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DYNAMICAL VISCOSITY IN POROUS MEDIA

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It is assumed that porous media consists of elastic cylindrical tubes filled with fluids, in which their radii have a certain distribution and the axes of the tubes can be oriented in various directions. The dynamical viscosity in such media is calculated. A comparison of the numerical results obtained by the author of this paper and by the Biot theory is given.

1. INTRODUCTION

A porous medium is composed of a solid frame and fluid-filled pores. Usually, it is difficult to model precisely, since it has a complex geometric structure. Several classical works have been published [1–3]. Using Kirchhoff's theory, Biot investigated sound propagation in these media and discovered that there are two dilatational waves and one rotational wave that propagate in isotropic porous media. In order to calculate their dynamical viscosity, he assumed that the pores were cylindrical, equal-radius tubes or same-width slits, and that the tubes or the slits were parallel to each other. If the actual conditions were more complicated, he further introduced some structural factors to take the complexity of the media into account. In the 1980s, the authors of some papers (see, e.g., reference [4]), also investigated dynamical permeability in porous media. A more detailed and comprehensive analysis of this can be found in references [5, 6].

In this paper, we begin with the modelling of media that consist of non-parallel cylindrical tubes, the radii of which have a certain distribution, and the dynamical viscosity is then calculated. The theoretical results show that the real and imaginary parts of the dynamical viscosity depend sensitively upon the size distribution of the pores.

2. SOUND PROPAGATION ALONG AN ARBITRARY DIRECTION

When a sound wave propagates along a direction that makes an angle α with the axis of the tube, one can decompose the vibration into a parallel component and a vertical component, and find the solutions corresponding to them, as was done in reference [7].

On the basis of the generalized Navier–Stokes equation, one has

$$\rho_f \frac{\partial}{\partial t} \vec{V} = -\nabla p + (\xi + \tfrac{1}{3}\eta)\nabla\nabla \cdot \vec{V} + \eta \nabla^2 \vec{V}, \tag{1}$$

where ξ and η are the bulk viscosity and the shear viscosity of the fluid respectively. Let

$$\vec{V} = \nabla\phi + \nabla \times \vec{\psi}. \tag{2}$$

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Substituting this into equation (1) yields

$$\rho_f \frac{\partial}{\partial t}(\nabla\phi) = -\nabla p + (\xi + \tfrac{4}{3}\eta)\nabla^2(\nabla\phi), \tag{3}$$

$$\rho_f \frac{\partial}{\partial t}(\nabla \times \vec{\psi}) = \eta\nabla^2(\nabla \times \vec{\psi}). \tag{4}$$

Let

$$\phi = \phi_1 + \phi_2, \tag{5}$$

where ϕ_1 and ϕ_2 are the acoustic component and viscous component of ϕ, respectively. Substituting equation (5) into equation (3) and writing

$$\rho_f \frac{\partial}{\partial t}(\nabla\phi_2) = (\xi + \tfrac{4}{3}\eta)\nabla^2(\nabla\phi_2), \tag{6}$$

one has

$$\rho_f \frac{\partial}{\partial t}(\nabla\phi_1) = -\nabla p - \frac{\mathrm{j}\phi_f\omega}{\beta^2}\nabla^2(\nabla\phi_1), \tag{7}$$

where

$$\beta = \mathrm{j}\sqrt{\mathrm{j}\rho_f\omega/(\xi + \tfrac{4}{3}\eta)} \tag{8}$$

is the complex wavenumber of the bulk viscous wave. Since $\nabla\phi_{\mathrm{j}}$ is an acoustic wave $\nabla^2(\nabla\phi_1) \sim k^2\nabla\phi_1$, where k is the wavenumber of the acoustic wave, it is easily seen that in a rather wide range of the frequency $k^2/|\beta|^2 \ll 1$, the second term on the right side of equation (7) is far less than the term on its left and, therefore, can be neglected. Thus,

$$\nabla\phi_1 = -\frac{1}{\mathrm{j}\rho_f\omega}\nabla p. \tag{9}$$

On the other hand, equations (4) and (6) can be rewritten as

$$\nabla^2(\nabla \times \vec{\psi}) + \beta_1^2\nabla \times \vec{\psi} = 0, \qquad \nabla^2(\nabla\phi_2) + \beta^2\nabla\phi_2 = 0, \tag{10, 11}$$

where

$$\beta_1 = \mathrm{j}\sqrt{\mathrm{j}\rho_f\omega/\eta}. \tag{12}$$

Thus, equations (10) and (11) can be regarded as the equations of the shear viscous wave and the bulk viscous waves respectively, the solutions of which were investigated by the authors of references [3, 7], except that here β and β_1 appear instead of the complex acoustic wavenumbers used in reference [7].

2.1. THE SOLUTION CORRESPONDING TO THE PARALLEL COMPONENT

In reference [3], Biot investigated the problem of sound vibration along the direction of the axis of the cylindrical tube, i.e., on the wall of the tube, the vibration can be described by $u_0 \cos\alpha\, \mathrm{e}^{\mathrm{j}\omega t}$. In this situation, the fluid in the tube will have an axis-symmetric vibration, which can be denoted by $u_z(r, z)$. In addition, there are only two kinds of waves in the fluid. The first is the sound wave, which propagates along the axial direction; and the second is the viscous wave, which propagates along the vertical direction of the axis. Thus, one has

$$\nabla\nabla \cdot \vec{V} = \vec{i}_r \frac{\partial^2 u_z}{\partial r\,\partial z} + \vec{i}_z \frac{\partial^2 u_z}{\partial z^2}, \qquad \nabla^2\vec{V} = \vec{i}_z\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right\}u_z.$$

Since

$$\partial u_z/\partial z \propto k u_z, \qquad \partial u_z/\partial r \propto \beta u_z,$$

and $k/\beta \ll 1$, the second term on the right side of equation (1) is far less than its third term, and equation (1) can be rewritten as

$$\rho_f \frac{\partial u_z}{\partial t} = -\frac{dp}{dz} + \eta\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{dr}\right\}u_z,$$

which is precisely the equation solved by Biot.

In this situation, the friction force of the fluid per unit length and the relative velocity with respect to the wall, which was averaged over the cross-section of the tube, should be written, respectively, as

$$\tau_{z0} = 2\pi\eta\left(u_0 + \frac{|\nabla p|}{\mathrm{j}\rho_f\omega}\right)\cos\alpha\beta_1 R\,\frac{\mathrm{J}_1(\beta_1 R)}{\mathrm{J}_0(\beta_1 R)}, \tag{13}$$

$$U_{z0} = -\left(u_0 + \frac{|\nabla p|}{\mathrm{j}\rho_f\omega}\right)\cos\alpha\left[1 - \frac{\mathrm{J}_1(\beta_1 R)}{\beta_1 R\mathrm{J}_0(\beta_1 R)}\right], \tag{14}$$

where ρ_f, η, R, ω, ∇p and $\mathrm{J}_n(x)$ ($n = 0, 1$) are the density, the shear viscosity of the fluid, the radius of the tubes, the angular frequency, the pressure gradient of the sound wave and Bessel functions, respectively.

2.2. THE SOLUTION CORRESPONDING TO THE VERTICAL COMPONENT

In reference [7], the authors investigated the solution in the situation that the vibration of the wall of the tube is in the direction vertical to its axis, the amplitude of which is $u_0 \sin\alpha$. However, their solutions are not expedient for this paper, since the acoustic component and the viscous component of the solution cannot be separated. In this section, a more expedient solution will be given.

Equations (10) and (11) can be regarded as the equations of the shear viscous wave and the volumetric viscous waves respectively. It is noted that there is no axi-symmetry, so that all of the velocities depend upon r and θ. Obviously, the solutions for $\nabla\phi_1$ can be obtained from equation (9), and ϕ_2 and Ψ can be obtained, as was done in reference [7], except that here we use the volume viscous wavenumber β instead of the complex acoustic wavenumber α. Thus, the components of the velocities in the r and the θ directions are

$$u_r = -\left(u_0 + \frac{|\nabla p|}{\mathrm{j}\rho_f\omega}\right)\sin\alpha\left\{1 + \left[\beta\mathrm{J}_0(\beta r) - \frac{1}{r}\,\mathrm{J}_1(\beta r)\right]B + \frac{1}{r}\,\mathrm{J}_1(\beta_1 r)C\right\}\cos\vartheta, \tag{15}$$

$$u_\vartheta = -\left(u_0 + \frac{|\nabla p|}{\mathrm{j}\rho_f\omega}\right)\sin\alpha\left\{1 - \frac{1}{r}\,\mathrm{J}_1(\beta r)B - \left[\beta_1\mathrm{J}_0(\beta_1 r) - \frac{1}{r}\,\mathrm{J}_1(\beta_1 r)\right]C\right\}\sin\vartheta, \tag{16}$$

respectively, where B and C are determined by the boundary conditions, which are

$$B = \frac{\beta_1 R\mathrm{J}_0(\beta_1 R) - 2\mathrm{J}_1(\beta_1 R)}{E}\,R, \qquad C = \frac{\beta R\mathrm{J}_0(\beta R) - 2\mathrm{J}_1(\beta R)}{E}\,R,$$

$$E = \{\beta\beta_1 R^2\mathrm{J}_0(\beta_1 R)\mathrm{J}_0(\beta R) - \beta R\mathrm{J}_0(\beta R)\mathrm{J}_1(\beta_1 R) - \beta_1 R\mathrm{J}_0(\beta_1 R)\mathrm{J}_1(\beta R)\}. \tag{17}$$

It should be noted that in reference [7] the last factor R on the right side of the first two expressions for B and C in equation (17) was omitted. Now, we calculate the projection of both u_r and u_θ to the vibration direction of the wall, which can be denoted as

$$u_x = u_r \cos\vartheta - u_\vartheta \sin\vartheta.$$

Averaging this on the section of the tube yields

$$U_x = \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} u_x r\,\mathrm{d}r\,\mathrm{d}\vartheta = U_{x0}\sin\alpha, \tag{18}$$

where

$$U_{x0} = \left(u_0 + \frac{1}{\mathrm{j}\rho_f\omega}|\nabla p|\right)\{1 - [\beta R\mathrm{J}_0(\beta R)\mathrm{J}_1(\beta_1 R) + \beta R\mathrm{J}_0(\beta_1 R)\mathrm{J}_1(\beta R)$$
$$-4\mathrm{J}_1(\beta R)\mathrm{J}_1(\beta_1 R)]/E\}. \tag{19}$$

As is already known, the viscous stresses are

$$\sigma_{rr} = -(\xi - \tfrac{2}{3}\eta)\nabla\cdot\vec{V} - 2\eta\frac{\partial u_r}{\partial r}, \qquad \sigma_{r\vartheta} = -\eta\left(\frac{1}{r}\frac{\partial u_r}{\partial\vartheta} + \frac{\partial u_\vartheta}{\partial r} - \frac{u_\vartheta}{r}\right),$$

the projection of which in the x direction is

$$\sigma_x = \sigma_{rr}\cos\vartheta - \sigma_{r\vartheta}\sin\vartheta.$$

Hence the frictional force on the tube per unit length is

$$\tau_x = \int_0^{2\pi}\sigma_x R\,\mathrm{d}\vartheta = \tau_{x0}\sin\alpha,$$

$$\tau_{x0} = \mathrm{j}\pi R\rho_f\omega\, R/E\left(u_0 + \frac{|\nabla p|}{\mathrm{j}\rho_f\omega}\right)\{\beta_1 R\mathrm{J}_2(\beta_1 R)\mathrm{J}_1(\beta R) + \beta R\mathrm{J}_2(\beta R)\mathrm{J}_1(\beta_1 R)\}. \tag{20}$$

2.3. THE COMPONENTS OF THE VELOCITY AND THE FRICTIONAL FORCE IN THE DIRECTION OF THE PRESSURE GRADIENT

From the previous results, one can calculate the components of both the flow velocity and the frictional force in the direction of the sound pressure gradient, which have been averaged over the section and on the surface of the wall, respectively. They are denoted by

$$U_n = U_{z0}\cos^2\alpha + U_{x0}\sin^2\alpha, \qquad \tau_n = \tau_{z0}\cos^2\alpha + \tau_{x0}\sin^2\alpha. \tag{21}$$

3. SIZE DISTRIBUTION OVER THE TUBES

Since the sizes of the tubes have a distribution, it is necessary to find an average value, but unfortunately, no suitable data are available to describe them. However, the author of reference [8] suggested that the diameters of the grains (in oceanic sediments) (in phi value) obey approximately a normal distribution, i.e.,

$$P(\phi) = \frac{1}{\sqrt{2\pi D}}\, e^{(\phi - \overline{\phi})^2/2D^2}, \tag{22}$$

where

$$\phi = -\log_2(d) \tag{23}$$

d is the diameter of a grain (in mm), $\overline{\phi}$ and D are the average of ϕ and the standard deviation respectively. It is easy to postulate that there may be a linear dependence of the diameters $2R$ of the pores in granular media upon the sizes of the grains d (for example, as was done in reference [9], $2R \approx (1/6 \text{ to } 1/7)d$). Thus, it is assumed here that the ϕ values of $2R$ also obey the distribution (22), so that all of the averages are denoted as

$$\hat{X} = \int_{-\infty}^{\infty} X(\phi)P(\phi)\, d\phi. \tag{24}$$

where X denotes one of the following quantities: U_{x0}, U_{z0}; τ_{x0} and τ_{z0}.

4. AVERAGE ON SPATIAL ORIENTATION OF THE TUBES

In general, the axes of the tubes are oriented in different directions in space, so that all the quantities have to be averaged over the solid angle Ω, which yield

$$\overline{U}_n = \int \hat{U}_n(\Omega)Q(\Omega)\, d\Omega, \qquad \bar{\tau}_n = \int \hat{\tau}_n(\Omega)Q(\Omega)\, d\Omega, \tag{25}$$

where $Q(\Omega)$ is the orientation distribution of the axes of the tubes. In this paper, a simple situation is considered, in which $Q(\Omega)$ is a uniform distribution. Hence

$$\overline{U}_n = \frac{1}{4\pi}\int_0^{\pi} \hat{U}_n 2\pi \sin\alpha\, d\alpha = \tfrac{2}{3}\hat{U}_x + \tfrac{1}{3}\hat{U}_z,$$

$$\hat{\tau}_n = \frac{1}{4\pi}\int_0^{\pi} \hat{\tau}_n 2\pi \sin\alpha\, d\alpha = \tfrac{2}{3}\hat{\tau}_{x0} + \tfrac{1}{3}\hat{\tau}_{z0}. \tag{26}$$

On the basis of the definition given in reference [3], the dynamical viscosity can be written as

$$F(\omega) = F_r(\omega) + jF_i(\omega) = \left[\frac{\overline{U}_n}{\bar{\tau}_n}\right]_{\omega=0} \frac{\bar{\tau}_n}{\overline{U}_n},$$

$$\left[\frac{\overline{U}_n}{\bar{\tau}_n}\right]_{\omega=0} = \frac{1}{24\pi\eta}\left\{1 + 16\,\frac{1 + E^2 + 294E}{1176E(1+E)}\right\},$$

$$E = (\xi + \tfrac{4}{3}\eta)/\eta.$$

Subsituting U_{x0}, U_{z0}, τ_{x0} and τ_{z0} into equation (24) to obtain $\hat{U}_{x0}$, $\hat{U}_{z0}$, $\hat{\tau}_{x0}$ and $\hat{\tau}_{z0}$, and then into equations (26) and (27), gives $F(\omega)$.

It can be noted that the results depend upon two viscosity coefficients, ξ and η. Unfortunately, there are few data available for the former coefficient. However, they can be evaluated for water. As is already known, the sound absorption due to viscosity can be denoted as [10]

$$\alpha_s = \frac{\omega^2 \zeta}{2\rho c^3}, \tag{28}$$

where $\zeta = \eta$ for an incompressible fluid and $\zeta = \xi + 4\eta/3$ for irrotational fluids. When $\xi = 0$, the latter situation corresponds to Stokes' classical theory. As is indicated by the data for water collected in reference [10],

$$\alpha_{ob}/\omega^2 = \text{constant},$$

where α_{ob} is the measured data of the sound absorption coefficients, and is almost three times the result of Stokes' absorption. On the other hand, the sound absorption can also be calculated from equation (1), and corresponds to $\zeta = \xi + 4\eta/3$ in equation (28); i.e.,

$$\alpha_{th} = \frac{\omega^2}{2\rho c^3}\left(\xi + \tfrac{4}{3}\eta\right), \tag{29}$$

where α_{th} is the theoretical value, in which two viscosity coefficients have been taken into account. In the frequency range far from the relaxation region (for example, lower than 10^8 Hz) [10], α_{th} is equal to α_{ob} approximately. Obviously,

$$\alpha_{th}/\alpha_s = \left(\xi + \tfrac{4}{3}\eta\right)/\zeta. \tag{30}$$

and

$$\zeta = \tfrac{4}{3}\eta, \qquad \alpha_{ob} \approx \alpha_{th}, \qquad \alpha_{ob}/\alpha_s \approx 3, \tag{31}$$

so that ξ/η is nearly 8/3. Similarly, ξ/η in air can also be obtained. However, since sound absorption depends upon the relative humidity, and particularly upon molecular relaxation effects, it is not proportional to ω^2 in general, unless the frequency is much lower than the relaxation frequency.

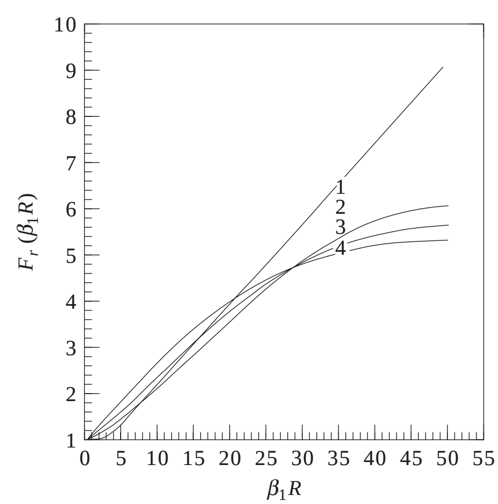

Figure 1. The real part of the dynamical viscosity versus $\beta_1 R$. 1, Biot's theory; 2, $D = 0{\cdot}5$; 3, $D = 0{\cdot}75$, 4, $D = 1{\cdot}0$. In 2, 3 and 4, $E = 4$.

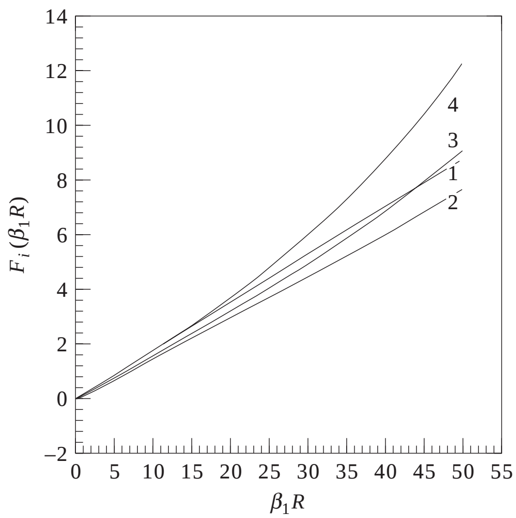

Figure 2. The imaginary part of the dynamical viscosity versus $\beta_1 R$. Key as Figure 1.

As for water, when $\xi/\eta = 8/3$, $E = 4$, and then the static viscosity is approximately $(\bar{\tau}_n/\bar{U}_n)_{\omega=0} \approx 13{\cdot}3\pi\eta$, instead of $8\pi\eta$ in Biot's theory, which means that the static viscosity is greatly increased. Only when $E = 1$ it is equal to $8\pi\eta$, which is the same as Biot's result.

5. NUMERICAL RESULTS

In Figures 1 and 3 is shown the relationship of the real parts of $F(\beta_1 R)$ versus $\beta_1 R$, while the standard deviation $D = 0{\cdot}5$, $0{\cdot}75$ and 1, in which $E = 4$ and $E = 4/3$, respectively. The relationship of the imaginary parts is shown in Figures 2 and 4. Obviously, for a greater $\beta_1 R$, they are both different from Biot's results.

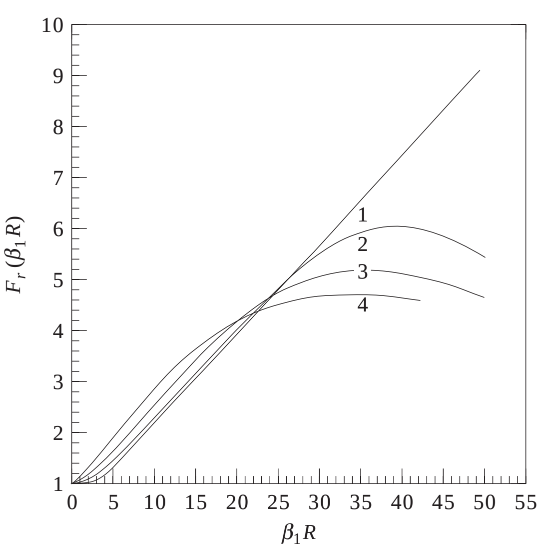

Figure 3. The real part of the dynamical viscosity versus $\beta_1 R$. 1, Biot's theory; 2, $D = 0{\cdot}5$; 3, $D = 0{\cdot}75$, 4, $D = 1{\cdot}0$. In 2, 3 and 4, $E = 4/3$.

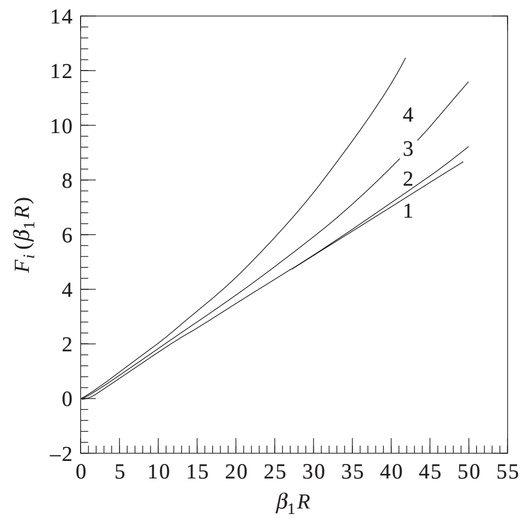

Figure 4. The imaginary part of the dynamical viscosity versus $\beta_1 R$. Key as Figure 3.

6. CONCLUSIONS

In porous media, if the axes of the tubes are distributed uniformly in the space and the sizes of the pores obey a normal distribution (in ϕ values), in the higher frequency range, their static viscosity will be higher than that given by Biot's theory, and the dynamical viscosity function will be different from Biot's results, both of which depend sensitively upon the average radius of the pores R and upon the standard deviation D, as well as on the ratio of both viscosity coefficients.

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