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STABILITY OF VIBRATIONS OF TWO OSCILLATORS MOVING UNIFORMLY ALONG A BEAM ON A VISCOELASTIC FOUNDATION

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The stability problem of two oscillators moving uniformly along an Euler–Bernoulli beam on a viscoelastic foundation has been studied. It is assumed that the masses and the beam are in continuous contact and that the velocity of the oscillators exceeds the minimum phase velocity of waves in the supported beam. Stability regions are found. It is shown that a range of velocities exists for which unstable vibrations of the two oscillators will occur for all elastic–inertial properties.

1. INTRODUCTION

The development of high speed trains has stimulated interest in studying the interaction of elastic systems with mechanical objects moving along them. The most important topic of this research is dealing with studies of critical velocities and resonance. However resonance is not the only 'dangerous' phenomenon for high speed trains. Another one is the instability of vibrations caused by a train interacting with the railroad track. This instability is important when a high speed train has passed the elastic wave barrier [1].

The results of previous studies on stability problems of a moving object interacting with an elastic system [2–5] indicate that there is a velocity after which the amplitude of the object vibrations can infinitely grow in time (in frames of a linear model) even for a damped supporting system. This result is quite different from the common sense about the effect of damping on vibrations at resonance (the amplitude of vibrations is finite due to damping).

Mathematically the instability implies that the roots of the characteristic equation in the Laplace domain (determining the eigenfrequencies of the system) possess a positive real part. The physical explanation of the instability has been given in [2] showing that it is caused by anomalous Doppler waves [6], radiated by the moving object.

In this paper the stability of two oscillators moving uniformly along a beam on a viscoelastic foundation is studied. Since trains have a few points of contact with the rails the motion of two oscillators can be considered as a basic model for the bogies of a locomotive. The developed theory can be extended to a system with more than two points of contact: i.e., a chain of oscillators. The goal of this investigation is to find stability

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regions depending on physical parameters such as the velocity of the train, the mass and the stiffness of train bogies.

As a method of investigation the problem is first transformed into the Laplace domain. Then the eigenfrequencies of vibrations of the two moving oscillators interacting with the beam are analysed with the help of the D-decomposition method [7]. This method gives an insight in the number of unstable roots for different intervals of the elastic–inertial properties of the system.

2. MODEL AND CHARACTERISTIC EQUATION

A system consisting of two oscillators, with constant relative distance d, moving uniformly along an Euler–Bernoulli on a viscoelastic foundation is considered, (see Figure 1). It is assumed that the masses and the beam are in continuous contact. Further the velocity of the oscillators exceeds the minimum phase velocity of waves in the supported beam. The governing equations of motion for the model are given as

$$\rho\frac{\partial^2\tilde{U}}{\partial t^2}+EI\frac{\partial^4\tilde{U}}{\partial x^4}+\varepsilon\frac{\partial\tilde{U}}{\partial t}+\kappa\tilde{U}=-\left(m\frac{\mathrm{d}^2\tilde{U}^{01}}{\mathrm{d}t^2}+K_0\,\tilde{U}^{01}\right)\delta(x-vt)$$

$$-\left(m\frac{\mathrm{d}^2\tilde{U}^{02}}{\mathrm{d}t^2}+K_0\,\tilde{U}^{02}\right)\delta(x-vt+d),$$

$$\tilde{U}^{01}(t)=\tilde{U}(vt,t),\qquad \tilde{U}^{02}(t)=\tilde{U}(vt-d,t),\qquad \lim_{x-vt\to\pm\infty}\tilde{U}(x,t)=0, \tag{2.1}$$

where $\tilde{U}(x,t)$ and $\tilde{U}^{0i}(t)$ $(i=1,2)$ are the vertical deflections of the beam and the masses respectively. In equations (2.1) ρ is the beam mass per unit length, EI is the bending stiffness of the beam, ε and κ are the viscosity and the stiffness of the foundation per unit length, m and K_0 are the mass and the stiffness of an oscillator (both oscillators are the same), v is the velocity of the oscillators and $\delta(.)$ is the Dirac delta function. The units of parameters are:

$$\rho(\mathrm{kg/m}),\ EI(\mathrm{Nm^2}),\ \varepsilon(\mathrm{kg/ms}),\ \kappa(\mathrm{kg/ms^2}),\ m(\mathrm{kg}),\ K(\mathrm{N/m}),\ v(\mathrm{m/s}),\ \delta(\mathrm{x})=(1/\mathrm{m}).$$

Introducing the following dimensionless variables and parameters

$$\bar{\tau}=t\cdot(\kappa/\rho)^{1/2},\qquad y=x(4EI/\kappa)^{-1/4},\qquad \alpha=v(4\kappa EI/\rho^2)^{-1/4},$$

$$\{U,U^{0i}\}=\{\tilde{U},\tilde{U}^{0i}\}(4EI/\kappa)^{-1/4}$$

$$D=d(4EI/\kappa)^{-1/4},\qquad M=4(m/\rho)\,(4EI/\kappa)^{-1/4},\qquad K=4(K_0/\kappa)\,(4EI/\kappa)^{-1/4},$$

$$\nu=\varepsilon/(\kappa\rho)^{1/2}$$

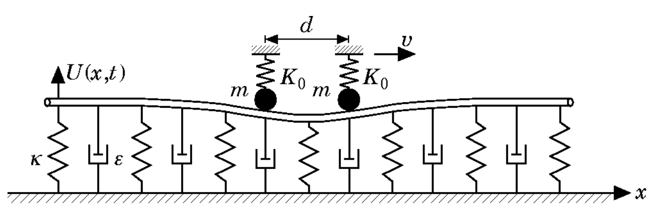

Figure 1. Uniform motion of two oscillators along an Euler–Bernoulli beam on a viscoelastic foundation.

one can rewrite equation (2.1) as follows

$$4U_{\bar\tau\bar\tau} + U_{yyyy} + 4vU_{\bar\tau} + 4U = -(M\,\mathrm{d}^2U^{01}/\mathrm{d}\bar\tau^2 + KU^{01})\,\delta(y-\alpha\bar\tau)$$
$$-(M\,\mathrm{d}^2U^{02}/\mathrm{d}\bar\tau^2 + KU^{02})\,\delta(y-\alpha\bar\tau+D),$$
$$U^{01}(\bar\tau) = U(\alpha\bar\tau,\bar\tau),\qquad U^{02}(\bar\tau) = U(\alpha\bar\tau - D,\bar\tau),\qquad \lim_{y-\alpha\bar\tau\to\pm\infty} U(y,\bar\tau) = 0. \tag{2.2}$$

Notice that the velocity $v_{ph}^{\min} = \sqrt[4]{4\kappa EI/\rho^2}$ ($\Leftrightarrow \alpha = 1$) is the minimum phase velocity of bending waves in the beam on an elastic foundation ($v\to 0$), see [8].

For further analysis it is convenient to perform the co-ordinate transformation

$$\xi = y - \alpha\bar\tau \;\wedge\; \tau = \bar\tau.$$

Then equations (2.2) take the following form

$$U_{\xi\xi\xi\xi} + 4(U_{\tau\tau} - 2\alpha U_{\xi\tau} + \alpha^2 U_{\xi\xi} + v(U_\tau - \alpha U_\xi) + U) = -(MU_{\tau\tau} + KU)\,\delta(\xi)$$
$$-(MU_{\tau\tau} + KU)\,\delta(\xi + D),\qquad \lim_{\xi\to\pm\infty} U(\xi,\tau) = 0. \tag{2.3}$$

Application of the following Laplace and Fourier transforms

$$V(\xi,p) = \int_0^\infty U(\xi,\tau)\cdot\exp(-p\tau)\,\mathrm{d}\tau,\qquad W(k,p) = \int_{-\infty}^\infty V(\xi,p)\cdot\exp(-\mathrm{i}k\xi)\,\mathrm{d}\xi, \tag{2.4}$$

to equation (2.3) with the trivial initial conditions $U(\xi,0) = U_\tau(\xi,0) = 0$ (the initial shape of the beam does not effect the stability of the system) results in the following equation

$$f(k,p)\;W(k,p) = -(Mp^2+K)\{V(0,p) + V(-D,p)\exp(\mathrm{i}kD)\},$$

where

$$f(k,p) = k^4 + 4(p^2 - 2\alpha\mathrm{i}kp - \alpha^2k^2 + vp - \mathrm{i}\alpha vk + 1). \tag{2.5}$$

Now inverting the Fourier transform one gets the solution in the Laplace domain. It yields

$$V(\xi,p) = -\frac{Mp^2+K}{2\pi}\left(V(0,p)\int_{-\infty}^\infty \frac{\exp(\mathrm{i}k\xi)}{f(k,p)}\,\mathrm{d}k + V(-D,p)\int_{-\infty}^\infty \frac{\exp(\mathrm{i}k(\xi+D))}{f(k,p)}\,\mathrm{d}k\right). \tag{2.6}$$

To determine $V(\xi,p)$ one has to know $V(0,p)$ and $V(-D,p)$ which are the Laplace images of $U^{01}(\tau)$ and $U^{02}(\tau)$ respectively. Assuming first $\xi = 0$ and second $\xi = -D$ in equation (2.6) it is found that

$$\begin{bmatrix} 1 + Z_0 I_0 & Z_0 I_+ \\ Z_0 I_- & 1 + Z_0 I_0 \end{bmatrix}\begin{bmatrix} V(0,p) \\ V(-D,p)\end{bmatrix} = \mathbf{0}, \tag{2.7}$$

with

$$Z_0 = Mp^2 + K,\qquad I_0 = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\mathrm{d}k}{f(k,p)},\qquad I_+ = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\exp(\mathrm{i}kD)\,\mathrm{d}k}{f(k,p)},$$

$$I_- = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\exp(-\mathrm{i}kD)\,\mathrm{d}k}{f(k,p)}. \tag{2.8}$$

The system of equations (2.7) has only a non-trivial solution if

$$(1 + Z_0\ I_0)^2 - Z_0^2 I_- I_+ = 0. \tag{2.9}$$

Equation (2.9) is the characteristic equation, the roots of which physically determine the eigenfrequencies of vibrations of the two moving oscillators interacting with the supported beam.

3. STABILITY OF VIBRATIONS

One can rewrite equation (2.9) in the form

$$K + Mp^2 + 1/[I_0(p) \pm \sqrt{I_+(p)I_-(p)}] = 0. \tag{3.1}$$

To analyse the stability of vibrations one should find out whether the roots of the characteristic equation (3.1) possess a positive real part. When one wants to study the stability depending on different physical parameters, as for example the velocity (α) and elastic–inertial properties of the system, then a straightforward approach to the determination of these roots and further checking for a positive real part is laborious. Therefore it is convenient here to use another method of root analysis, namely the D-decomposition method [7], which first was used for solving these type of problems in [4]. The idea of this method is to map the imaginary axis of the complex (p)-plane (= border between stability and instability) onto the plane of a complex parameter K (neglecting the physical meaning of this parameter temporarily). The mapped line will divide the K-plane into domains with different number of roots with a positive real part.

Substituting $p = \mathrm{i}\Omega$ into equation (3.1) gives the following rule for the mapping

$$K = M\Omega^2 - 1/[I_0(\Omega) \pm \sqrt{I_+(\Omega)I_-(\Omega)}], \tag{3.2}$$

where Ω is a real value which has to be varied from minus to plus infinity. According to (2.8) the expressions for the integrals are given as

$$I_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{F(k,\Omega)}, \qquad I_+ = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}kD)\,\mathrm{d}k}{F(k,\Omega)}, \qquad I_- = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(-\mathrm{i}kD)\,\mathrm{d}k}{F(k,\Omega)}, \tag{3.3}$$

with (see equation (2.5))

$$F(k,\Omega) = k^4 + 4(-\Omega^2 + 2\alpha\Omega k - \alpha^2 k^2 + \mathrm{i}v\Omega - \mathrm{i}\alpha v k + 1). \tag{3.4}$$

The integrals (3.3) can be elaborated with the help of the method of contour integration. Using for I_0 and I_+ the contour shown in Figure 2a and for I_- the contour given in Figure 2b (the choice of the contour in the upper or the lower for I_0 is arbitrary but for I_+ and I_- it follows with Jordan's lemma) one obtains the following expressions

Figure 2. (a) Contour of integration in case of I_0 and I_+; (b) contour of integration in case I_-.

$$I_0 = \mathrm{i} \sum_n \left. \frac{k - k_n}{(k - k_1)(k - k_2)(k - k_3)(k - k_4)} \right|_{k = k_n},$$

$$I_+ = \mathrm{i} \sum_n \left. \frac{(k - k_n) \exp(\mathrm{i} k_n D)}{(k - k_1)(k - k_2)(k - k_3)(k - k_4)} \right|_{k = k_n},$$

$$I_- = -\mathrm{i} \sum_m \left. \frac{(k - k_m) \exp(-\mathrm{i} k_m D)}{(k - k_1)(k - k_2)(k - k_3)(k - k_4)} \right|_{k = k_m}, \tag{3.5}$$

where k_n are the roots of the equation $F(k, \Omega) = 0$ with a positive imaginary part and k_m those with a negative imaginary part.

Now according to the D-decomposition method one has to plot two curves (following from equation (3.2)) on the complex K-plane using Ω as the parameter for this curve. This can be done numerically with the help of any standard program for the determination of complex roots of the polynomials. Since each of the two branches for K following from the mapping is symmetrical with respect to the real axis (see Appendix B) one only has to vary Ω from zero to infinity to get the complete D-decomposition picture. So $K(\Omega = 0)$ can be seen as a starting point for each curve and therefore it is useful to analyse this point, which can be done analytically for $v \to 0$.

The roots of the equation $F(k, 0) = 0$ are given as $(v = 0)$

$$k_{1,2} = \pm\sqrt{2\alpha^2 + 2\sqrt{\alpha^4 - 1}} \wedge k_{3,4} = \pm\sqrt{2\alpha^2 - 2\sqrt{\alpha^4 - 1}},$$

where $\alpha \geqslant 1$ (the velocity of motion is larger than or equal to the minimum phase velocity of waves in the supported beam). Introducing a small viscosity the roots k_1 and k_2 will move from the real k-axis into the upper half-plane and the poles k_3 and k_4 into the lower one as depicted in Figure 3. Then equations (3.5) can be elaborated to give the following expressions ($\Omega = 0$ and $v \to 0$)

$$I_0 = \mathrm{i}\left(\frac{1}{(k_1 - k_2)(k_1 - k_3)(k_1 - k_4)} + \frac{1}{(k_2 - k_1)(k_2 - k_3)(k_2 - k_4)} \right) = 0,$$

$$I_+ = \mathrm{i}\left(\frac{\exp(\mathrm{i} k_1 D)}{(k_1 - k_2)(k_1 - k_3)(k_1 - k_4)} + \frac{\exp(\mathrm{i} k_2 D)}{(k_2 - k_1)(k_2 - k_3)(k_2 - k_4)} \right) = -\frac{\sin(k_1 D)}{k_1 (k_1^2 - k_3^2)},$$

$$I_- = -\mathrm{i}\left(\frac{\exp(-\mathrm{i} k_3 D)}{(k_3 - k_1)(k_3 - k_2)(k_3 - k_4)} + \frac{\exp(-\mathrm{i} k_4 D)}{(k_4 - k_1)(k_4 - k_2)(k_4 - k_3)} \right) = \frac{\sin(k_3 D)}{k_3 (k_3^2 - k_1^2)}.$$

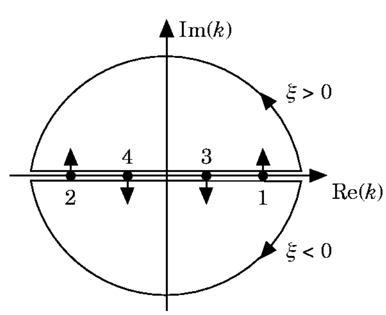

Figure 3. Translation of the roots due to a small viscosity and the contours of integration for $\xi > 0$ and $\xi < 0$.

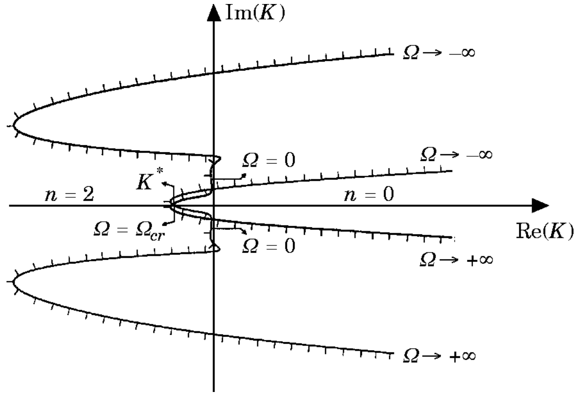

Figure 4. Separation of the complex K-plane in domains with a different number of unstable roots for $\alpha = 1{\cdot}1$.

Using these results equation (3.2) is rewritten as

$$K(\Omega = 0) = \pm(\sqrt{-\sin(k_1 D)\sin(k_3 D)/k_1 k_3 (k_1^2 - k_3^2)^2})^{-1}. \quad (3.6)$$

Equation (3.6) shows that depending on the values of the parameters k_1, k_3 and D it is possible that the starting point for the D-decomposition curves is either imaginary or real for $v \to 0$. Oscillator velocities for which

$$k_1 D = n\pi \text{ or } k_3 D = n\pi \Leftrightarrow \alpha_{c,n} = \sqrt{\frac{1}{4}\left(\frac{\pi n}{D}\right)^2 + \left(\frac{D}{\pi n}\right)^2}, \qquad n = 1, 2, 3, \ldots . \quad (3.7)$$

determine such a transition. Physically these velocities can be understood to be the resonance velocities for the two oscillators if they are subjected to a constant load. Then in steady state the load generates waves and for $\alpha = \alpha_{c,n}$ the distance between the oscillators is divisible by half of a wavelength. From now on these velocities are called critical velocities.

Now the D-decomposition curves according to (3.2) can be depicted. The following parameters are used for all curves: $v = 0{\cdot}01$ (small viscosity), $M = 5$ and $D = 0{\cdot}4$. In Figures 4–6 the D-decomposition curves for three different velocities are depicted. In Figure 4 $\alpha = 1{\cdot}1$: i.e., the velocity is smaller than the first critical velocity ($\alpha_{c,1} = 3{\cdot}9$).

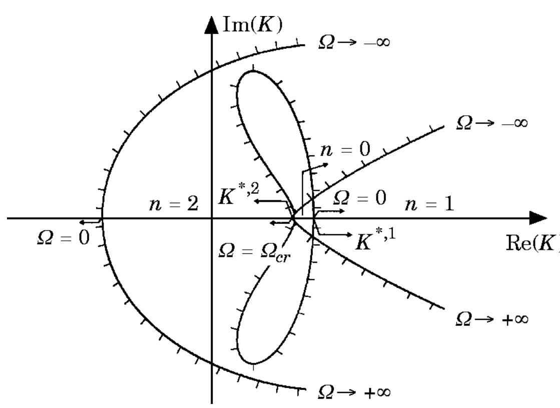

Figure 5. Separation of the complex K-plane in domains with different number of unstable roots for $\alpha = 4{\cdot}5$.

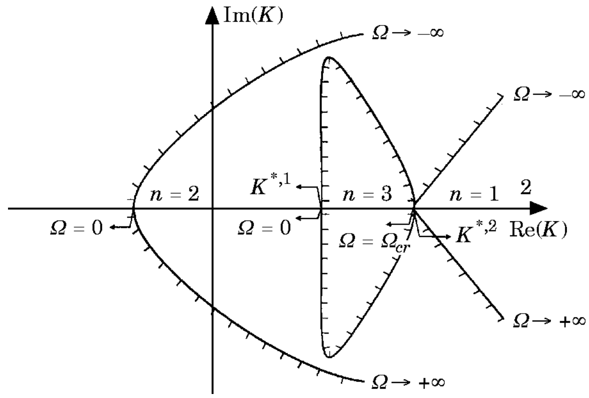

Figure 6. Separation of the complex K-plane in domains with different number of unstable roots for $\alpha = 6{\cdot}8$.

In Figure 5: $\alpha = 4{\cdot}5$ and in Figure 6: $\alpha = 6{\cdot}8$: i.e., the velocities are larger than the first and smaller than the second critical velocity ($\alpha_{c,2} \approx 7{\cdot}9$).

One side of the lines in all figures is shaded. This side is related to the right side of the imaginary axes in the complex p-plane. Crossing of a line in the direction of the shading implies an additional root with a positive real part [7].

Now the number of unstable roots for some particular value of K has to be derived, because so far only the relative number of unstable roots in the different domains of the K-plane is known, but not the number itself. This particular case, for $K = 0$, has been evaluated in Appendix A. Using these results and starting in the domains which include the point $K = 0$ the number of unstable roots ($= n$) for arbitrary K is determined (see Figures 4–6).

The complex parameter K has only a physical meaning when it is positive and real (the stiffness of an oscillator is real and positive). So the stability of the system is determined by the number of unstable roots for $K \in \{\mathrm{Re}(K) > 0,\ \mathrm{Im}(K) = 0\}$. This number can be changed when the D-decomposition curve has a crossing point with the positive part of the real axis.

From Figure 4 ($1 \leqslant \alpha < \alpha_{c,1}$) it is seen that the mapped line crosses the real axis only once in $(-K^*, 0)$. This point corresponds with $\Omega = \Omega_{cr}$. The parameter Ω_{cr} is related to the appearance of anomalous Doppler waves (see Appendix C and [2]). The crossing point in Figure 4 is located in the negative part of the real axis and with increasing velocity (still $\alpha < \alpha_{c,1}$) it moves to the positive part (this situation is not depicted since only a horizontal translation of the mapped lines takes place). So it is seen from the figure that for velocities somewhat larger than $\alpha = 1$ the vibrations of the system are always stable. However with increasing velocity an interval of the oscillator stiffnesses ($0 < K < K^*$) exists for which the vibrations of the system are unstable.

The qualitative difference between Figures 5, 6 ($\alpha_{c,1} < \alpha < \alpha_{c,2}$) and Figure 4 is that now three crossing points of the mapped lines with the real axis occur. The extra two points are related to $\Omega = 0$ (starting points of the D-decomposition curves are in Figures 5 and 6 located on the real axis). Two cases are distinguished. First, in Figure 5 the crossing point related to $\Omega = \Omega_{cr}$ is located between the starting points and there exists an interval ($K^{*,2} < K < K^{*,1}$) where the vibrations of the system are stable. Second, in Figure 6 the crossing point related to $\Omega = \Omega_{cr}$ is located to the right of both starting points and the vibrations of the system are always unstable.

It should be noted that velocities of the oscillators for which $\alpha_{c,n} < \alpha < \alpha_{c,n+1}$ with $n \in \{2, 4, 6, \ldots\}$ give qualitatively the same results as follow from Figure 4. Oscillator

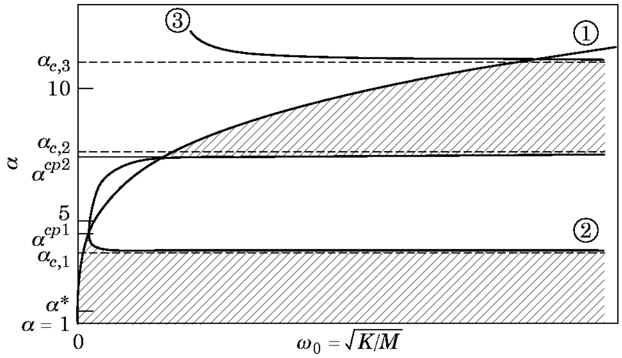

Figure 7. Domains of (in)stability (hatched domains are stable) for *two* oscillators moving uniformly along an E–B beam on a viscoelastic foundation ($D = 0{\cdot}4$ and $v = 0{\cdot}01$).

velocities for which $\alpha_{c,n} < \alpha < \alpha_{c,n+1}$ with $n \in \{1,3,5,\ldots\}$ give qualitatively the same results as follow from Figures 5 and 6.

In Figure 7 the domains of stable vibrations, depicted on the plane of the eigenfrequency of the oscillator ($\omega_0 = \sqrt{K/M}$) versus the velocity α, are hatched. Line number 1 is related to $\Omega = \Omega_{cr}$ and line numbers 2 and 3 to $\Omega = 0$. The following conclusions can be drawn from this figure:

1. Instability can only take place if the velocity is larger than α^* (when $v = 0 \to \alpha^* = 1$).
2. There exists a range of velocities for which for all values of the elastic-inertial parameters unstable vibrations results (for example, $\alpha^{cp1} < \alpha < \alpha^{cp2}$).

The second conclusion differs from that for one oscillator system moving uniformly along the same system. Its stable domain is hatched in Figure 8. This domain can be obtained using the results of [2] or with the help of D-decomposition procedure with the mapping rule: $K = M\Omega^2 - 1/I_0(\Omega)$. It is easily seen that in case of one oscillator interacting with the beam the vibrations can be stable at any velocity by choosing the elastic–inertial properties of the moving oscillator in an appropriate way. This is the main difference between a system with one and with two oscillators.

Now one asks where the energy comes from to increase the amplitude of vibrations of the oscillators. The only external force which acts on the system is the horizontal force maintaining the uniform motion of oscillators [9]. As it was shown in [2] this force delivers the energy to the system when the oscillators radiate anomalous Doppler waves.

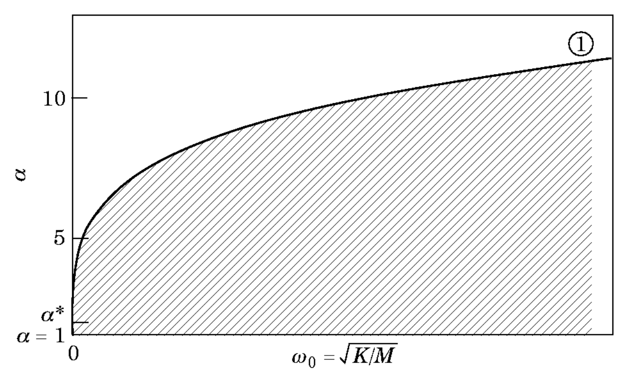

Figure 8. Domains of (in)stability (hatched domains are stable) for *one* oscillator moving uniformly along an E–B beam on a viscoelastic foundation ($v = 0{\cdot}01$).

The important question still remains of what is the minimum value of the velocity α^* for which instability in practice can occur? The simplest way to estimate α^* is to use the expression for the minimum phase velocity of waves in the beam, $v_{ph}^{\min} = \sqrt[4]{4\kappa EI/\rho^2}$ ($\Leftrightarrow \alpha = 1$), and substitute into it for the beam parameters the parameters of the rail, and for the stiffness of the elastic foundation the stiffness of the track subsoil measured statically. In this case one obtains $v_{ph}^{\min} = 500$–1000 km/h (the value 500 is obtained when the beam models the rails, sleepers and the ballast [10]). So from first sight one can conclude that this instability is not of importance for present day trains. However, more sophisticated research on three dimensional models [10, 11] shows that this velocity α^* is approximately equal to the velocity of Rayleigh waves, which in peat and wet soil (for example, in the western part of Holland) can be in the order of 200 km/h.

Thus, results presented here should be considered as qualitative ones. To obtain realistic values for the borders of instability domains one has to consider more complicated models of the train track system, where the subsoil is described by a three-dimensional solid.

3. CONCLUSIONS

Regions of (un)stable vibrations for two oscillators moving uniformly along an Euler–Bernoulli beam on a viscoelastic foundation are derived. It turned out that two frequencies are important related to the instability: the frequency for which anomalous Doppler waves are starting to be radiated and the zero frequency. It is shown that a range of velocities exists for which unstable vibrations of the two oscillators will occur for any values of the elastic–inertial properties (for the chosen distance between the oscillators). This result is different from the one oscillator problem. The way of analysis in this paper can be applied to models with more points of contact, resulting in an insight into their stability regions.

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APPENDIX A

The number of unstable roots for $K = 0$ and $M = 5$ is derived and this result will be used as an "initial" point for determining the number of unstable roots for all domains of the D-decomposition pictures in the complex K-plane of section 3. Therefore curves in the complex M-plane for the same parameters and same velocities as in the three Figures 4–6 from section 3 will be derived and then the number of unstable roots in the point $M = 5$ (real) will be determined.

The equation for the mapping in the M-plane for $K = 0$ is (it follows from equation (3.2))

$$M = 1/[\Omega^2(I_0(\Omega) \pm \sqrt{I_+(\Omega)I_-(\Omega)})] \tag{A1}$$

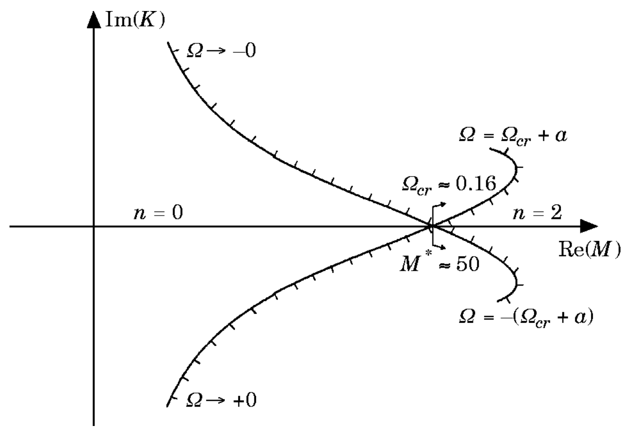

Figure A1. Separation of the complex M-plane in domains with a different number of unstable roots for $\alpha = 1{\cdot}1$.

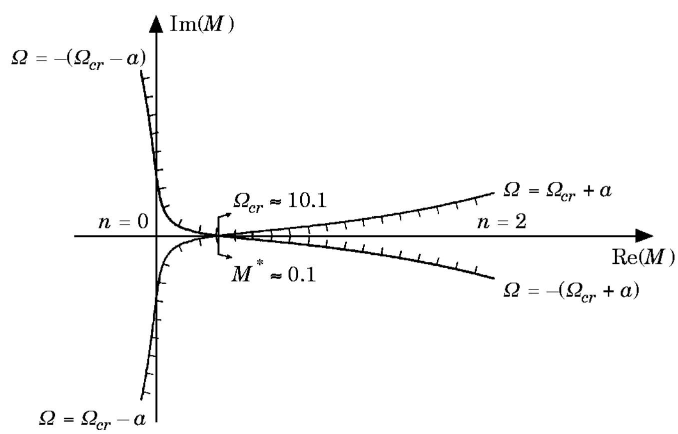

Figure A2. Separation of the complex M-plane in domains with a different number of unstable roots for $\alpha = 4{\cdot}5$.

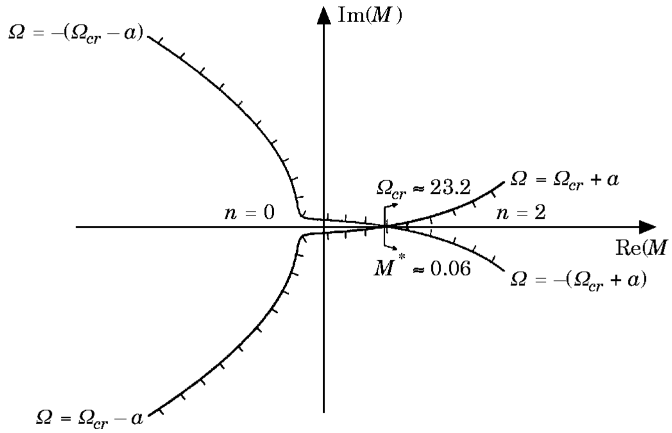

Figure A3. Separation of the complex M-plane in domains with different number of unstable roots for $\alpha = 6{\cdot}8$.

where I_0, I_+ and I_- are given by equation (3.5). The D-decomposition curves for $\alpha = 1{\cdot}1$, $\alpha = 4{\cdot}5$ and $\alpha = 6{\cdot}8$ are depicted in the following three figures (Figures A1–A3) in the vicinity of the single crossing point of the mapped lines with the real axis (a is a small real and positive value).

Crossing of the D-decomposition lines in the direction of the shading implies an additional root with a positive real part; i.e., an extra unstable root. Now the number of unstable roots for some particular value of M has to be derived, because so far only the relative number of unstable roots in the different domains of the M-plane is known, but not the number itself. It can be done numerically for $M = 0$. Then calculations show that no roots with a positive real part exist. Physically this result is evident since the instability is the result of an interaction between the moving body and the beam [2]. Thus, if the body is absent ($M = 0$, $K = 0$) instability can not occur and the number of unstable roots is zero $\Rightarrow n = 0$. Using this result and starting in the domains which include the point $M = 0$ one can determine the number of unstable roots for $M = 5$. The results have been summarised in Table A1:

Table A1

Number of unstable roots, n, for different velocities α and for $M = 5$ and $K = 0$

α	1·1	4·5	6·8
n	0	2	2

APPENDIX B

Theorem. Consider the following rule for the mapping

$$K = M\Omega^2 - 1/[I_0(\Omega) \pm \sqrt{I_+(\Omega)I_-(\Omega)}], \tag{B1}$$

where Ω is a real value which has to be varied from minus to plus infinity and where the expressions for the integrals are given as

$$I_0(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{F(k,\Omega)}, \qquad I_+(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}kD)\,\mathrm{d}k}{F(k,\Omega)},$$

$$I_-(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(-\mathrm{i}kD)\,\mathrm{d}k}{F(k,\Omega)}, \tag{B2}$$

with

$$F(k,\Omega) = k^4 + 4(-(\Omega-\alpha k)^2 + \mathrm{i}\nu(\Omega-\alpha k) + 1). \tag{B3}$$

Then each of the two branches for K itself (K_1 *or* K_2 *following from equation* (B1)) *is symmetrical with respect to the real K-axis, so*

$$\mathrm{Re}\,(K_j(\Omega)) = \mathrm{Re}\,(K_j(-\Omega)) \wedge \mathrm{Im}\,(K_j(\Omega)) = -\mathrm{Im}\,(K_j(-\Omega)), \quad \textit{with } j = 1, 2. \tag{B4}$$

Proof. One can rewrite the integrals from equation (B2) as follows

$$I_0(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F^r - \mathrm{i}F^i}{f}\,\mathrm{d}k,$$

$$I_+(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(F^r - \mathrm{i}F^i)\exp(\mathrm{i}kD)\,\mathrm{d}k}{f}, \qquad I_-(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(F^r - \mathrm{i}F^i)\exp(-\mathrm{i}kD)\,\mathrm{d}k}{f}, \tag{B5}$$

with

$$F(k,\Omega) = F^r + \mathrm{i}F^i \quad \text{and} \quad f = (F^r)^2 + (F^i)^2.$$

where F^r is the real and F^i the imaginary part of equation (B3). From equation (B3) it yields

$$F^r(k,\Omega) = F^r(-k,-\Omega) \wedge F^i(k,\Omega) = -F^i(-k,-\Omega). \tag{B6}$$

Changing the variable of integration along $k = -k$ and using equation (B6) the integrals I_0, I_+ and I_- from equation (B5) are rewritten for $\Omega = -\Omega$ as follows

$$I_0(-\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F^r + \mathrm{i}F^i}{f}\,\mathrm{d}k,$$

$$I_+(-\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(F^r + \mathrm{i}F^i)\exp(-\mathrm{i}kD)}{f}\,\mathrm{d}k,$$

$$I_-(-\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(F^r + \mathrm{i}F^i)\exp(\mathrm{i}kD)}{f}\,\mathrm{d}k, \tag{B7}$$

Comparing equations (B5) and (B7) shows that the integral $I_0(\Omega) = \overline{I_0(-\Omega)}$ (complex conjugated). Using equations (B5) and (B7) the product of the integrals I_+ and I_- is rewritten as

$$I_+(\Omega)I_-(\Omega) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(ac+bd)+\mathrm{i}(bc-ad)}{f^2}\,\mathrm{d}k\,\mathrm{d}\kappa,$$

$$I_+(-\Omega)I_-(-\Omega) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(ac+bd)-\mathrm{i}(bc-ad)}{f^2}\,\mathrm{d}k\,\mathrm{d}\kappa, \tag{B8}$$

where

$$a = F^r\cos(kD) + F^i\sin(kD), \qquad b = F^r\sin(kD) - F^i\cos(kD),$$

$$c = F^r\cos(\kappa D) - F^i\sin(\kappa D), \qquad d = F^r\sin(\kappa D) + F^i\cos(\kappa D).$$

From equations (B8) one can see that $I_+(\Omega)\ I_-(\Omega) = \overline{I(-\Omega)I(-\Omega)}$.

Evidently the square root of the product of I_+ and I_- will give two values which can be taken complex conjugated by an appropriate choice of the branches. Hereby equation (B4) has been proved. □

APPENDIX C

In order to determine the physical meaning of the parameter Ω_{cr} (section 3) one oscillator moving with a constant velocity (α) along an E–B beam on an elastic foundation is considered. It is assumed that its mass and the beam are in continuous contact. Further the velocity of the oscillator exceeds the minimum phase velocity in the beam ($\alpha \geqslant 1$). The governing dimensionless equations for this model are given as (see [2 or 12])

$$4U_{\bar{\tau}\bar{\tau}} + U_{yyyy} + 4U = 0, \qquad U^{01}(\bar{\tau}) = U(\alpha\bar{\tau}, \bar{\tau}), \qquad [U]_{y=\alpha\bar{\tau}} = [U_y]_{y=\alpha\bar{\tau}} = [U_{yy}]_{y=\alpha\bar{\tau}} = 0,$$

$$[U_{yyy}]_{y=\alpha\bar{\tau}} = -(M\ddot{U}^{01} + KU^{01}), \qquad \lim_{y-\alpha\bar{\tau}\to\pm\infty} U(y,\bar{\tau}) = 0, \tag{C1}$$

where $U(y,\bar{\tau})$ and $U^{01}(y,\bar{\tau})$ are the vertical deflections of the beam and the mass respectively. The parameters used in equations (C1) are the same as those used in section 2.

It is assumed that the object oscillates harmonically with a frequency Ω and an amplitude A ($\to U^{01}(t) = A\exp(\mathrm{i}\Omega t)$). Seeking for the solution of equations (C1) in the form of travelling waves ($\to U(y,\bar{\tau}) = B\exp(\mathrm{i}(\omega\bar{\tau} - ky))$) one obtains the following system of equations determining the frequencies ω and wave numbers k of the radiated waves in the beam

$$k^4 + 4(-\omega^2 + 1) = 0. \qquad \omega = \alpha k + \Omega \tag{C2}$$

The first equation in (C2) is the dispersion equation and the second one is the so-called 'kinematic invariant', see [12 or 13]. This system of equations is equivalent to the equation $F(k,\Omega) = 0$, see equation (3.4) for $v = 0$, which determines the poles of the integrals I_0, I_+ and I_- in equations (3.3). The solution of the system (C2) (and thus also the solution of $F(k,\Omega) = 0$) can be determined graphically with the help of Figure C1.

The curve $\omega = \omega(k)$ in this figure represents the dispersion relation of the elastic system and the straight line $\omega = \alpha k + \Omega$ is the kinematic invariant. The crossing points of the kinematic invariant with the dispersion curve are the frequencies and wavenumbers of the four waves radiated by the moving oscillator. Two of these waves are normal Doppler waves (related to point 1 and 3) and two of them are anomalous (related to point 2 and 4). The phase velocity v_{ph} of an anomalous Doppler wave satisfies the inequality, see [6] $1 - v/v_{ph} < 0$.

A special property of the anomalous Doppler waves is that being radiated by an moving object they increase the energy of vertical vibrations of the object [2]. The frequency Ω_{cr}

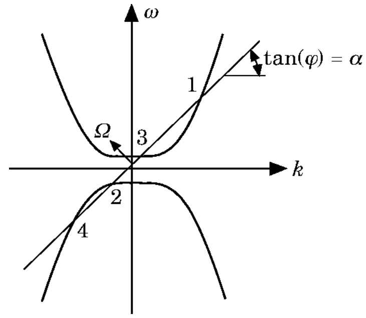

Figure C1. Graphical evaluation of frequencies and wavenumbers radiated by a source with frequency Ω moving with the velocity α.

for which anomalous Doppler waves are originating takes place when the kinematic invariant is tangential to the dispersion curve for negative frequencies. This happens when $k_2 \to k_4$ (see Figure C1). For this situation the behaviour of the integrals I_0, I_+ and I_- from equation (3.3) is analysed.

Since it is known that by the introduction of a small viscosity the poles k_1 and k_2 possess a positive imaginary part and the poles k_3 and k_4 a negative imaginary part, elaboration of equations (3.5) results for $v \to 0$ and $\Omega = \Omega_{cr}$ ($k_2 \to k_4$) in

$$I_0 = \lim_{k_2 \to k_4} \left(\frac{-\mathrm{i}}{(k_1 - k_2)(k_2 - k_3)(k_2 - k_4)} \right) \to \infty,$$

$$I_+ I_- = \lim_{k_2 \to k_4} \left(\frac{1}{(k_2 - k_1)(k_2 - k_3)(k_3 - k_1)(k_2 - k_4)} \times \left(\frac{\exp(\mathrm{i}D(k_2 - k_3))}{(k_2 - k_3)^2} + \frac{\exp(\mathrm{i}D(k_2 - k_1))}{(k_2 - k_1)^2} \right) \right) \to \infty.$$

Using these results at least one of the curves following from the rule for the mapping (see equation (3.2)) is reduced to $K = M\Omega_{cr}^2$. Now it is clear that this curve possesses a common point (K^*, 0) with the real axis for the frequency Ω_{cr}.

In section 3 it is shown that this point influences the stability of the system. So it happens that the "important" frequency Ω_{cr} for one oscillator moving along a beam, which physically is related to the appearance of anomalous Doppler waves [2], is also "important" when related to the (in)stability for two oscillators along the same beam.