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VIBRATION EIGENFREQUENCY ANALYSIS OF A SINGLE-LINK FLEXIBLE MANIPULATOR

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We analyse the vibration eigenfrequencies of a flexible slewing beam with a payload attached at one end. A wave propagation method (WPM) is used. There are four types of waves which propagate along a beam—two dispersive waves travelling in opposite directions, and two evanescent waves near the endpoints. We add a fifth time-harmonic function corresponding to oscillation of the beam at the payload end. We show that the large frequencies are asymptotically identical to those for the clamped–free beam, independent of the payload. For small eigenfrequencies, we incorporate WPM with a perturbation iteration procedure, the results of which agree well with "exact" values which result from solving a transcendental equation cited elsewhere in the literature.

1. INTRODUCTION

Resonant eigenfrequency analysis is important in the design and control of vibrating structures. In this paper, a theoretical analysis is performed, as well as numerical computations, for a distributed parameter structure—a flexible robotic manipulator. In particular, the entire range (low, medium and high) of the vibration spectrum is calculated for this particular structure to a high degree of accuracy and with a minimal amount of numerical computation. The author believes that such a comprehensive combined study has not been done previously.

The simplest example of a flexible manipulator is the so-called flexible slewing beam, which consists of a single flexible beam with a link at one end, the hub. The slewing beam has applications in many fields, including areas such as robotics and aerodynamics and, especially recently, in the study of large, flexible space structures (such as the planned space station).

A rigorous model for the dynamics of a flexible slewing beam, with a rotor located at the hub and a payload at the free end, has been derived in both references [1] and [2], in both cases using a variational approach. These models are more complete than the classical Euler–Bernoulli model in that they allow for the effects of the payload, as well as those of the inertia of the beam and the hub, on the motion.

In the study of structural dynamics it is essential to be able to calculate the natural vibration frequencies of the structures under study. Even for the Euler–Bernoulli beam, such eigenfrequency analysis involves finding the zeros of quite complicated transcendental functions (see references [3, 4]). For the case of the flexible slewing beam, the model in references [1] and [2] leads to an even more complicated expression (reference [2], equation (48), p. 302). It is the author's understanding that an analysis of the eigenfrequencies has not been done.

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The wave propagation method (WPM) is a physically intuitive asymptotic method for the estimation of the eigenfrequencies of certain physical systems which are modelled by PDEs. WPM initially was developed by Keller and Rubinow [5] for second-order systems and was later generalized by Chen and Zhou [6] and Chen *et al*. [7] to systems modelled by fourth-order equations, such as the Euler–Bernoulli beam and Kirchhoff thin plate. Finally, in reference [8], Chen and Coleman developed a formal perturbation approach for improving the accuracy of WPM for the lowest few eigenmodes in the case of the Euler–Bernoulli beam.

The purpose of this paper is two-fold. First, WPM is applied along with a perturbation procedure to the model of a flexible slewing beam derived in references [1] and [2]. The presence of the payload attached to the tip of the flexible robotics arm adds certain intricacies to the boundary conditions and the ensuing transcendental equations. The novelty here is that the payload effects can be handled by adding an extra time-oscillatory term (see w_p in equation (12)) to the usual four types of waves on a beam. It is also proven that the eigenfrequencies of the slewing beam are asymptotically equivalent to those of the classical clamped–free Euler–Bernoulli beam, independent of the payload.

Second, in the process of performing the above, it has been found that the results given in reference [2] do not seem to satisfy the characteristic equation given in references [1] and [2] (reference [2], equation (48))—this seems only to be a matter of units, and possibly a misunderstanding by the author of the units used there, as their results are a constant multiple of the results given here (see Table 4). In order to be complete, solutions to that characteristic equation are provided, along with a comparison between these values and the WPM results, for a significant portion of the spectrum, thereby providing benchmarks which do not seem to appear elsewhere in the literature, to the best of the author's knowledge. The data given here also exhibit the asymptotic convergence of these values to those for the clamped–free Euler–Bernoulli beam, as mentioned above.

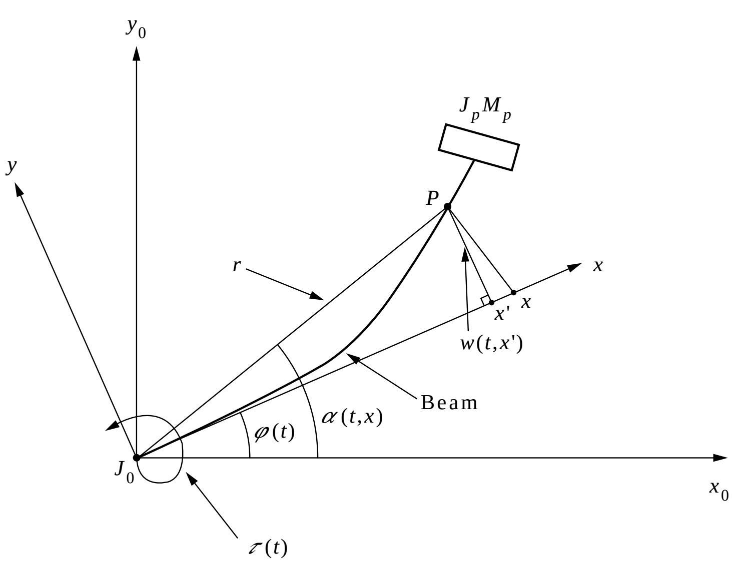

Figure 1. The flexible slewing beam. (Reprinted with permission from Morris and Taylor ([2], p. 295) Copyright 1996 by the Society for Industrial and Applied Mathematics. All rights reserved.)

2. THE PROBLEM

Following reference [2], the flexible slewing beam shown in Figure 1 is treated. A rotor is located at the hub of the beam (labelled J_0 in the figure), while a payload is attached at the opposite, free end. The beam deflects transversally only, and its movement is restricted to the plane. Again, following reference [2], (x_0, y_0) denotes co-ordinates in the inertial frame in which the beam rotates, while (x, y) represents the non-inertial frame in which the x-axis is tangent to the beam at its hub. The physical constants necessary for describing the system are E: Young's modulus; I: area moment of inertia; L: length of beam; ρ: linear mass density; J_0: inertia of rotor; J_p: inertia of payload; and M_p: mass of payload.

In reference [2], variational methods are used to derive the equations of motion and associated boundary conditions for the system. These are

PDEs:

$$EIw_{xxxx} + \rho(x\varphi_{tt} + w_{tt}) = 0, \tag{1}$$

$$J\varphi_{tt} + u(t) - \tau(t) = 0, \qquad 0 < x < L,\ t > 0; \tag{2}$$

BCs:

$$w(0, t) = 0, \qquad w_x(0, t) = 0, \tag{3, 4}$$

$$EIw_{xx}(L, t) = -J_p[\varphi_{tt}(t) + w_{xtt}(L, t)], \tag{5}$$

$$EIw_{xxx}(L, t) = M_p[L\varphi_{tt}(t) + w_{tt}(L, t)], \qquad t > 0, \tag{6}$$

where $w = w(x', t)$ = displacement of point P with x-value $= x'$ in the frame (x, y); $\varphi = \varphi(t)$ = angle between the frames (x, y) and (x_0, y_0); $\tau = \tau(t)$ = applied torque at hub; $u = u(t) = \int_0^L \rho x w_{tt}(x, t)\,\mathrm{d}x + M_p L w_{tt}(L, t) + J_p w_{xtt}(L, t)$; J = total inertia $= J_0 + J_p + M_p L^2 + \int_0^L \rho x^2\,\mathrm{d}x$.

Eliminating φ from the system, and setting $\tau \equiv 0$ (as one is dealing with the natural frequencies of vibration), one arrives at

PDE:

$$EIw_{xxxx} + \rho w_{tt} - x\frac{\rho}{J}u = 0, \qquad 0 < x < L,\ t > 0; \tag{7}$$

BCs:

$$w(0, t) = 0, \qquad w_x(0, t) = 0, \tag{8, 9}$$

$$EIw_{xx}(L, t) - \frac{J_p}{J}u(t) + J_p w_{xtt}(L, t) = 0, \tag{10}$$

$$EIw_{xxx}(L, t) + \frac{M_p L}{J}u(t) - M_p w_{tt}(L, t) = 0, \qquad t > 0. \tag{11}$$

3. APPLICATION OF WPM

It is easy to see that the PDE (7) has the so-called wave solution

$$w(x,t) = \underbrace{A\,\mathrm{e}^{-ik(ax+kt)}}_{\text{wave I}} + \underbrace{B\,\mathrm{e}^{-ik(-ax+kt)}}_{\text{wave II}} + \underbrace{C\,\mathrm{e}^{-k(ax+ikt)}}_{\text{wave III}}$$

$$+ \underbrace{D\,\mathrm{e}^{-k[a(L-x)+ikt]}}_{\text{wave IV}} + \underbrace{F\,\mathrm{e}^{-ik^2t}x}_{w_p}, \tag{12}$$

where $a^4 = \rho/EI$, and where waves I and II are *dispersive waves* travelling to the left and right, respectively; waves III and IV are *evanescent waves* near the endpoints $x = 0$ and $x = L$, respectively; and w_p is a particular solution of the non-homogeneous equation. (Physically, w_p corresponds to the payload attached at $x = L$). A calculation (similar to that in reference [2]) shows that F must satisfy

$$J_0F = \left[\left(-\frac{\rho L}{ika} + \frac{\rho}{k^2a^2} + M_pL - ikaJ_p\right)\mathrm{e}^{-ikaL} - \frac{\rho}{k^2a^2}\right]A$$

$$+ \left[\left(\frac{\rho L}{ika} + \frac{\rho}{k^2a^2} + M_pL + ikaJ_p\right)\mathrm{e}^{ikaL} - \frac{\rho}{k^2a^2}\right]B$$

$$+ \left[\left(-\frac{\rho L}{ka} - \frac{\rho}{k^2a^2} + M_pL - kaJ_p\right)\mathrm{e}^{-kaL} + \frac{\rho}{k^2a^2}\right]C$$

$$+ \left[\frac{\rho L}{ka} - \frac{\rho}{k^2a^2} + M_pL + kaJ_p + \frac{\rho}{k^2a^2}\,\mathrm{e}^{-kaL}\right]D. \tag{13}$$

Now, the WPM estimates entail applying the BCs (8)–(11) to the wave solution (12) and neglecting terms of exponentially small order. Thus, near the endpoint $x = 0$, wave IV is neglected and BCs (8) and (9) are applied to

$$A\,\mathrm{e}^{-ik(ax+kt)} + B\,\mathrm{e}^{-ik(-ax+kt)} + C\,\mathrm{e}^{-k(ax+ikt)} + F\,\mathrm{e}^{-ik^2t}x,$$

resulting in

$$A + B + C = 0,$$

$$(u_1 + v_1\,\mathrm{e}^{ikaL})A + (u_2 + v_2\,\mathrm{e}^{ikaL})B + u_3C + v_4D = 0, \tag{14}$$

where, both presently and below, the notation used is

$$u_1 = -ikaJ_0 - \frac{\rho}{k^2a^2}, \qquad v_1 = -\frac{\rho L}{ika} + \frac{\rho}{k^2a^2} + M_pL - ikaJ_p,$$

$$u_2 = ikaJ_0 - \frac{\rho}{k^2a^2}, \qquad v_2 = \frac{\rho L}{ika} + \frac{\rho}{k^2a^2} + M_pL + ikaJ_p,$$

$$u_3 = -kaJ_0 + \frac{\rho}{k^2a^2}, \qquad v_3 = -\frac{\rho L}{ka} - \frac{\rho}{k^2a^2} + M_pL - kaJ_p,$$

$$u_4 = kaJ_0 + \frac{\rho}{k^2a^2}, \qquad v_4 = \frac{\rho L}{ka} - \frac{\rho}{k^2a^2} + M_pL + kaJ_p. \tag{15}$$

One may rewrite equation (14) as the *reflection relation*

$$\underbrace{\begin{bmatrix} 1 & 0 \\ u_3 & v_4 \end{bmatrix}}_{R_1}\begin{bmatrix} C \\ D \end{bmatrix} = -\underbrace{\begin{bmatrix} 1 & 1 \\ u_1 + v_1\,\mathrm{e}^{-ikaL} & u_2 + v_2\,\mathrm{e}^{ikaL} \end{bmatrix}}_{R_2}\begin{bmatrix} A \\ B \end{bmatrix}. \tag{16}$$

Similarly, near the endpoint $x = L$, BCs (10) and (11) are applied to

$$A\,\mathrm{e}^{-ik(ax+kt)} + B\,\mathrm{e}^{-ik(-ax+kt)} + D\,\mathrm{e}^{-k[a(L-x)+ikt]} + F\,\mathrm{e}^{-ik^2t}x,$$

which leads to the reflection relation

$$\underbrace{\begin{bmatrix} 0 & aEI - k^3J_p \\ 0 & a^3EI + kM_p \end{bmatrix}}_{R_3}\begin{bmatrix} C \\ D \end{bmatrix} = \underbrace{\begin{bmatrix} aEI - ik^3J_p & aEI + ik^3J_p \\ -ia^3EI - kM_p & ia^3EI - kM_p \end{bmatrix}}_{R_4}\begin{bmatrix} \mathrm{e}^{-ikaL} & 0 \\ 0 & e^{ikaL} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}. \tag{17}$$

Combining equations (16) and (17) one arrives at

$$R\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{18}$$

where

$$R = R_3R_1^{-1}R_2 + R_4\begin{bmatrix} \mathrm{e}^{-ikaL} & 0 \\ 0 & \mathrm{e}^{ikaL} \end{bmatrix}, \tag{19}$$

and equation (18) will hold for non-trivial $\begin{bmatrix} A \\ B \end{bmatrix}$ if and only if k is such that $\det R = 0$. After much simplification, one finds $\det R = 0$ if and only if

$$z\,\mathrm{e}^{ikaL} - \bar{z}\,\mathrm{e}^{-ikaL} + iy = 0, \tag{20}$$

where the real quantity y and the complex quantity z are given by

$$\begin{aligned} y = y(k) &= 2k^3a^3J_p[a^4E^2I^2 + aEIM_pk] + 2k^2a^2M_pL[a^4E^2I^2 - a^3EIJ_pk^3] \\ &\quad - 2\rho[a^3EIJ_pk^3 + J_pM_pk^4] + 2ka\rho L[-aEIM_pk + J_pM_pk^4], \end{aligned} \tag{21}$$

$$\begin{aligned} z = z(k) &= k^3a^3J_0[ia^4E^2I^2 - (1-i)aEIM_pk - (1+i)a^3EIJ_pk^3 - iJ_pM_pk^4] \\ &\quad + \rho[(1-i)a^4E^2I^2 + 2aEIM_pk + 2ia^3EIJ_pk^3 - (1-i)J_pM_pk^4]. \end{aligned} \tag{22}$$

In turn, equation (20) is equivalent to

$$(\mathrm{Re}\,z)\sin kaL + (\mathrm{Im}\,z)\cos kaL + y = 0. \tag{23}$$

At this point, the following can be seen:

Theorem 3.1. *For large* k, *one has*

$$kaL \approx \frac{(2n+1)\pi}{2}, \qquad n = 1, 2, \ldots .$$

Therefore, k is asymptotically distributed the same as for the clamped–free Euler–Bernoulli beam

$$EIw_{xxxx} + \rho w_{tt} = 0, \qquad 0 < x < L,\ t > 0;$$

$$w(0, t) = w_x(0, t) = w_{xx}(L, t) = w_{xxx}(L, t) = 0, \qquad t > 0.$$

Proof: Dividing equation (23) by k^7 and neglecting terms of order $1/k$ or smaller, one gets

$$\cos kaL = 0$$

which implies that $kaL = (2n + 1)\pi/2$. (Note that the above still holds when there is no payload, i.e., when $J_p = M_p = 0$.) Therefore, for large k, the dominant term in the asymptotic expansion for k is the same as that for the clamped–free beam (see reference [8], for example).

4. A PERTURBATION PROCEDURE FOR IMPROVING THE ACCURACY OF WPM

A perturbation method is now developed formally, as in reference [8], in order to improve the accuracy of the WPM calculations. First note that, if the BCs (8)–(11) are applied to the wave solution (12) without neglecting terms of exponentially small order, the result can be written in matrix form as

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & e^{-kaL} \\ u_1 + v_1\, e^{-ikaL} & u_2 + v_2\, e^{ikaL} & u_3 + v_3\, e^{-kaL} & u_4\, e^{-kaL} + v_4 \\ (-aEI + ik^3J_p)\, e^{-ikaL} & -(aEI + ik^3J_p)\, e^{ikaL} & (aEI + k^3J_p)\, e^{-kaL} & aEI - k^3J_p \\ (ia^3EI + kM_p)\, e^{-ikaL} & (-ia^3EI + kM_p)\, e^{ikaL} & (-a^3EI + kM_p)\, e^{-kaL} & a^3EI + kM_p \end{bmatrix}}_{M} \cdot \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{24}$$

where u_i, v_i, $i = 1, 2, 3, 4$, are given by equations (15). Now, the exact natural frequencies are those k for which $\det M = 0$, and this equation is equivalent to equation (48) in reference [2]. However, rather than solving $\det M = 0$, one defines the matrix

$$M_\varepsilon = \begin{bmatrix} 1 & 1 & 1 & \varepsilon \\ u_1 + v_1\, e^{-ikaL} & u_2 + v_1\, e^{ikaL} & u_3 + v_3\varepsilon & u_4\varepsilon + v_4 \\ (-aEI + ik^3J_p)\, e^{-ikaL} & -(aEI + ik^3J_p)\, e^{ikaL} & (aEI + k^3J_p)\varepsilon & aEI - k^3J_p \\ (ia^3EI + kM_p)\, e^{-ikaL} & (ia^3EI + kM_p)\, e^{ikaL} & (-a^3EI + kM_p)\varepsilon & a^3EI + kM_p \end{bmatrix}. \tag{25}$$

Note that when $\varepsilon = e^{-kaL}$, M_ε is the matrix M in equation (24); when $\varepsilon = 0$, $\det M_0 = 0$ leads to the WPM approximation.

Now, the perturbation method proceeds as follows. Let

$$k = k_0 + \varepsilon k_1 + O(\varepsilon^2), \tag{26}$$

in which case one also has

$$\mathrm{e}^{\pm ikaL} = \mathrm{e}^{\pm ik_0 aL}(1 \pm ik_1 aL\varepsilon) + O(\varepsilon^2), \tag{27}$$

$$k^n = k_0^n + nk_0^{n-1}k_1\varepsilon + O(\varepsilon^2), \qquad n \in \mathbb{Z}^+. \tag{28}$$

Then calculate det M_ε:

$$\det M_\varepsilon = f_1(k_0) + \varepsilon[f_2(k_0) + k_1 f_3(k_0)] + O(\varepsilon^2), \tag{29}$$

where

$$f_1(k_0) = \det M_0 \text{ (so } \det M_0 = 0 \Rightarrow k_0 \text{ results from WPM)},$$

$$f_2(k_0) = 8ik_0^3(a^5 J_0 E^2 I^2 + k_0^4 a J_0 J_p M_p) + z_1\,\mathrm{e}^{ik_0 aL} - \bar{z}_1\,\mathrm{e}^{-ik_0 aL},$$

$$f_3(k_0) = 4i\Bigg(5k_0^4\frac{\rho L J_p M_p}{a} - 3k_0^2 a\rho J_p EI - 4k_0^3\frac{\rho J_p M_p}{a^2} + 2k_0 a^4 M_p L E^2 I^2 - 5k_0 a^3 M_p L J_p EI$$
$$+ 3k_0^2 a^5 J_p E^2 I^2 - 4ik_0^3 a^2 EIJ_p M_p\Bigg) + z_2\,\mathrm{e}^{ik_0 aL} - \bar{z}_2\,\mathrm{e}^{-ik_0 aL}, \tag{30}$$

where z_1 and z_2 are given by

$$z_1 = 4k_0^2\Bigg(-\rho LEIM_p - ik_0^3\frac{\rho L J_p M_p}{a} - ik_0 a\rho EIJ_p + k_0^2\frac{\rho J_p M_p}{a^2} + a^4 M_p L E^2 I^2$$
$$+ ik_0^3 a^3 M_p LEIJ_p + ik_0 a^5 J_p E^2 I^2 - k_0^2 a^2 J_p EIM_p\Bigg),$$

$$z_2 = 2\Bigg[3ik_0^2 a^5 J_0 E^2 I^2 - 6(1+i)k_0^5 a^4 J_0 J_p EI + 4(-1+i)k_0^3 a^2 J_0 EIM_p - 7ik_0^6 a J_0 J_p M_p$$
$$+ 6ik_0^2 a\rho J_p EI + 2\frac{\rho EIM_p}{a} + 4(-1+i)k_0^3\frac{\rho J_p M_p}{a^2}\Bigg] + 2iaL\Bigg[ik_0^3 a^5 J_0 E^2 I^2$$
$$- (1+i)k_0^6 a^4 J_0 J_p EI + (-1+i)k_0^4 a^2 J_0 EIM_p - ik_0^7 J_0 J_p M_p + (1-i)a^2\rho E^2 I^2$$
$$+ 2ik_0^3 a\rho J_p EI + 2k_0\frac{\rho EIM_p}{a} + (-1+i)k_0^4\frac{\rho J_p M_p}{a^2}\Bigg]. \tag{31}$$

Next, as noted above, $f_1(k_0) = 0$ if and only if k_0 results from the application of WPM. Then, the perturbation coefficient k_1 is determined by requiring the coefficient of ε to vanish, i.e., one requires

$$k_1 = -\frac{f_2(k_0)}{f_3(k_0)}. \tag{32}$$

Finally, an appropriate choice for ε is needed:

$$\varepsilon = \varepsilon_0 = e^{-k_0 aL},$$

is chosen from which the first improvement of WPM is obtained:

$$k_{01} = k_0 + \varepsilon_0 k_1.$$

One may now update the improvement by letting

$$\varepsilon = \varepsilon_1 = e^{-k_{01} aL},$$

with the improved result

$$k_{02} = k_0 + \varepsilon_1 k_1.$$

Proceeding recursively, and letting $k_{00} = k_0$, one has

$$k_{0,n+1} = k_0 + \varepsilon_n k_1 = k_0 + k_1 \exp(-k_{0n} aL), \qquad n = 0, 1, 2, \ldots .$$

5. RESULTS AND COMPARISONS

The methods developed above are now applied to the two specific cases treated in reference [2] (which were chosen because they were used to come up with the experimental results in references [1] and [9]). First note that, in each of these cases, $J_p = M_p = 0$; the WPM and perturbation calculations then become much simpler. WPM equation (20) becomes

$$\tan k_0 aL = \frac{\rho - a^3 J_0 k_0^3}{\rho} \tag{33}$$

or

$$k_0 aL + n\pi + \tan^{-1}\left(\frac{-\rho + a^3 J_0 k_0^3}{\rho}\right) = 0, \qquad n \in \mathbb{Z}. \tag{34}$$

As for equation (32), one obtains

$$k_1 = -\frac{2k_0^3 a^2 J_0}{(3k_0^2 a^2 J_0 + \rho L)\cos k_0 aL + L(-k_0^3 a^3 J_0 + \rho)\sin k_0 aL}. \tag{35}$$

Table 1 lists the values of the physical constants used for the calculations in reference [2] and also for the calculations given below.

TABLE 1

The values of the physical constants from reference [2] *and used for the calculations in Tables* 2 *and* 3

Constant	Value (Table 2)	Value (Table 3)
E	$2{\cdot}1 \times 10^{11}$ N/m^2	$6{\cdot}9 \times 10^{10}$ N/m^2
I	$1{\cdot}167 \times 10^{-11}$ m^4	$8{\cdot}31934 \times 10^{-11}$ m^4
L	0·7 m	1·0 m
ρ	2·646 kg/m	0·233172 kg/m
J_0	$1{\cdot}3 \times 10^{-3}$ (kg)m^2	$5{\cdot}176 \times 10^{-3}$ (kg)m^2
J_p	0 (kg)m^2	0 (kg)m^2
M_p	0 kg	0 kg

TABLE 2

Comparison of the first 30 *eigenfrequencies* $\beta = \alpha k$ *for the slewing beam with physical data given in column two of Table* 1

"Exact" (β)	WPM (β_0)	β_{01}	β_{02}	β_{03}	Classical clamped	Classical clamped
5·5504	5·5474	5·5510	5·5509	5·5509	2·6789	5·6094
9·6955	9·6966	9·6955	9·6955		6·7059	10·098
13·258	13·258				11·221	14·586
16·660	16·660				15·709	19·074
20·616	20·616				20·196	23·561
24·899	24·899				24·684	28·050
29·297	29·297				29·172	32·538
33·740	33·740				33·660	37·026
38·202	38·202				38·148	41·514
42·674	42·674				42·646	46·002
47·152	47·152				47·124	50·490
51·633	51·633				51·612	54·978
56·117	56·117				56·100	59·466
60·601	60·601				60·588	63·954
65·086	65·086				65·076	68·442
69·573	69·573				69·564	72·930
75·059	74·059				74·052	77·418
78·546	74·546				78·540	81·906
83·033	83·033				83·028	86·394
87·520	87·520				87·516	90·882
92·008	92·008				92·004	95·370
96·495	96·495				96·492	99·858
100·98	100·98				100·98	104·35
105·47	105·47				105·47	108·83
109·96	109·96				109·96	113·32
114·45	114·45				114·44	117·81
118·93	118·93				118·93	122·30
123·42	123·43				123·42	126·79
127·91	127·91				127·91	131·27
132·40	132·40				132·40	135·77

In each of Tables 2 and 3, the first 30 eigenfrequencies, $\beta = \alpha k$, are listed for the slewing beam with data listed in Table 1 (note that the tables in reference [2] actually list the values for β^2). The first column contains the "exact" frequencies, i.e., those which result from solving the characteristic equation derived in references [1] and [2] (again, reference [2], equation (48)), namely

$$\rho^3 \cos \beta L \sinh \beta L - \rho^3 \sin \beta L \cosh \beta L - 2\rho^2 M_p \beta \sin \beta L \sinh \beta L - 2\rho^2 J_p \beta^3 \cos \beta L \cosh \beta L - \rho^2 J_0 \beta^3 (1 + \cos \beta L \cosh \beta L) - \rho M_p \beta^4 (J_0 + J_p) \cos \beta L \sinh \beta L - \sin \beta L \cosh \beta L) + \rho J_0 J_p \beta^6 (\cos \beta L \sinh \beta L + \sin \beta L \cosh \beta L) - J_0 J_p M_p \beta^7 (1 - \cos \beta L \cosh \beta L) = 0. \tag{36}$$

TABLE 3

Comparison of the first 30 eigenfrequencies $\beta = \alpha k$ for the slewing beam with physical data given in column three of Table 1

"Exact" (β)	WPM (β_0)	WPM β_{01}	WPM β_{02}	WPM β_{03}	Classical clamped	Classical clamped
3·3563	3·3242	3·3568	3·3568	3·3568	1·8752	3·9266
5·1531	5·1647	5·1533	5·1531	4·4941	4·6941	7·0686
7·9528	7·9521	7·9528	7·9528		7·8548	10·210
11·030	11·030				10·996	13·352
14·153	14·153				14·137	16·493
17·288	17·288				17·279	19·635
20·426	20·426				20·420	22·777
23·565	23·565				23·562	25·918
26·706	26·706				26·704	29·060
29·847	29·847				29·845	32·201
32·988	32·988				32·987	35·343
36·129	36·129				36·128	38·485
39·271	39·271				39·270	41·626
42·412	42·412				42·412	44·768
45·554	45·554				45·553	47·909
48·695	48·695				48·695	51·051
51·837	51·837				51·836	54·192
54·978	54·978				54·978	57·334
58·120	58·120				58·119	60·476
61·261	61·261				61·261	63·617
64·403	64·403				64·403	66·759
67·544	67·544				67·544	69·900
70·686	70·686				70·686	73·042
73·828	73·828				73·828	76·184
76·969	76·969				76·969	79·325
80·111	80·111				80·111	82·467
83·252	83·252				83·252	85·608
86·394	86·394				86·394	88·750
89·535	89·535				89·535	91·892
92·677	92·677				92·677	95·033

The equation was solved using the IMSL routine DNEQNF, and all results converge to at least eight digits. All computations were performed on the DEC Alpha 2100 at Fairfield University.

The second column of each table contains the WPM calculations $\beta_0 = \alpha k_0$, and these are followed by the perturbation calculations $\beta_{01} = \alpha k_{01}$, $\beta_{02} = \alpha k_{02}$, etc. (until agreement or near-agreement is reached with the corresponding "exact" eigenfrequency in column one). Again, the WPM equation was solved using DNEQNF.

The final two columns contain, respectively, the corresponding frequencies for the classical Euler–Bernoulli clamped–free and simply-supported (pinned)–free beams. These frequencies were calculated by the author using the Legendre-tau spectral method, and they agree with the values given elsewhere in the literature (e.g., in reference [10]).

One sees that the exact and the WPM frequencies are in agreement except for the first few, for which the perturbation calculations give agreement or near-agreement. Also, in each table, the convergence of the exact frequencies to those of the clamped–free beam can be seen, as predicted by the theorem, above. Finally, the tables seem to suggest that one always has

$$\beta_c \leqslant \beta \leqslant \beta_p, \tag{37}$$

TABLE 4

Comparison of the squares (β^2) of the first five "exact" eigenfrequencies from Table 2, and the first two from Table 3, with those (α) obtained in reference [2]

β	β^2	α	β^2/α
5·5504	30·807	4·719	6·528
9·6955	94·003	14·40	6·528
13·258	175·77	26·92	6·529
16·660	277·56	42·51	6·529
20·616	425·02	65·10	6·529
3·3563	11·265	8·896	1·266
5·1531	26·554	20·97	1·266

where β_c and β_p are the corresponding clamped–free and pinned–free frequencies, respectively. This result is also suggested by WPM, as follows. The WPM equation for β_c is

$$\cos \beta_c L = 0 \Rightarrow \beta_c L = (2n + 1)\pi/2, \qquad n \in \mathbb{Z}, \tag{38}$$

while for β_s it is

$$\tan \beta_s L = 1 \Rightarrow \beta_s L = (4n + 1)\pi/4, \qquad n \in \mathbb{Z}, \tag{39}$$

(see reference [8], for example). First, rewrite equation (33) as

$$\cos \beta_0 L = \frac{\rho}{\rho - \beta_0^3 J_0} \sin \beta_0 L. \tag{40}$$

One sees that, if $\beta_0 L \approx (2n + 1)\pi/2$, then we must have $\beta_0 L \geqslant (2n + 1)\pi/2$ in order for the signs to "work out". Therefore, $\beta_0 \geqslant \beta_c$.

Similarly, if one rewrites equation (33) as

$$\tan \beta_0 L = 1 - \beta_0^2 J_0/\rho, \tag{41}$$

one sees that, if $\beta_0 L \approx (4n + 1)\pi/4$, then one must have $\beta_0 L \leqslant (4n + 1)\pi/4$, i.e., $\beta_s \geqslant \beta_0$.

In conclusion, it is seen that WPM allows for accurate computation of the vibration spectrum of the flexible slewing beam for all but the few lowest eigenfrequencies, for which it is found that WPM combined with the perturbation procedure gives accurate values. Therefore, it is possible to compute the entire spectrum to a high degree of accuracy, with a minimal amount of computation.

Finally, it should be mentioned here that Table 4 constitutes a justification of the author's earlier statement that the results obtained in reference [2] are a constant multiple of those presented here.

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