the phenomenon of eigenvalue veering and modal sensitivity of the associated eigenvectors [10, 13, 22]. Many investigations into these problems have been previously reported in the literature (e.g., see references [10, 22, 23]).

Since rotor bladed disk assemblies constitute one of the most important engineering examples of cyclic symmetric systems, we consider a simple model of a bladed disk assembly. It consists of a strongly coupled n-degree-of-freedom cyclic system with nominally identical masses, and ground and coupling springs (Figure 1). Two deterministic perturbations ε_1 and ε_k are applied to any two stiffness and mass elements of the system. The subscript k identifies the blade to which the second perturbation is applied. It is well known that the eigenvalues of such a strongly coupled system ($k_c \sim O(k_t)$) are at most degenerate with multiplicity two, and that only two independent (say, diagonal) symmetry-breaking perturbations, ε_1 and ε_k ($k > 1$), are sufficient for unfolding the system's singularities completely [18, 21].

The primary objectives of this work are to investigate the effects of mistuning on the dynamical properties of strongly coupled cyclic systems in terms of the modal parameters. The analysis here has been done by applying singular perturbation methods along with modal expansions to gain a physical insight into the effects of perturbations on the eigenvalues, eigenvectors and forced response amplitudes, respectively. The primary advantage of this approach is that it provides asymptotically valid algebraic expansions for the eigenvalues and eigenvectors, thus making it possible to determine qualitative and quantitative information on the dynamical properties for any given perturbed parameter value. Although, in the present work, the method has been applied only to strongly coupled systems, the algebraic expansions obtained can be used for weakly coupled cyclic systems with the restriction that the perturbations are sufficiently smaller than the corresponding coupling parameters. Usually, in singular perturbation methods, the inner and outer solutions are matched where their domains of validity overlap to obtain composite or uniformly valid asymptotic expansions. Many times, obtaining composite expansions is cumbersome and is really not needed as the inner and outer solutions can be plotted or used simultaneously to determine the approximate eigenvalue loci. In this work, we have

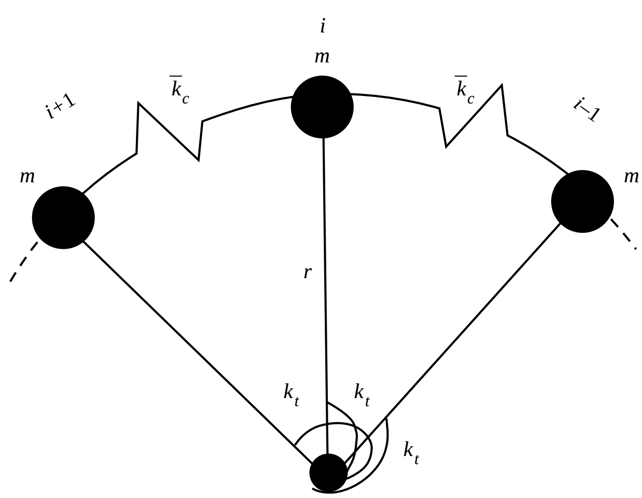

Figure 1. Model of an n-degree-of-freedom bladed disk assembly.

used inner expansions as an approximation to the composite expansions for the purpose of determining the forced response, as the inner expansions provide sufficiently accurate results in the domain of interest.

2. SINGULAR PERTURBATION ANALYSIS

In this section, the standard or regular perturbation methodology is first used to find algebraic expressions for the eigenvalues and eigenvectors of any finite-dimensional linear dynamical system dependent on at most two parameters. These expressions are termed "outer expansions" or straightforward expansions. These expansions fail to be analytic in the neighborhood of a singularity and break down at the singular point. Points where the outer solutions are unbounded are referred to as singular points. It is shown that the singularity causing the breakdown of the straightforward expansions can be analyzed by the well-developed singular perturbation techniques [24, 25] and that appropriate uniformly valid asymptotic expansions for the eigenvalues and eigenvectors can be constructed which provide a correct qualitative and good quantitative approximation. The basic idea behind analysis by the singular perturbation technique is the following: suppose the square matrix $\mathbf{A}(\varepsilon_1, \varepsilon_k)$ has elements that depend on two parameters ε_1 and ε_k. By applying the regular perturbation technique to the eigenvalue problem

$$\mathbf{A}(\varepsilon_1, \varepsilon_k)\boldsymbol{\phi}(\varepsilon_1, \varepsilon_k) = \lambda(\varepsilon_1, \varepsilon_k)\boldsymbol{\phi}(\varepsilon_1, \varepsilon_k),$$

we obtain algebraic expressions for the eigenvalues and eigenfunctions of $\mathbf{A}(\varepsilon_1, \varepsilon_k)$ as a power series in one of the two small parameters or perturbations (say ε_k). The coefficients of the power series are dependent on the second parameter ε_1, and these expansions are valid for sufficiently small ε_k, for all values of ε_1, so long as no singularities arise. Singularities in expansions occur for values of ε_1 where the eigenfrequencies and the eigenfunctions lose their smoothness, and it is then said that the expansion is not uniformly valid for all ε_1. Away from the singular values of the parameter ε_1, the straightforward expansions are a good approximation and are called the "outer expansions" [24, 25]. The neighborhood of the singular parameter point in ε_1 is then stretched or rescaled in terms of a new parameter so as to remove the singularity. The expansion in terms of the new parameter is valid only in the neighborhood of the singular point in ε_1 and is called the "inner expansion" [24, 25]. The inner and the outer solutions can be matched where their domains of validity overlap, and then a composite expansion can be constructed which is valid uniformly throughout the function domain for all values of the parameter ε_1. This technique is called the method of matched asymptotic expansions in the literature [24, 25] and is now applied to a general perturbed cyclic system.

2.1. PERTURBED CYCLIC SYSTEMS

Consider n identical particles of mass m each, arranged in a ring and interconnected by identical springs of stiffness $\bar{k}_c$. Assume that all the masses are hinged to the ground by torsional springs of stiffness k_t and that the radius of the ring is r, as shown in Figure 1. For perturbed systems we consider the cases where two of the torsional springs, two of the coupling springs or any two coupling and torsional springs are perturbed by ε_1' and ε_k'. The structure of the equations of motion is somewhat different in the various cases, and these are described in the following development.

The equations of motion for the perturbed cyclic system, when two ground springs are perturbed, can be written as

$$\ddot{x}_1 + \left(\frac{k_t + \varepsilon_1'}{mr^2} + \frac{2\overline{k}_c}{m}\right)x_1 - \frac{\overline{k}_c}{m}(x_2 + x_n) = 0,$$

$$\ddot{x}_2 + \left(\frac{k_t}{mr^2} + \frac{2\overline{k}_c}{m}\right)x_2 - \frac{\overline{k}_c}{m}(x_3 + x_1) = 0,$$

$$\cdot$$

$$\ddot{x}_k + \left(\frac{k_t + \varepsilon_k'}{mr^2} + \frac{2\overline{k}_c}{m}\right)x_k - \frac{\overline{k}_c}{m}(x_{k+1} + x_{k-1}) = 0, \quad (1)$$

$$\cdot$$

$$\ddot{x}_i + \left(\frac{k_t}{mr^2} + \frac{2\overline{k}_c}{m}\right)x_i - \frac{\overline{k}_c}{m}(x_{i+1} + x_{i-1}) = 0,$$

$$\cdot$$

$$\ddot{x}_n + \left(\frac{k_t}{mr^2} + \frac{2\overline{k}_c}{m}\right)x_n - \frac{\overline{k}_c}{m}(x_1 + x_{n-1}) = 0.$$

Defining a new time scale $\tau = \alpha t$, where $\alpha^2 = k_t/mr^2$, the equations (1) can be written in non-dimensional form as

$$\begin{aligned} &x_1'' + (1 + 2k_c + \varepsilon_1)x_1 - k_c(x_2 + x_n) = 0,\\ &x_2'' + (1 + 2k_c)x_2 - k_c(x_3 + x_1) = 0,\\ &x_k'' + (1 + 2k_c + \varepsilon_k)x_k - k_c(x_{k+1} + x_{k-1}) = 0,\\ &x_n'' + (1 + 2k_c)x_n - k_c(x_1 + x_{n-1}) = 0, \end{aligned} \quad (2)$$

where $\varepsilon_1 = \varepsilon_1'/k_t$, $k_c = \overline{k}_c r^2/k_t$ and $\varepsilon_k = \varepsilon_k'/k_t$.

The eigenvalue problem corresponding to the system of equations (2), when two ground springs, the first and the kth one, are perturbed, is given by

$$\mathbf{A}(\varepsilon_1, \varepsilon_k)\boldsymbol{\phi}(\varepsilon_1, \varepsilon_k) = \lambda(\varepsilon_1, \varepsilon_k)\boldsymbol{\phi}(\varepsilon_1, \varepsilon_k),$$

where

$$\mathbf{A} = \begin{bmatrix} a+\varepsilon_1 & -k_c & 0 & \cdot & \cdot & 0 & -k_c \\ -k_c & a & -k_c & \cdot & \cdot & 0 & 0 \\ \cdot & & & & & & \\ \cdot & & & & & & \\ 0 & \cdot & \cdot & k_c & a+\varepsilon_k & -k_c & 0 \\ \cdot & & & & & & \\ \cdot & & & & & & \\ -k_c & 0 & \cdot & \cdot & 0 & -k_c & a \end{bmatrix}. \quad (3)$$

In the case when the first and the kth coupling springs are perturbed, the matrix $\mathbf{A}$ has the form

$$\mathbf{A} = \begin{bmatrix} a+\varepsilon_1 & -(k_c+\varepsilon_1) & 0 & \cdot & \cdot & 0 & \cdot & -k_c \\ -(k_c+\varepsilon_1) & a+\varepsilon_1 & -k_c & \cdot & \cdot & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & 0 & \cdot & \cdot \\ 0 & 0 & 0 & -k_c & a+\varepsilon_k & -(k_c+\varepsilon_k) & \cdot & 0 \\ 0 & 0 & 0 & 0 & -(k_c+\varepsilon_k) & a+\varepsilon_k & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ -k_c & 0 & 0 & \cdot & \cdot & \cdot & -k_c & a \end{bmatrix}. \tag{4}$$

Similarly, for the case of perturbations in the first coupling and the kth ground spring, the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} a+\varepsilon_1 & -(k_c+\varepsilon_1) & 0 & 0 & \cdot & \cdot & 0 & -k_c \\ -(k_c+\varepsilon_1) & a+\varepsilon_1 & -k_c & 0 & \cdot & \cdot & 0 & 0 \\ 0 & -k_c & a & -k_c & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & -k_c & a+\varepsilon_k & -k_c & 0 & 0 \\ 0 & \cdot & \cdot & 0 & -k_c & a & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ -k_c & \cdot & \cdot & 0 & 0 & \cdot & -k_c & a \end{bmatrix}. \tag{5}$$

In equations (3)–(5), $a = 1 + 2k_c$.

Consider the eigenvalue problem

$$\mathbf{A}(\varepsilon_1, \varepsilon_k)\boldsymbol{\phi}(\varepsilon_1, \varepsilon_k) = \lambda(\varepsilon_1, \varepsilon_k)\boldsymbol{\phi}(\varepsilon_1, \varepsilon_k). \tag{6}$$

The interest here is in the development of asymptotic expansions for the eigenvalues and eigenvectors in terms of the perturbation parameters ε_1 and ε_k.

2.2. Regular perturbation expansions (outer expansions)

For small values of ε_k, it is natural to expand the eigenvalues and eigenvectors in a regular expansion as a power series in ε_k, regarding ε_1 as a constant or fixed parameter in the range of interest. Thus, we write $\mathbf{A}$, λ, and $\boldsymbol{\phi}$ in powers of ε_k as

$$\mathbf{A} = \bar{\mathbf{A}}_0 + \bar{\mathbf{A}}_1\,\varepsilon_k + \bar{\mathbf{A}}_2\,\varepsilon_k^2 + O(\varepsilon_k^3), \qquad \bar{\mathbf{A}}_j = 0, \qquad \forall j \geqslant 2, \tag{7}$$

$$\lambda_i = \lambda_{0i} + \lambda_{1i}\,\varepsilon_k + \lambda_{2i}\,\varepsilon_k^2 + O(\varepsilon_k^3), \tag{8}$$

and

$$\boldsymbol{\phi}_i = \boldsymbol{\phi}_{0i} + \boldsymbol{\phi}_{1i}\,\varepsilon_k + \boldsymbol{\phi}_{2i}\,\varepsilon_k^2 + O(\varepsilon_k^3). \tag{9}$$

It is well known from the perturbation theory for linear operators or matrices [16, 20] that, for systems dependent on a single parameter, simple eigenvalues and the corresponding eigenvectors are a smooth function of the parameter. In the case of multiple eigenvalues, however, the eigenvalues and eigenvectors are only continuous and are not smoothly dependent on the parameter. Note that in equations (7)–(9), we have suppressed the dependence on the second parameter ε_1 and we will check for the uniformity with respect to ε_1 after the formal expansion is determined. Substituting equations (7)–(9) into the eigenvalue problem, equation (6), equating coefficients of each power of ε_k to zero, andsolving the resulting sequence of homogeneous and non-homogeneous linear equations, gives the following expressions for the eigenvalues and the corresponding eigenvectors:

$$\lambda_i = \lambda_{0i} + \left(\frac{\boldsymbol{\phi}_{0i}^T \bar{\mathbf{A}}_1 \boldsymbol{\phi}_{0i}}{\boldsymbol{\phi}_{0i}^T \boldsymbol{\phi}_{0i}} \right) \varepsilon_k + \frac{\boldsymbol{\phi}_{0i}^T \bar{\mathbf{A}}_1}{\boldsymbol{\phi}_{0i}^T \boldsymbol{\phi}_{0i}} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\boldsymbol{\phi}_{0j}^T \bar{\mathbf{A}}_1 \boldsymbol{\phi}_{0i}}{(\lambda_{0i} - \lambda_{0j}) \boldsymbol{\phi}_{0j}^T \boldsymbol{\phi}_{0j}} \boldsymbol{\phi}_{0j} \right] \varepsilon_k^2 + O(\varepsilon_k^3), \tag{10}$$

$$\boldsymbol{\phi}_i = \boldsymbol{\phi}_{0i} + \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\boldsymbol{\phi}_{0i}^T \bar{\mathbf{A}}_1 \boldsymbol{\phi}_{0i}}{(\lambda_{0i} - \lambda_{0j}) \boldsymbol{\phi}_{0j}^T \boldsymbol{\phi}_{0j}} \right] \varepsilon_k + O(\varepsilon_k^2), \qquad i = 1, 2, \ldots, n. \tag{11}$$

The expansions in equations (10) and (11) are the regular or standard (outer) expansions [20] of the eigenvalue problem (6). Here, λ_{0i} and $\boldsymbol{\phi}_{0i}$ are, respectively, the eigenvalues and eigenvectors of the matrix $\bar{\mathbf{A}}_0$ which is only dependent on the parameter ε_1. The appearance of $(\lambda_{0i} - \lambda_{0j})$ terms in the denominators in equations (10) and (11) clearly implies that these expansions are valid only for simple eigenvalues, and the singularities in the eigenvalue and eigenvector expansions arise in the second order and first order terms, respectively, as the eigenvalues approach each other. As is well known, and is also shown in the next subsection, most of the eigenvalues of the cyclic system obtained in the limit of matrix $\bar{\mathbf{A}}_0$, as $\varepsilon_1 \to 0$, appear as double eigenvalues. Thus, when $\varepsilon_1 \to 0$, some $\lambda_{0i} \to \lambda_{0j}$. This implies that the regular perturbation technique fails and λ_i and $\boldsymbol{\phi}_i$ become unbounded, that is, the continuity of the eigenvalues and eigenvectors with respect to the perturbation ε_1 breaks down. Thus, in the neighborhood of $\varepsilon_1 = 0$ the expansions in equations (10) and (11) become non-uniform or singular for double eigenvalues, and non-analytic points in parameter ε_1 have been identified.

Note that since ε_1 and ε_k have been treated as two independent parameters, there is no control over the expansions in equations (10) and (11) in the limit process when $\varepsilon_1 \to 0$ and $\varepsilon_k \to 0$. By stretching the neighborhood of the singular parameter value $\varepsilon_1 = 0$, and taking into consideration the nature of the singularity, we can find an exact relation between the limit process for these two parameters. Then, ε_1 and ε_k become dependent in a well defined manner in the neighborhood of the singular parameter value, called the "inner region". The solutions of the eigenvalue problem in the inner region are called the "inner expansions". By asymptotically matching the "inner" and the "outer expansions" and combining them appropriately, one obtains the so called composite expansions which, for sufficiently small ε_k, are uniformly valid throughout the interval of interest in ε_1. It should be pointed out that the outer expansions in expressions (10) and (11) are valid for all fixed values of ε_1, $\varepsilon_1 \neq 0$, and as ε_1 approaches $\varepsilon_1 = 0$, their validity radius in ε_k shrinks faster to zero.

2.3. SINGULAR PERTURBATION METHOD (INNER EXPANSIONS)

In order to develop the inner expansions, we assume that the perturbation parameters ε_1 and ε_k are related (dependent) by a set of mathematical parameters ξ_1, ξ_2, $\xi_2 \cdots$ and μ through a "stretching" transformation of the form

$$\varepsilon_1 = \varepsilon_0 + \xi \mu^a + \sum_{j=2}^{\infty} \xi_j \, (\mu^a), \tag{12}$$

where μ is a new small parameter defined by

$$\varepsilon_k(\mu) = (\operatorname{sgn} \varepsilon_k)\mu^b, \tag{13}$$

and ε_0 is the singular point for ε_1. In the following development we have set ξ_j, $j \geqslant 2$ to zero. For fixed μ, the quantity ξ serves as the internal variable. The positive constants a and b are to be determined by the nature of the singularities of the characteristic equation for the eigenvalue problem (6) near $\varepsilon_1 = 0$, since $\varepsilon_1 = 0$ is the singular point of interest. The eigenfunctions $\boldsymbol{\phi}$ and the eigenvalues λ for the inner region can be written as the expansions

$$\boldsymbol{\phi}(\varepsilon_1, \varepsilon_k) = \mathbf{Z}(\varepsilon_1(\mu), \varepsilon_k(\mu)) = \mathbf{Z}(\mu) = \sum_{j=0}^{\infty} \mathbf{Z}_j \mu^j, \tag{14}$$

$$\lambda(\varepsilon_1, \varepsilon_k) = \lambda(\varepsilon_1(\mu), \varepsilon_k(\mu)) = \Omega(\mu) = \sum_{j=0}^{\infty} \Omega_j \mu^j. \tag{15}$$

The expansions (14) and (15) are the "inner expansions" and the coefficients $\mathbf{Z}_j$ and Ω_j are called the "inner coefficients". Substituting equations (12) and (13) into any of the matrices $\mathbf{A}$, as defined in equation (3), (4) or (5), we get elements of the matrix $\mathbf{A}$ as functions of the parameter μ. By considering the characteristic equation for the matrix $\mathbf{A}$, and by utilizing dominance balance arguments, it can be shown that for each of the above perturbed cases, $a = b = 1$. With these values of a and b and for positive ε_k, the matrix $\mathbf{A}(\mu)$ as a power series in μ takes the form

$$\mathbf{A}(\mu) = \mathbf{A}_0 + \mathbf{A}_1 \mu + \mathbf{A}_2 \mu^2 + O(\mu^3), \tag{16}$$

where $\mathbf{A}_0$ is the matrix corresponding to the tuned state ($\varepsilon_1 = \varepsilon_k = 0$), $\mathbf{A}_1$ gives the perturbation matrix, and

$$\mathbf{A}_j = 0, \qquad \forall j > 1. \tag{17}$$

Substituting expressions (12)–(16) into the eigenvalue problem (6) and equating the μ^0 order terms yields

$$\mathbf{A}_0 \mathbf{Z}_0 = \Omega_0 \mathbf{Z}_0. \tag{18}$$

The eigenvalues and eigenvectors of the tuned system are then given by, when n is odd:

$$\Omega_{0i} = 1 + 2k_c \left[1 - \cos \frac{2\pi}{n}(i-1)\right], \qquad i = 1, \dots, \frac{n+1}{2},$$

$$\mathbf{Z}_{0i} = [1, \cos \alpha_i, \dots, \cos(n-1)\alpha_i]^T, \qquad i = 1, \dots, \frac{n+1}{2}, \tag{19}$$

$$\mathbf{Z}_{0n+2-i} = [0, \sin \alpha_i, \dots, \sin(n-1)\alpha_i]^T, \qquad i = 2, \dots, \frac{n+1}{2},$$

where $\alpha_i = (2\pi/n)\,(i-1)$, and when n is even:

$$\Omega_{0i} = 1 + 2k_c\left[1 - \cos\frac{2\pi}{n}\,(i-1)\right], \qquad i = 1, 2, \ldots, \frac{n}{2}+1,$$

$$\mathbf{Z}_{0i} = [1, \cos\alpha_i, \ldots, \cos(n-1)\alpha_i]^T, \qquad i = 1, \ldots, \frac{n}{2}+1, \tag{20}$$

$$\mathbf{Z}_{0n+2-i} = [0, \sin\alpha_i, \ldots, \sin(n-1)\alpha_i]^T, \qquad i = 1, \ldots, \frac{n}{2}+1,$$

where $\alpha_i = (2\pi/n)\,(i-1)$.

Note that the unperturbed matrix $\mathbf{A}_0$ is symmetric and real Hermitian, so that its eigenvalues are real, and even double eigenvalues have non-defective eigenvectors. When n is odd, there is one simple eigenvalue and the remaining $(n-1)$ eigenvalues appear as double eigenvalues. Each of the double eigenvalues Ω_{0i} has two linearly independent eigenvectors defined by $\mathbf{Z}_{0i}$ and $\mathbf{Z}_{0n+2-i}$, the so-called "cosine" and the "sine" modes. In fact, any linear combination of the "cosine" and "sine" modes is also an eigenvector. Similar behavior holds when n is even except that now the smallest and the largest eigenvalues are simple. Thus, in each case, there are n linearly independent eigenvectors and these eigenvectors span an n-dimensional vector space. In other words, the zeroth order eigenvectors can be regarded as a basis while the higher order eigenvectors can be expressed as linear combinations of the zeroth order eigenvectors. Since equation (18) has double (multiple) eigenvalues, a perturbation theory for multiple eigenvalues has to be developed. The crucial step in the theory is based on the observation that the eigensubspace corresponding to a multiple eigenvalue is complete except that the eigenvectors are non-unique. In order to obtain eigenvectors for the perturbed system which are continuous in the limits as $\mu\to 0$, the eigenvectors of the unperturbed problem (equation (18)) should be chosen appropriately, taking the perturbation matrix $\mathbf{A}_1$ into account. This isaccomplished by suitably rotating the initial orthonormal basis of the unperturbed or tuned system, given in equations (19) and (20), to obtain a basis which is the limit of eigenvectors for the perturbed system as $\mu\to 0$.

Such a perturbation theory for real, symmetric and positive definite matrices has already been developed in Courant and Hilbert [16]. Applying this perturbation theory to the eigenvalue problem for $\mathbf{A}(\mu)$, we obtain the following expressions for the eigenvalues and eigenvectors:

$$\Omega_i = \Omega_{0i} + d_{ii}\,\mu + \left[\sum_{\substack{m=\alpha+1\\ i\leqslant\alpha}}^{n} \frac{d_{im}^2}{(\Omega_{0i}-\Omega_{0m})}\right]\mu^2 + O(\mu^3) \qquad \text{(for doublets)}, \tag{21}$$

$$\Omega_i = \Omega_{0i} + (\mathbf{Z}_{0i}^T\mathbf{A}_1\,\mathbf{Z}_{0i})\mu + (\mathbf{Z}_{0i}^T\mathbf{A}_1\,\mathbf{Z}_{1i})\mu^2 + O(\mu^3) \qquad \text{(for simple eigenvalues)}, \tag{22}$$

$$\mathbf{Z}_i = \mathbf{Z}_{0i}^* + \left[\sum_{\substack{j=1\\ j\neq i}}^{\alpha}\left\{\frac{1}{\Omega_{1i}-\Omega_{1j}}\sum_{\substack{m=\alpha+1\\ i\leqslant\alpha}}^{n}\frac{d_{im}\,d_{mj}}{\Omega_{0i}-\Omega_{0m}}\right\}\mathbf{Z}_{0j}^* + \sum_{\substack{j>\alpha\\ j\neq\alpha}}^{n}\frac{d_{ij}}{\Omega_{0i}-\Omega_{0j}}Z_{0j}^*\right]\mu + O(\mu^2)$$

$$\text{(for doublets)}, \tag{23}$$

$$\mathbf{Z}_i = \mathbf{Z}_{0i} + \left(\sum_{\substack{m=1\\ m\neq i}}^{n}\frac{d_{im}}{\Omega_{0i}-\Omega_{0m}}\mathbf{Z}_{0m}^*\right)\mu + O(\mu^2) \qquad \text{(for simple eigenvalues)}, \tag{24}$$

where $d_{jm} = \mathbf{Z}_{0j}^{*T}\,\mathbf{A}_1\,\mathbf{Z}_{0m}^{*}$ and$\mathbf{Z}_{0i}^{*} = \sum_{m=1}^{\alpha} a_{im}\,\mathbf{Z}_{0m}$,$\alpha$ being the multiplicity of the degenerate eigenvalues. Here the coefficients a_{im} are determined by solving the equations

$$\sum_{m=1}^{\alpha} (D_{mj} - \Omega_{1i}\,\delta_m^j) a_{im} = 0,$$

where $D_{mj} = \mathbf{Z}_{0m}^{T}\mathbf{A}_1\,\mathbf{Z}_{0j}$. Note that for strongly coupled cyclic systems, the integer α equals 2. The above expansions in simplified and explicit form can be written for an odd number of blades system as

$$\mathbf{Z}_{0i}^{*} = a_{ii}\left[\mathbf{Z}_{0i} - \frac{(\xi + \cos(2\pi(i-1)k/n) - q_i)}{\sin(2\pi(i-1)k/n)}\,\mathbf{Z}_{0n+2-i}\right],$$

$$\mathbf{Z}_{0n+2-i}^{*} = a_{n+2-ii}\left[\mathbf{Z}_{0i} - \frac{(\xi + \cos(2\pi(i-1)k/n) + q_i)}{\sin(2\pi(i-1)k/n)}\,\mathbf{Z}_{0n+2-i}\right],$$

$$\Omega_{1i} = (1 + \xi + q_i)/n, \qquad \Omega_{1n+2-i} = (1 + \xi - q_i)/n,$$

where

$$a_{ii} = 1\bigg/\sqrt{1 + \left[\frac{\xi + \cos(2\pi(i-1)k/n) - q_i}{\sin[2\pi(i-1)k/n]}\right]^2},$$

$$a_{n+2-ii} = 1\bigg/\sqrt{1 + \left[\frac{\xi + \cos(2\pi(i-1)k/n) + q_i}{\sin[2\pi(i-1)k/n]}\right]^2},$$

$$q_i = \sqrt{(\xi+1)^2 - 4\xi\sin^2(2(i-1)\pi(k-1)/n)}, \qquad i = 2, 3, \ldots, (n+1)/2,$$

$$d_{jm} = \xi(\mathbf{Z}_0^{*})_{1j}\,(\mathbf{Z}_0^{*})_{1m} + (\mathbf{Z}_0^{*})_{kj}\,(\mathbf{Z}_0^{*})_{km}.$$

$(\mathbf{Z}_0^{*})_{kj}$ defines the kth row element of the jth eigenvector, $\xi = \varepsilon_1/\varepsilon_k$ and $\varepsilon_k = \mu$. Higher order coefficients can be calculated easily by using these coefficients.

The "inner expansions" for the eigenvalues and the eigenvectors, as defined by the expressions in equations (21)–(24), are valid for sufficiently small μ, that is, in theneighborhood of the singular point $\varepsilon_1 = 0$, and for small ε_k. The expansions, uniformly valid over the domain of the parameter ε_1, can be obtained by suitably matching the "inner expansions" with the "outer expansions" over an overlapping domain and by then constructing the appropriate "composite expansions". This is somewhat difficult to accomplish in the present case as the expressions for the ''outer expansions'' are not known explicitly. For small enough ε_1 and ε_k, however, we can use the inner expansions themselves as composite or complete expansions for the eigenvalues and eigenvectors. This is verified to be the case for reasonable values of parameters ε_1 and ε_k by comparing the exact (numerically determined) eigenvalues with those given by the expansions in equations (21)–(24).

3. FORCED RESPONSE ANALYSIS

The eigenvalues and eigenvectors derived in section 2 can now be used to study the forced response of a perturbed cyclic system. For this purpose, a harmonic forcing, which differs only in phase from blade to blade, is introduced on the n-degree-of-freedom strongly coupled system given in Figure 1. This excitation, called an engine-order excitation, is commonly used for cyclic systems in the literature [4, 11, 13, 14]. For the forced response analysis we introduce a small hysteretic damping in order to obtain bounded amplitudes of vibration for the blades at resonance. The forced response of the mistuned system is determined by using the modal expansion analysis and only the final result is presented in this section. This result is used to generate the numerical frequency response curves. However, in order to gain more insight into the effects of mistuning on the perturbed system, a first order approximated force response is obtained by combining the model expansion analysis with the perturbation technique. The procedure is summarized in Appendix A. This type of forced response analysis can also be found in Part II of the work of Wei and Pierre [13]. The equations of motion with hysteretic damping h are given by

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}(1 + jh)\mathbf{x} = \mathbf{Q}(t), \tag{25}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices, respectively. The forcing function $\mathbf{Q}(t)$ is defined as

$$\mathbf{Q}(t) = [(Q_m)_1, (Q_m)_2, \ldots, (Q_m)_i, \ldots, (Q_m)_n]^T. \tag{26}$$

For harmonic response, equations (25) can be solved easily by using modal expansion analysis. The forced response amplitudes of vibration are given by magnitudes of the steady state solution

$$\mathbf{x}(t) = \sum_{i=1}^{n} \frac{(\mathbf{Z}_i^T\mathbf{Q})\mathbf{Z}_i\, \mathrm{e}^{j\theta_i}\, \mathrm{e}^{j\omega t}}{\sqrt{(\Omega_i^2 - \omega^2)^2 + (\Omega_i^2 h)^2}}, \tag{27}$$

where

$$\theta_i = \tan^{-1}\left(\frac{-h}{1 - \omega^2/\Omega_i^2}\right), \qquad (Q_m)_i = Q_m\, \mathrm{e}^{j\beta_i},$$

and where $(Q_m)_i$ is the excitation on the ith blade due to the mth engine-order force, Q_m is the amplitude of the force, β_i is the inter blade phase angle $= 2\pi m(i - 1)/n$, h is the hysteretic damping coefficient, n is the number of blades, and $j = \sqrt{-1}$.

Note that the integer m signifies the sequence in which the different blades are excited as the harmonic excitation travels around the structure. The first order approximated forced response results, combining the perturbation method with modal expansion analysis, can be written as follows (see Appendix A for details):

$$\mathbf{x}(t) = \sum_{i=1}^{n} \frac{(\mathbf{Z}_{0i}^T\mathbf{Q})\mathbf{Z}_{0i}\, \mathrm{e}^{j(\omega t + \theta_{0i})}}{\sqrt{(\Omega_{0i}^2 - \omega^2)^2 + (h\Omega_{0i})^2}} + \varepsilon_k \sum_{i=1}^{n} \left\{ \frac{\left[\mathbf{Z}_{1i}^T\mathbf{Q} - \dfrac{2(1+jh)\Omega_{0i}\,\Omega_{1j}\,(\mathbf{Z}_{0i}^T\mathbf{Q})\,\mathrm{e}^{j\theta_{0i}}}{\sqrt{(\Omega_{0i}^2 - \omega^2)^2 + (h\Omega_{0i}^2)^2}} \right]}{\sqrt{(\Omega_{0i}^2 - \omega^2)^2 + (h\Omega_{0i}^2)^2}} \mathbf{Z}_{0i} + \frac{(\mathbf{Z}_{0i}^T\mathbf{Q})}{\sqrt{(\Omega_{0i}^2 - \omega^2)^2 + (h\Omega_{0i}^2)^2}} \right\} \mathrm{e}^{j(\omega t + \theta_{0i})}, \quad (28)$$

where $\theta_{0i} = \tan^{-1}(-h/(1-\omega^2/\Omega_{0i}^2))$. $\mathbf{Z}_{0i}$, $\mathbf{Z}_{1i}$, Ω_{0i} and Ω_{1i} are already defined in section 2.3. Note, $\mathbf{Z}_{0i}$ has been used here as the tuned system eigenfunction instead of the rotated eigenfunction $\mathbf{Z}_{0i}^*$, since, it can easily be shown that the transformation matrix associated with the zeroth order eigenfunctions of the mistuned system $\mathbf{Z}_{0i}^*$, is a pure rotational matrix. One method of verifying this fact is to show that the determinant of the transformation matrix simplifies to one. The first term in equation (28) is the response of the tuned system, and the remaining terms represent the effects of mistuning. The amplitude of response of the ith blade, to the first order, can be written using equation (28) as

$$|x_i| = \left| \sum_{l=1}^{n} \frac{(\mathbf{Z}_{0i}^T\mathbf{Q})\,(\mathbf{Z}_{0i})_l\, \mathrm{e}^{j\theta_{0l}}}{\sqrt{(\Omega_{0l}^2 - \omega^2)^2 + (h\Omega_{0l})^2}} + \varepsilon_k \sum_{l=1}^{n} \left\{ \frac{\left[\mathbf{Z}_{1l}^T\mathbf{Q} - \dfrac{2(1+jh)\Omega_{0l}\,\Omega_{1l}\,(\mathbf{Z}_{0l}^T\mathbf{Q})\,\mathrm{e}^{j\theta_{0l}}}{\sqrt{(\Omega_{0l}^2 - \omega^2)^2 + (h\Omega_{0l}^2)^2}} \right]}{\sqrt{(\Omega_{0l}^2 - \omega^2)^2 + (h\Omega_{0l}^2)^2}} (\mathbf{Z}_{0i})_l + \frac{(\mathbf{Z}_{0l}^T\mathbf{Q})}{\sqrt{(\Omega_{0l}^2 - \omega^2)^2 + (h\Omega_{0l}^2)^2}} \right\} \mathrm{e}^{j\theta_{0l}} \right|, \qquad i = 1, 2, \ldots, n, \quad (29)$$

where $(\mathbf{Z}_{0i})_l = i$th row element in the lth eigenvector. It is clear from equation (29) that the amplitude of the ith blade $|x_i|$ depends upon the damping h, the forcing $\mathbf{Q}$, the excitation frequency ω and the mistuning perturbations. The maximum amplitude of the ith blade occurs at resonant frequencies. These expressions allow us to explicitly study the effects of mistuning parameters on the maximum amplitudes of response. These results are discussed in the next section.

4. RESULTS AND DISCUSSION

In this section, the inner expansions for the eigenvalues and eigenvectors, as developed in section 2.3, are applied to a five-degree-of-freedom model of a cyclic system with perturbations (ε_1, ε_k) in two ground springs. Here, the subscripts 1 and k appearing on the

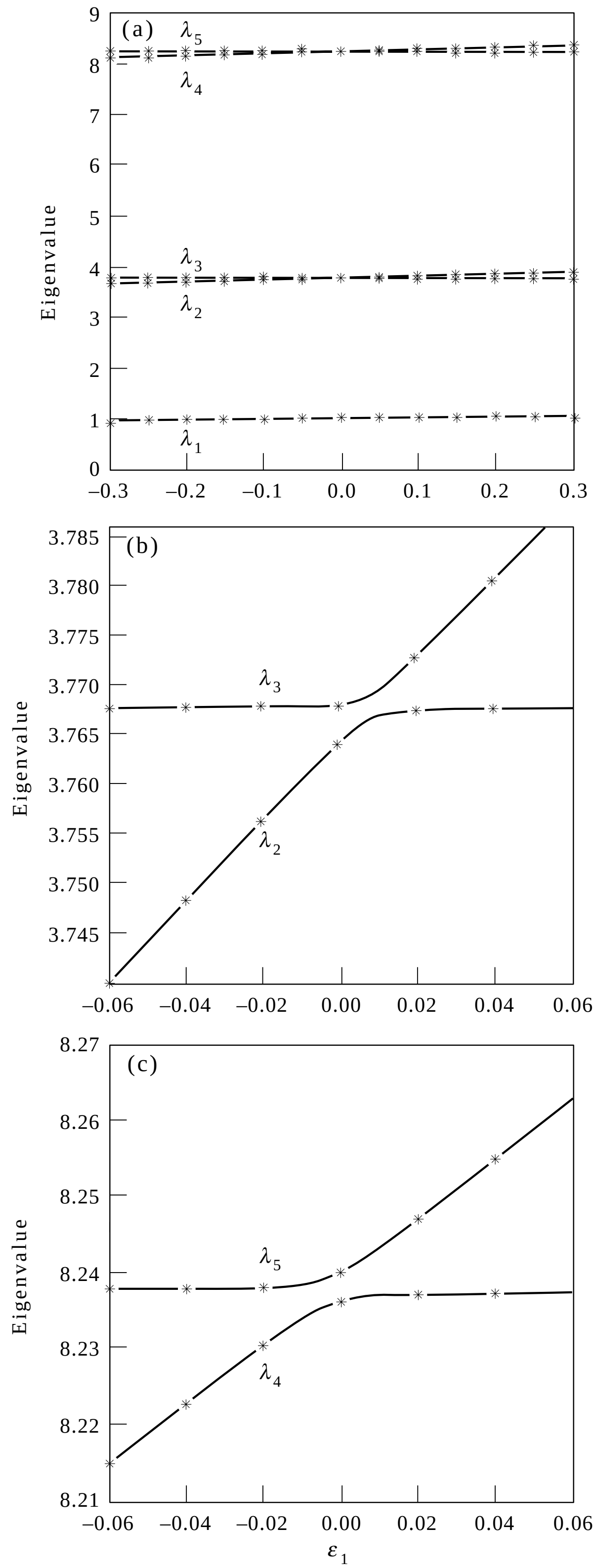

Figure 2. Comparison of the exact eigenvalues ($*$) with those obtained from the singular perturbation method (——) for strongly coupled five blade system; $\varepsilon_2=0{\cdot}01$, $k_c=2$. (a) All eigenvalues; (b) eigenvalues λ_2 and λ_3; (c) eigenvalues λ_4 and λ_5.

ε's denote the blade number to which the perturbations are applied. For all the results shown, we have used the index k to be 2, that is, two adjacent ground springs were perturbed to mistune the system. Preliminary calculations with cases where other combination of perturbations are induced, that is, $k = 3$, or the coupling springs are perturbed, do not seem to give any qualitatively different results. However, this aspect of where the two perturbations arise needs to be more thoroughly explored before any complete answers can be given.

For the numerical results, two sets of coupling spring constant values, $k_c = 2$ and $k_c = 0{\cdot}01$, have been considered. These k_c values are representative of strongly and weaklycoupled systems respectively. The details of explicit computations for the five-blade system are given in Appendix B. The eigenvalues of the five-degree-of-freedom mistuned system, as predicted by the inner expansions, are shown in Figures 2 and 3 as a function of the mistuning ε_1. The corresponding exact solutions of the eigenvalue problem (denoted by stars) have also been plotted for comparison, and the two results are clearly in excellent agreement. It should be noted, however, that the analysis in section 2.3 assumed the eigenvalues to be isolated doublets which are well separated, that is, the tuned system is strongly coupled. Thus, the generated eigenvalues and eigenfunctions are not valid for weakly coupled cyclic systems where all the eigenvalues are clustered, or the cyclic system is nth order degenerate. Consequently, the forced response analysis for weakly coupled cyclic systems cannot be performed by utilizing these eigenfunctions.

The power of the singular perturbation technique in obtaining uniformly valid solutions for the eigenvalues can be put in a better perspective if Figures 2 and 3 are compared with the solutions obtained from the traditional perturbation method (the outer expansions),

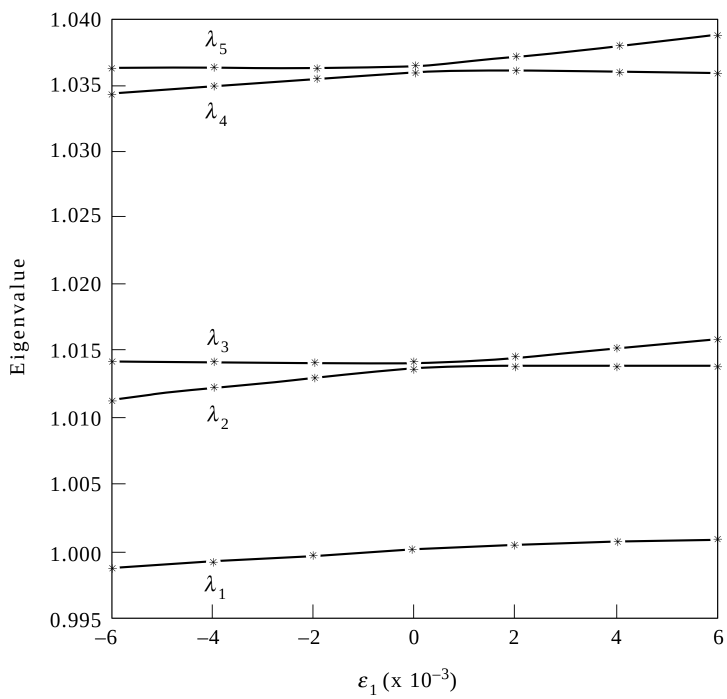

Figure 3. Comparison of the exact eigenvalues ($*$) with those obtained from the singular perturbation method (——) for a weakly coupled five blade system; $\varepsilon_2 = 0{\cdot}001$, $k_c = 0{\cdot}01$.

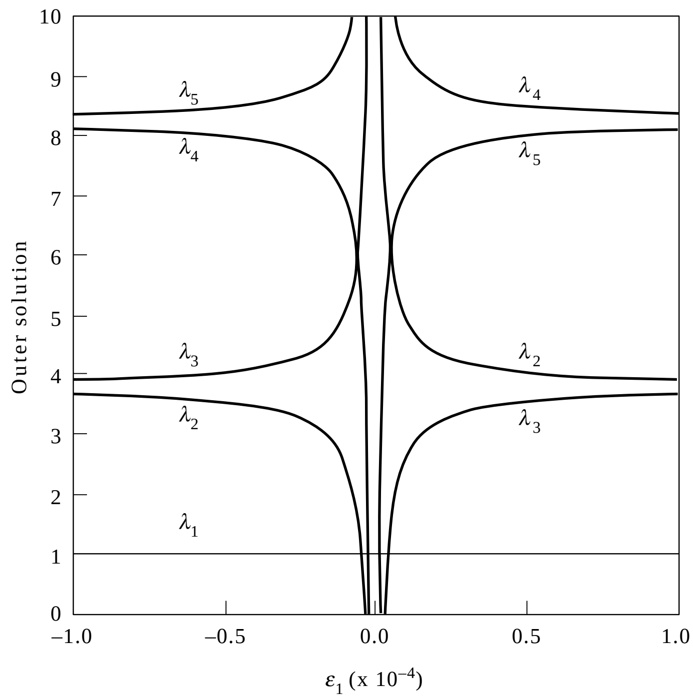

Figure 4. Eigenvalues of the mistuned cyclic system as predicted by the outer expansions; $\varepsilon_2 = 0{\cdot}01$, $k_c = 2$.

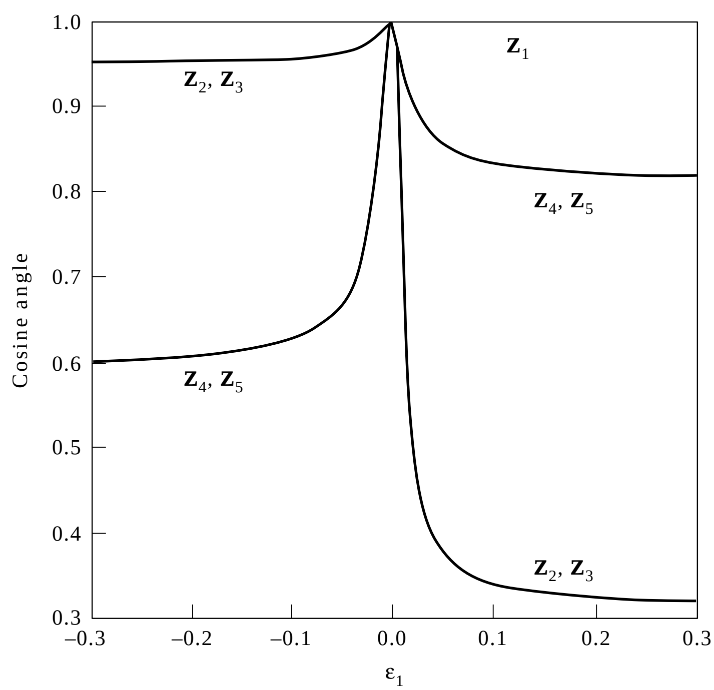

Figure 5. Eigenvector rotations corresponding to the three nominal eigenvalues for the mistuned system; $\varepsilon_2 = 0{\cdot}01$, $k_c = 2$.

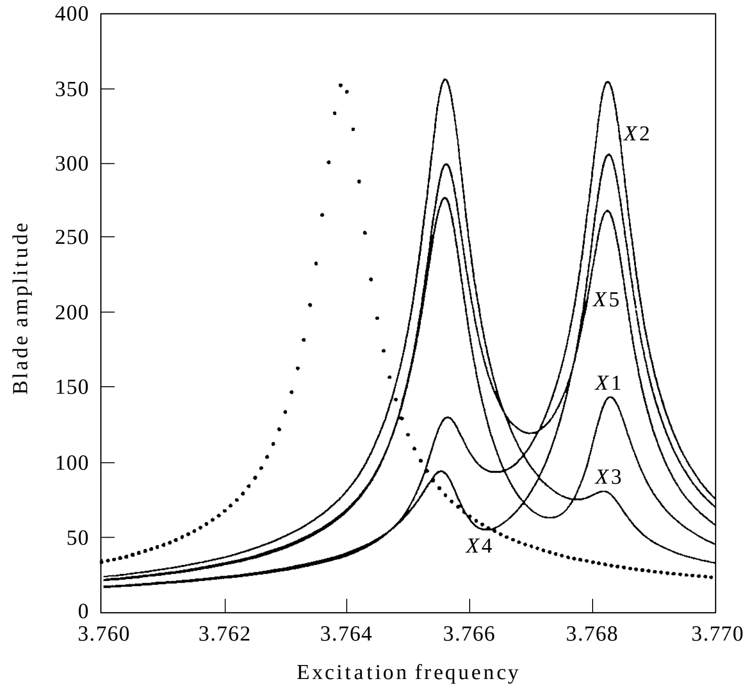

Figure 6. Blade amplitudes versus the excitation frequency for tuned (· · · ·) and mistuned five blade system; $k_c = 2$, $\varepsilon_1 = 0{\cdot}005$, $\varepsilon_2 = 0{\cdot}01$, $h = 0{\cdot}0002$, engine-order excitation $m = 1$ or 4.

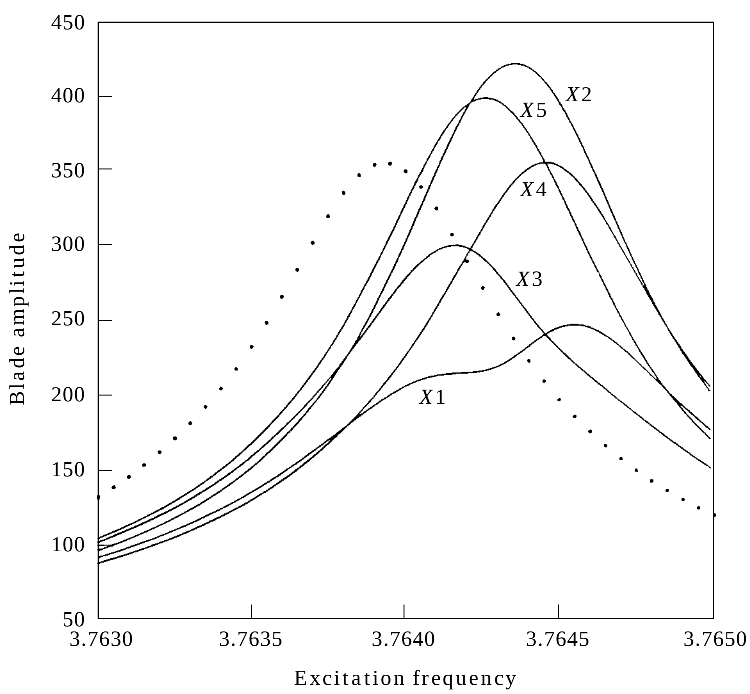

Figure 7. Blade amplitudes versus the excitation frequency for tuned (· · · ·) and mistuned five blade system; $k_c = 2$, $\varepsilon_1 = 0{\cdot}001$, $\varepsilon_2 = 0{\cdot}001$, $h = 0{\cdot}0002$, engine-order excitation $m = 1$ or 4.

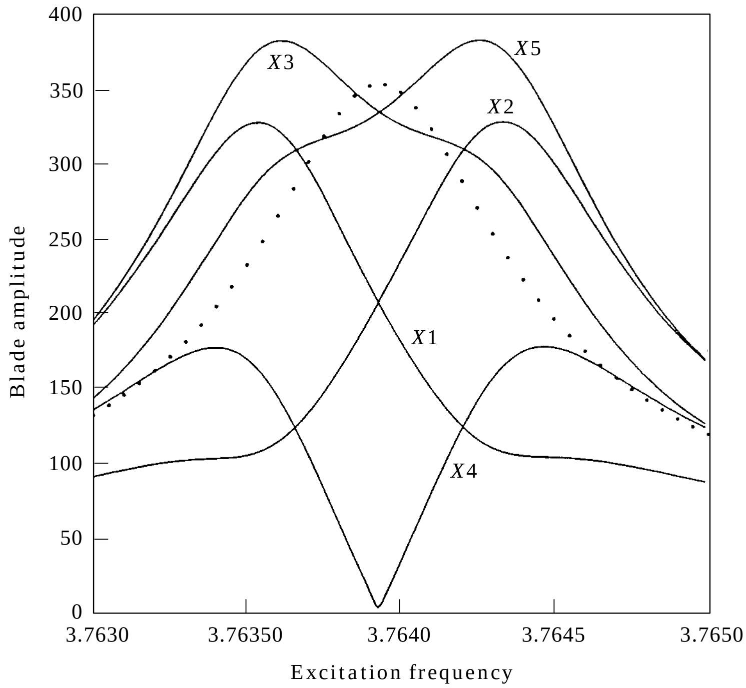

Figure 8. Blade amplitudes versus the excitation frequency for tuned (· · · ·) and mistuned five blade system; $k_c = 2$, $\varepsilon_1 = -0{\cdot}001$, $\varepsilon_2 = 0{\cdot}001$, $h = 0{\cdot}0002$, engine-order excitation $m = 1$ or 4.

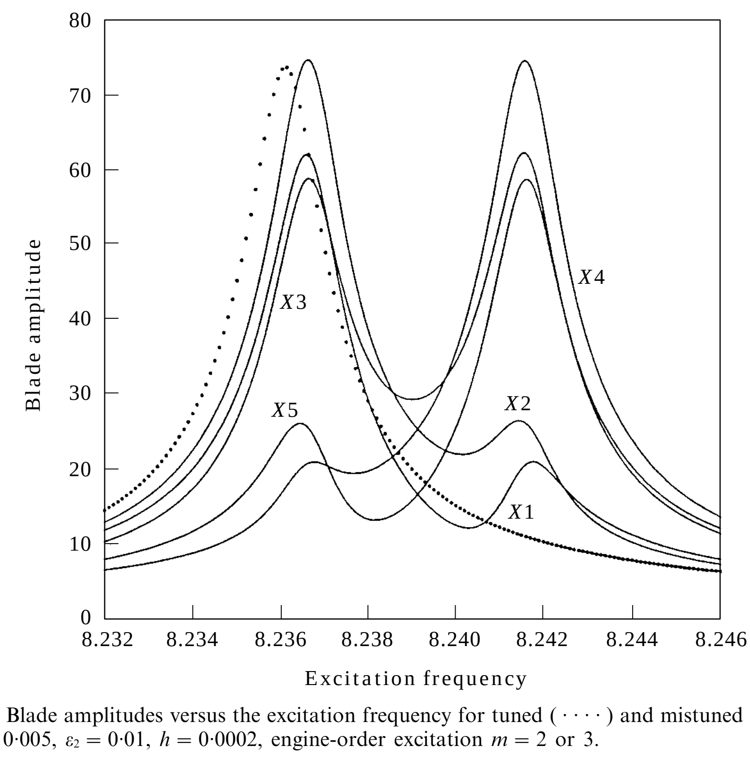

Figure 9. Blade amplitudes versus the excitation frequency for tuned (· · · ·) and mistuned five blade system; $k_c = 2$, $\varepsilon_1 = 0{\cdot}005$, $\varepsilon_2 = 0{\cdot}01$, $h = 0{\cdot}0002$, engine-order excitation $m = 2$ or 3.

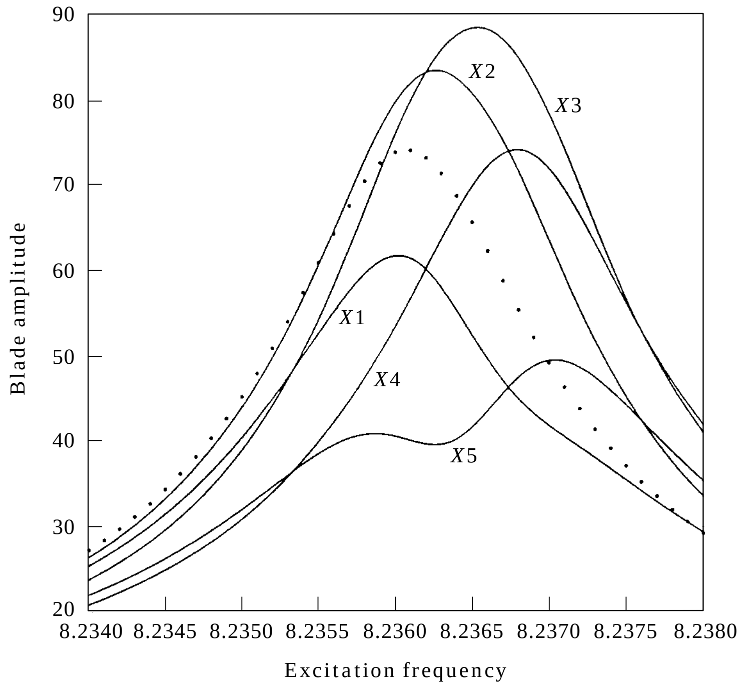

Figure 10. Blade amplitudes versus the excitation frequency for tuned (· · · ·) and mistuned five blade system; $k_c = 2$, $\varepsilon_1 = 0{\cdot}001$, $\varepsilon_2 = 0{\cdot}001$, $h = 0{\cdot}0002$, engine-order excitation $m = 2$ or 3.

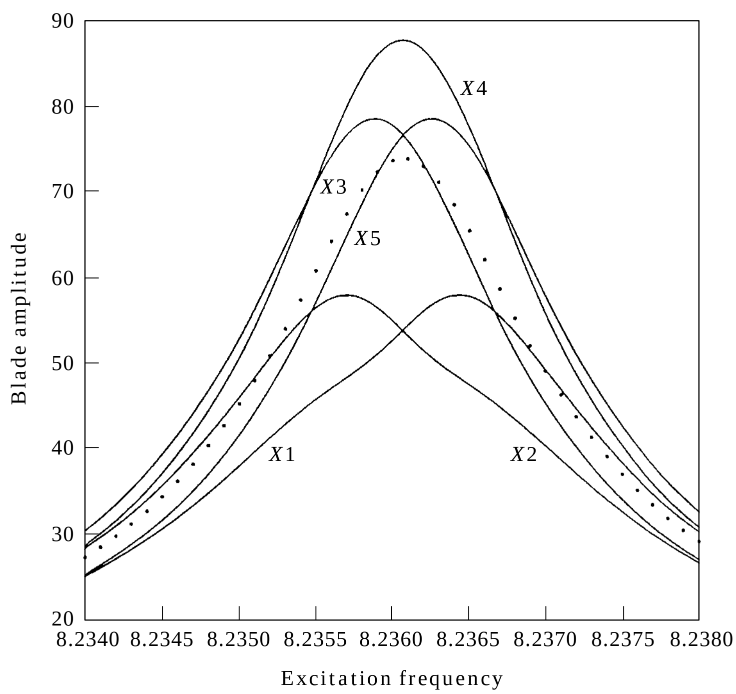

Figure 11. Blade amplitudes versus the excitation frequency for tuned (· · · ·) and mistuned five blade system; $k_c = 2$, $\varepsilon_1 = -0{\cdot}001$, $\varepsilon_2 = 0{\cdot}001$, $h = 0{\cdot}0002$, engine-order excitation $m = 2$ or 3.

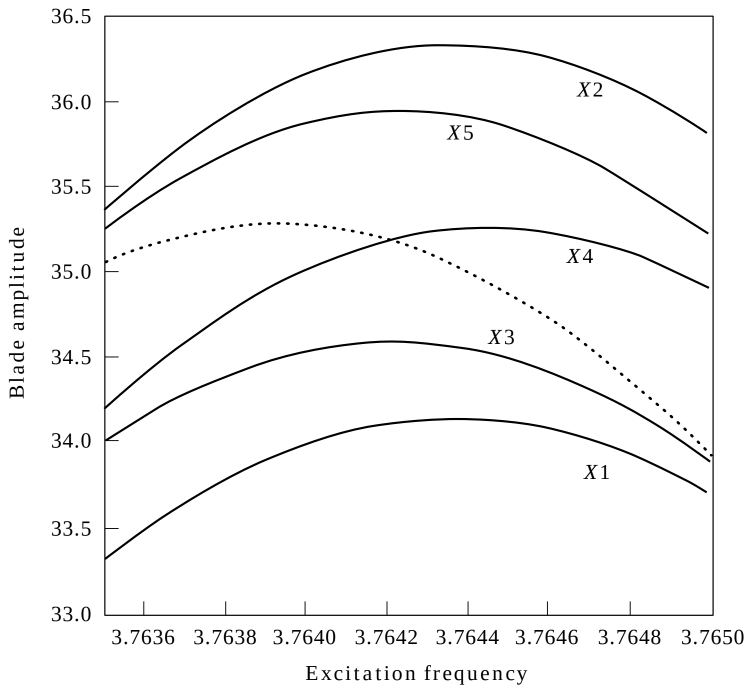

Figure 12. Blade amplitudes versus the excitation frequency for tuned (· · · ·) and mistuned five blade system; $k_c = 2$, $\varepsilon_1 = -0{\cdot}001$, $\varepsilon_2 = 0{\cdot}001$, $h = 0{\cdot}0002$, engine-order excitation $m = 1$ or 4.

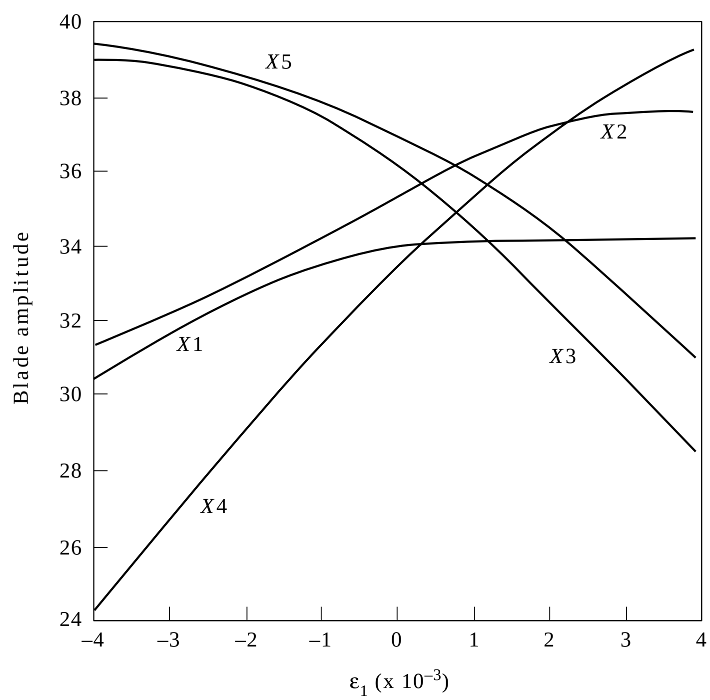

Figure 13. Maximum blade amplitudes versus ε_1 with the excitation frequency at resonance; $k_c = 2$, $\varepsilon_2 = 0{\cdot}002$, $h = 0{\cdot}002$, excitation frequency $= \Omega_3$, engine-order excitation $m = 1$ or 4.

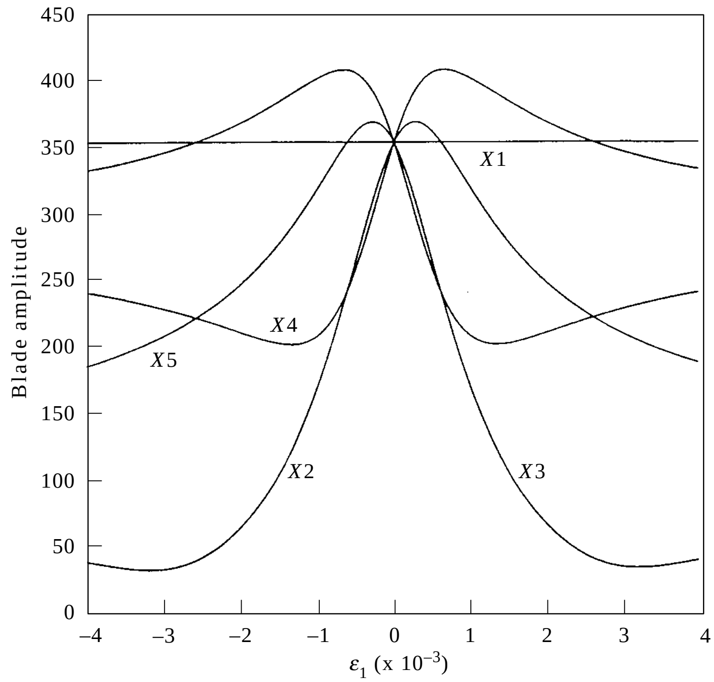

Figure 14. Maximum blade amplitudes versus ε_1 with the excitation frequency at resonance; $k_c = 2$, $\varepsilon_2 = 0$, $h = 0{\cdot}0002$, excitation frequency $= \Omega_3$, engine-order excitation $m = 1$ or 4.

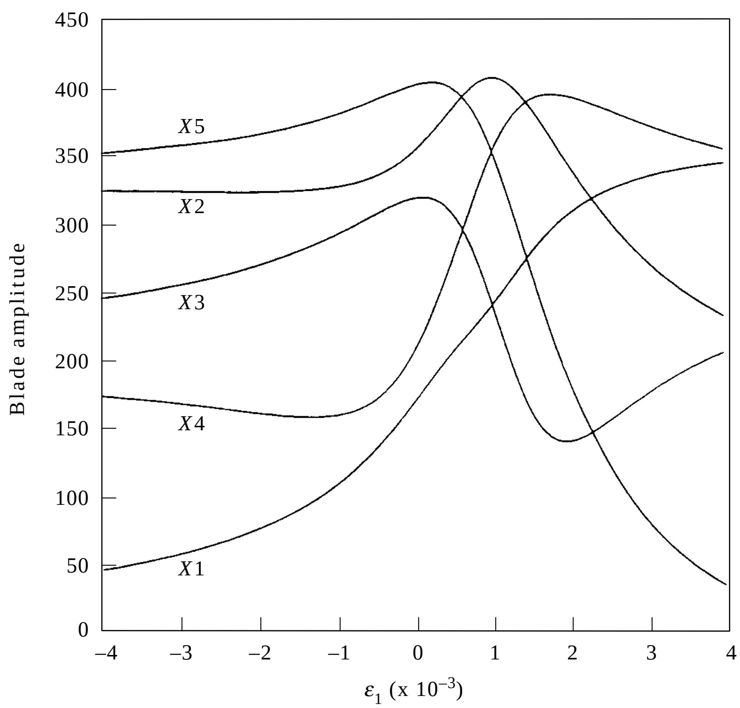

Figure 15. Maximum blade amplitudes versus ε_1 with the excitation frequency at resonance; $k_c = 2$, $\varepsilon_2 = 0{\cdot}001$, $h = 0{\cdot}0002$, excitation frequency $= \Omega_3$, engine-order excitation $m = 1$ or 4.

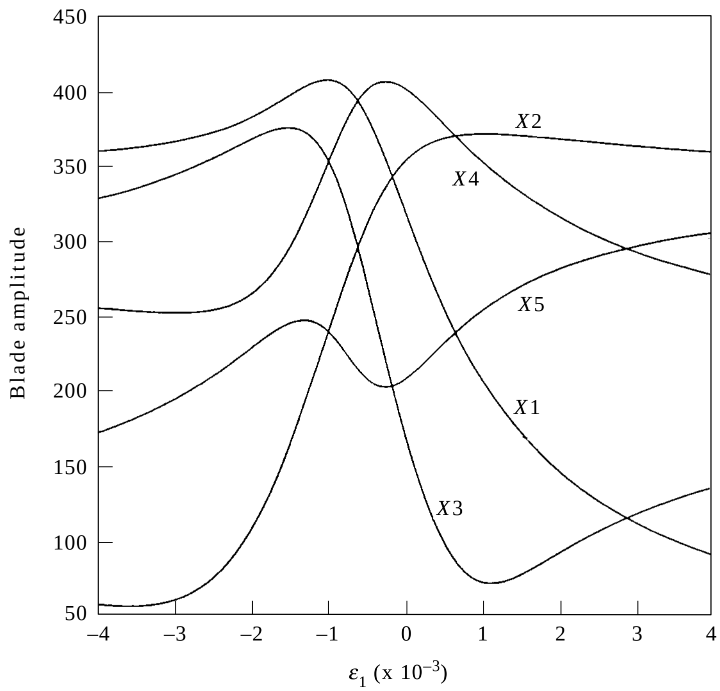

Figure 16. Maximum blade amplitudes versus ε_1 with the excitation frequency at resonance; $k_c = 2$, $\varepsilon_2 = -0{\cdot}001$, $h = 0{\cdot}0002$, excitation frequency $= \Omega_3$, engine-order excitation $m = 1$ or 4.

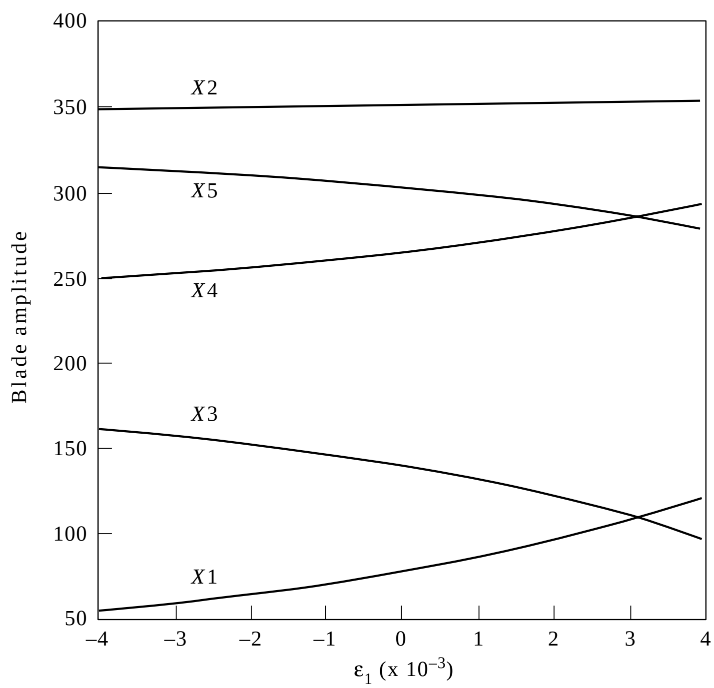

Figure 17. Maximum blade amplitudes versus ε_1 with the excitation frequency at resonance; $k_c = 2$, $\varepsilon_2 = 0{\cdot}01$, $h = 0{\cdot}0002$, excitation frequency $= \Omega_3$, engine-order excitation $m = 1$ or 4.

as shown in Figure 4. These expansions break down in the neighborhood of $\varepsilon_1 = 0$. Figures 2 and 3 also confirm the already known results [13] that eigenvalue curve veering occurs both for the strongly coupled and for the weakly coupled cyclic systems.

The eigenvalue curve veering in the presence of mistuning also leads to a rapid but continuous change in the eigenfunctions in the veering region, as noted by Perkins and Mode [23], and also shown by Happawana *et al*. [10]. In reference [10], the authors need the modal sensitivity function as a measure of these changes. In the present study, also of a system with five elements, it is found that the eigenvectors corresponding to double eigenvalues for the tuned system undergo a rapid change across the singular point ($\varepsilon_1 = \varepsilon_2 = 0$). Figure 5 shows the plots of rotation angles between the perturbed and the nominal eigenvectors for $k_c = 2{\cdot}0$ and $\varepsilon_2 = 0{\cdot}01$. Rapid changes in the eigenvectors take place only for the doublets being perturbed. Here the rotation angle between the eigenvector for the nominal system ($\varepsilon_1 = 0$), $\boldsymbol{\phi}_{0i}$, and the one for the perturbed system $\boldsymbol{\phi}_i$ is defined by $\cos\theta = \langle \boldsymbol{\phi}_{0i}, \boldsymbol{\phi}_i \rangle / \|\boldsymbol{\phi}_{0i}\| \, \|\boldsymbol{\phi}_i\|$.

Frequency response curves for the five-degree-of-freedom strongly coupled cyclic system ($k_c = 2$), are plotted for various engine-order excitations, and are given in Figures 6–12. The three distinct natural frequencies for the tuned cyclic system for the chosen value of k_c are 1·0, 3·7639 and 8·2361, respectively. These plots clearly exhibit the significance of different values of perturbation parameters as well as the damping constant. In these figures, the perturbation values relative to the damping constant as well as the order of excitation have been varied. In Figures 13–17 are plotted the maximum blade amplitudes as a function of the perturbation to show their variation near resonant frequencies.

One observes from Figures 6 and 9 that, for $\varepsilon_1 = 0{\cdot}005$ and $\varepsilon_2 = 0{\cdot}01$, there is no significant change in the maximum amplitudes of vibration for the mistuned system when compared to the response for the tuned systems. Note that the damping constant used here is quite small ($h = 0{\cdot}0002$). The two plots show that different engine-order excitations not only influence resonant motions near different coincident frequencies because of the spatial structure of the excitation, but also the amplitudes of the peaks significantly. Furthermore, the engine-order excitation has a very significant effect on both the tuned and the mistuned strongly coupled systems. Response amplitude plots (Figures 7 and 10) for $\varepsilon_1 = 0{\cdot}001$ and $\varepsilon_2 = 0{\cdot}001$ do, however, show a significant change in some mistuned blades amplitudes relative to the tuned system. This simply exhibits and underscores the fact that under appropriate conditions, even for strongly coupled cyclic systems, symmetry-breaking perturbations in the eigenvalue veering region can lead to high amplitudes of vibration. Such an anomalous behavior is due to an instability inherent in the cyclicity of the tuned system not encountered in strongly coupled linear periodic systems.

It is evident from Figures 7 and 8 as well as from Figures 10 and 11, that for only a small change in ε_1 from 0·001 to $-0{\cdot}001$ across the singular point $\varepsilon_1 = 0$, a considerable change in amplitude of vibration of some blades has occurred. For instance, the vibration amplitude of the second blade, X_2, in Figure 7 compared to that in Figure 8 has increased about 30%. That is to say, a very small parametric change in the tuned system could result in a considerable difference in the vibration response at the individual nodes. However, it can be observed from Figures 12 and 13 that sufficiently large damping (ten times larger than used in the other figures) present in the system not only removes the small perturbation effects, but also significantly lowers the peaks in the forced vibratory response.

Figures 13–17 show the effects of perturbations ε_1 and ε_2 on the "maximum" amplitudes of blades at resonant frequencies. It is observed that the maximum amplitudes of blades depend on perturbations. The effect of single perturbation ε_1 on the maximum amplitudes of blades, displayed in Figure 14, shows that the amplitudes of some blades vary rapidly with small change in ε_1, especially around $\varepsilon_1 = 0$. However, it is seen from Figure 13 that a larger damping introduced into the system controls the small perturbation effects. Comparing Figure 15 or 16 with Figure 17, it can be observed that the maximum amplitudes of response vary much more rapidly in the vicinity of the singularity ($\varepsilon_1 = 0{\cdot}0$, $\varepsilon_2 = 0{\cdot}0$). Thus, closer to the singular point in the parameter space, the response is much more sensitive and hence difficult to predict when random perturbations are present in the system.

5. CONCLUSIONS

The method of matched asymptotic expansions has been shown to lead to qualitatively correct and asymptotically valid algebraic solutions for the eigenvalues, eigenvectors as well as the forced response amplitudes of mistuned cyclic systems. It is shown that inner expansions, which are very much easier to develop, can be used as reasonable approximations to the composite expansions for computing the forced response. This notonly reduces the work involved, but also reduces the associated computational cost. The approach, although applied here to a strongly coupled cyclic system, is valid for unfolding any double eigenvalue.

The analysis and numerical results in this work clearly show that, even without mode localization, the amplitudes of vibration of some blades in a strongly coupled but slightly perturbed system could be significantly more than would be predicted on the basis of a perfectly tuned system. This is a consequence of the splitting of double eigenvalues and rapid variation of eigenvectors for small parameter mistunings. The implications of these observations for vibration abatement in structural systems is a subject for future studies.

ACKNOWLEDGMENTS

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APPENDIX A: FIRST ORDER APPROXIMATED FORCED RESPONSE

The equations of motion for an n-degree-of-freedom system with hysteretic damping h and forcing $\mathbf{Q}(t)$ can be written as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}(1 + jh)\mathbf{x} = \mathbf{Q}(t). \tag{A1}$$

Equation (A1) can easily be solved using modal expansions. Let

$$\mathbf{x} = \sum_{i=1}^{n} \mathbf{Z}_i \, h_i(t), \tag{A2}$$

where $\mathbf{Z}_i$ and $h_i(t)$ are the eigenfunctions and modal participation factors, respectively. Substituting equation (A2) into equation (A1) and simplifying, we obtain

$$\ddot{h}_i(t) + \omega_{di}^2 h_i(t) = \mathbf{Z}_i^T \mathbf{Q}(t), \quad i = 1, 2, \ldots, n, \tag{A3}$$

where $\omega_{di}^2 = \Omega_i^2(1 + jh)$.

For small ε_k as defined in the paper, we can write Ω_i, $\mathbf{Z}_i$ and $h_i(t)$ in powers of ε_k as

$$\Omega_i^2 = \Omega_{0i}^2 + 2\Omega_{0i}\,\Omega_{1i}\,\varepsilon_k + O(\varepsilon_k^2), \tag{A4}$$

$$\mathbf{Z}_i = \mathbf{Z}_{0i} + \mathbf{Z}_{1i}\,\varepsilon_k + O(\varepsilon_k^2), \qquad h_i(t) = h_{0i}(t) + h_{i1}(t)\varepsilon_k + O(\varepsilon_k^2). \tag{A5, A6}$$

Substituting equations (A4)–(A6) into equations (A3) and grouping zeroth and first order terms of ε_k, we get

$$\ddot{h}_{0i}(t) + (1 + jh)\Omega_{0i}^2 h_{0i}(t) = \mathbf{Z}_{0i}^T \mathbf{Q}(t), \tag{A7}$$

$$\ddot{h}_{1i} + (1 + jh)\Omega_{0i}^2 h_{1i}(t) = \mathbf{Z}_{1i}^T\,\mathbf{Q}(t) - (1 + jh)\,(2\Omega_{0i}\,\Omega_{1i}\,h_{0i}(t)), \qquad i = 1, 2, \ldots, n. \tag{A8}$$

Solving equations (A7) and (A8), we get the first order approximation to the forced response,

$$\mathbf{x}(t) = \sum_{i=1}^{n} \frac{(\mathbf{Z}_{0i}^T\,\mathbf{Q}(t))\mathbf{Z}_{0i}\,\mathrm{e}^{j\omega t}}{\Omega_{0i}^2(1 + jh) - \omega^2}$$

$$+ \varepsilon_k \sum_{i=1}^{n} \left\{ \mathbf{Z}_{0i} \frac{\left[\mathbf{Z}_{1i}^T\,\mathbf{Q} - \dfrac{(1 + jh)2\Omega_{0j}\,\Omega_{1i}\,\mathbf{Z}_{0i}^T\,\mathbf{Q}}{\Omega_{0i}^2(1 + jh) - \omega^2} \right]}{\Omega_{0i}^2(1 + jh) - \omega^2}\,\mathrm{e}^{j\omega t} + \mathbf{Z}_{1i} \left[\frac{\mathbf{Z}_{0i}^T\,\mathbf{Q}}{\Omega_{0i}^2(1 + jh) - \omega^2} \right] \mathrm{e}^{j\omega t} \right\}. \tag{A9}$$

APPENDIX B: FIVE-DEGREE-OF-FREEDOM CYCLIC SYSTEM WITH TWO PERTURBED GROUND SPRINGS

The appropriate eigenvalue problem for the system is given by

$$\mathbf{A}(\varepsilon_1, \varepsilon_2)\boldsymbol{\phi}(\varepsilon_1, \varepsilon_2) = \lambda(\varepsilon_1, \varepsilon_2)\boldsymbol{\phi}(\varepsilon_1, \varepsilon_2), \tag{B1}$$

where

$$\mathbf{A}(\varepsilon_1, \varepsilon_2) = \begin{bmatrix} a + \varepsilon_1 & -k_c & 0 & 0 & -k_c \\ -k_c & a + \varepsilon_2 & -k_c & 0 & 0 \\ 0 & -k_c & a & -k_c & 0 \\ 0 & 0 & -k_c & a & -k_c \\ -k_c & 0 & 0 & -k_c & a \end{bmatrix}. \tag{B2}$$

Here, the perturbations are introduced in two adjacent ground springs. As noted in section 2, for strongly coupled cyclic systems, the inner expansions can be used as composite expansions for sufficiently small ε_1 and ε_2. The inner expansions for the eigenvalues and eigenvectors of this system can be obtained by solving the eigenvalue problem for matrix **A** defined in equation (B2).

In the inner region, the mistuning parameters ε_1 and ε_2 are related by

$$\varepsilon_1 = \xi\mu, \qquad \varepsilon_2 = (\operatorname{sgn} \varepsilon_2)\mu. \tag{B3, B4}$$

Substituting equations (B3) and (B4) into equation (B1), and evaluating the μ^0 term gives

$$\mathbf{A}_0 = \begin{bmatrix} a & -k_c & 0 & 0 & -k_c \\ -k_c & a & -k_c & 0 & 0 \\ 0 & -k_c & a & -k_c & 0 \\ 0 & 0 & -k_c & a & -k_c \\ -k_c & 0 & 0 & -k_c & a \end{bmatrix}, \tag{B5}$$

where $a = 1 + 2k_c$.

Subsequently, the μ^1 term gives

$$\mathbf{A}_1 = \begin{bmatrix} \xi & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{B6}$$

and the remaining matrices are all zero, that is,

$$\mathbf{A}_j = 0, \qquad \forall j \geqslant 2. \tag{B7}$$

Solving the eigenvalue problem for matrix $\mathbf{A}_0$, which corresponds to the tuned cyclic system, gives the eigenvalues and eigenvectors as

$$\Omega_{0i} = 1 + 2k_c\,[1 - \cos(2\pi(i-1)/5)], \qquad i = 1, 2, 3, \tag{B8}$$

$$\mathbf{Z}_{0i} = [1, \cos\alpha_i, \ldots, \cos 4\alpha_i]^T, \qquad i = 1, 2, 3, \tag{B9}$$

$$\mathbf{Z}_{0n+2-1} = [0, \sin\alpha_i, \ldots, \sin 4\alpha_i], \qquad i = 2, 3, \tag{B10}$$

where

$$\alpha_i = 2\pi(i-1)/5. \tag{B11}$$

Following the general theory presented in the paper, the coefficients Ω_{1j} and a_{ik} are found by solving

$$\sum_{k=1}^{\alpha_1} (D_{kj} - \Omega_{1j}\,\delta_k^j)a_{jk} = 0, \qquad \sum_{k=\alpha_1+1}^{\alpha_2} (D_{kj} - \Omega_{1j}\,\delta_k^j)a_{jk} = 0, \tag{B12, B13}$$

where $\alpha_1 = 2$ and $\alpha_2 = 4$, that is, by solving the characteristic equation

$$|D_{kj} - \Omega_{1j}\,\delta_k^j| = 0, \tag{B14}$$

for the eigenvalues Ω_{1j}, $j = 1, 2, 3, 4$, where $D_{ij} = \mathbf{Z}_{0i}^T \mathbf{A}_1 \mathbf{Z}_{0j}$. The matrix $\mathbf{A}_1$ and the vectors $\mathbf{Z}_{0i}$ are given in equations (B6), (B9) and (B10), respectively. The calculated expressions for the coefficients D_{ij} are

$$D_{11} = \tfrac{2}{5}(\xi + p\cos^2 4\pi/5), \qquad D_{21} = D_{12} = \tfrac{2}{5}p\cos 4\pi/5 \sin 4\pi/5, \qquad D_{22} = \tfrac{2}{5}p\sin^2 4\pi/5,$$

$$D_{33} = \tfrac{2}{5}(\xi + p\cos^2 2\pi/5), \qquad D_{34} = D_{43} = \tfrac{2}{5}p\sin 2\pi/5 \cos 2\pi/5, \qquad D_{44} = \tfrac{2}{5}p\sin^2 2\pi/5,$$

where $p = \operatorname{sgn} \varepsilon_2$.

Substituting the values of D_{kj} into equation (B14) and solving the characteristic equation gives

$$\Omega_{11} = (p + \xi + q_1)/5, \qquad \Omega_{12} = (p + \xi - q_1)/5, \qquad \Omega_{13} = (p + \xi + q_2)/5, \quad \text{(B15–B17)}$$

$$\Omega_{14} = (p + \xi - q_2)/5, \qquad \Omega_{15} = \mathbf{Z}_{05}^T \mathbf{A}_1 \mathbf{Z}_{05} = (\xi + p)/5, \quad \text{(B18, B19)}$$

where

$$q_1 = \sqrt{(\xi + p)^2 - 4p\xi \sin^2 4\pi/5}, \qquad q_2 = \sqrt{(\xi + p)^2 - 4p\xi \sin^2 2\pi/5}.$$

Substituting these expressions for the coefficients Ω_{1j} into equations (B12) and (B13), and applying the orthogonality condition, we find the coefficients a_{ik}. These a_{ik} are given by

$$a_{11} = 1/\sqrt{1 + [(\xi + p\cos 8\pi/5 - q_1)/p\sin 8\pi/5]^2}, \quad \text{(B20)}$$

$$a_{12} = -\frac{(\xi + p\cos 8\pi/5 - q_1)}{p\sin 8\pi/5}\, a_{11}, \quad \text{(B21)}$$

$$a_{21} = 1/\sqrt{1 + [(\xi + p\cos 8\pi/5 + q_1)/p\sin 8\pi/5]^2}, \quad \text{(B22)}$$

$$a_{22} = -\frac{(\xi + p\cos 8\pi/5 + q_1)}{p\sin 8\pi/5}\, a_{21}, \quad \text{(B23)}$$

$$a_{33} = 1/\sqrt{1 + ((\xi + p\cos 4\pi/5 - q_2)/p\sin 4\pi/5)^2}, \quad \text{(B24)}$$

$$a_{43} = 1/\sqrt{1 + [(\xi + p\cos 4\pi/5 + q_2)/p\sin 4\pi/5]^2}, \quad \text{(B25)}$$

$$a_{34} = -\frac{(\xi + p\cos 4\pi/5 - q_2)}{p\sin 4\pi/5}\, a_{33}, \qquad a_{44} = -\frac{(\xi + p\cos 4\pi/5 + q_2)}{p\sin 4\pi/5}\, a_{43}. \quad \text{(B26, B27)}$$

Having determined the coefficients a_{ik}, we can now find the rotated eigenvectors of the tuned system, $\mathbf{Z}_{0i}^*$, as follows:

$$\mathbf{Z}_{01}^* = a_{11}\left[\mathbf{Z}_{01} - \frac{(\xi + p\cos 8\pi/5 - q_1)}{p\sin 8\pi/5}\mathbf{Z}_{02}\right] = [AL11\ AL12\ AL13\ AL14\ AL15]^T, \quad \text{(B28)}$$

$$\mathbf{Z}_{02}^* = a_{21}\left[\mathbf{Z}_{01} - \frac{(\xi + p\cos 8\pi/5 + q_1)}{p\sin 8\pi/5}\mathbf{Z}_{02}\right] = [AL21\ AL22\ AL23\ AL24\ AL25]^T, \quad \text{(B29)}$$

$$\mathbf{Z}_{03}^{*} = a_{33}\left[\mathbf{Z}_{03} - \frac{(\xi + p\cos 4\pi/5 - q_2)}{p\sin 4\pi/5}\mathbf{Z}_{04}\right] = [AL31\ AL32\ AL33\ AL34\ AL35]^T, \quad \text{(B30)}$$

$$\mathbf{Z}_{04}^{*} = a_{43}\left[\mathbf{Z}_{03} - \frac{(\xi + p\cos 4\pi/5 + q_2)}{p\sin 4\pi/5}\mathbf{Z}_{04}\right] = [AL41\ AL42\ AL43\ AL44\ AL45]^T, \quad \text{(B31)}$$

$$\mathbf{Z}_{05}^{*} = \mathbf{Z}_{05} = [AL51\ AL52\ AL53\ AL54\ AL55]^T. \quad \text{(B32)}$$

The first-order correction to the eigenvectors, $\mathbf{Z}_{1i}$, are then given by

$$\mathbf{Z}_{11} = \frac{1}{\Omega_{11} - \Omega_{12}}\left[\frac{d_{13}\,d_{32}}{\Omega_{01} - \Omega_{03}} + \frac{d_{14}\,d_{42}}{\Omega_{01} - \Omega_{04}} + \frac{d_{15}\,d_{52}}{\Omega_{01} - \Omega_{05}}\right]\mathbf{Z}_{02}^{*}$$

$$+\left[\frac{d_{13}\,\mathbf{Z}_{03}^{*}}{\Omega_{01} - \Omega_{03}} + \frac{d_{14}\,\mathbf{Z}_{04}^{*}}{\Omega_{01} - \Omega_{04}} + \frac{d_{15}\,\mathbf{Z}_{05}^{*}}{\Omega_{01} - \Omega_{05}}\right] = [BT11\ BT12\ BT13\ BT14\ BT15]^T, \quad \text{(B33)}$$

and similar expressions:

$$\mathbf{Z}_{12} = [BT21\ BT22\ BT23\ BT24\ BT25]^T, \quad \text{(B34)}$$

$$\mathbf{Z}_{13} = [BT31\ BT32\ BT33\ BT34\ BT35]^T, \quad \text{(B35)}$$

$$\mathbf{Z}_{14} = [BT41\ BT42\ BT43\ BT44\ BT45]^T, \quad \text{(B36)}$$

$$\mathbf{Z}_{15} = [BT51\ BT52\ BT53\ BT54\ BT55]^T. \quad \text{(B37)}$$

The next terms in the eigenvalue expansions, Ω_{2i}, are given by using the second-order correction terms in equations (21) and (22). These are

$$\Omega_{21} = \frac{d_{13}^2}{\Omega_{01} - \Omega_{03}} + \frac{d_{14}^2}{\Omega_{01} - \Omega_{04}} + \frac{d_{15}^2}{\Omega_{01} - \Omega_{05}}, \quad \text{(B38)}$$

$$\Omega_{22} = \frac{d_{23}^2}{\Omega_{02} - \Omega_{03}} + \frac{d_{24}^2}{\Omega_{02} - \Omega_{04}} + \frac{d_{25}^2}{\Omega_{02} - \Omega_{05}}, \quad \text{(B39)}$$

$$\Omega_{23} = \frac{d_{31}^2}{\Omega_{03} - \Omega_{01}} + \frac{d_{32}^2}{\Omega_{03} - \Omega_{02}} + \frac{d_{35}^2}{\Omega_{03} - \Omega_{05}}, \quad \text{(B40)}$$

$$\Omega_{24} = \frac{d_{41}^2}{\Omega_{04} - \Omega_{01}} + \frac{d_{42}^2}{\Omega_{04} - \Omega_{02}} + \frac{d_{45}^2}{\Omega_{04} - \Omega_{05}}, \quad \text{(B41)}$$

$$\Omega_{25} = \mathbf{Z}_{05}^{*T}\,\mathbf{A}_1\,\mathbf{Z}_{15} = \xi(AL51)\,(BT51) + p(AL52)\,(BT52), \quad \text{(B42)}$$

where $d_{jm} = \mathbf{Z}_{0j}^{*T}\,\mathbf{A}_1\,\mathbf{Z}_{0m}^{*}$ are given by

$$d_{jm} = \xi(ALj1)\,(ALm1) + p(ALj2)\,(ALm2), \tag{B43}$$

and $p = \operatorname{sgn} \varepsilon_2$.

The forced response analysis is performed simply by using these inner expansions and the expressions developed in section 3.