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IN-PLANE VIBRATION OF CIRCULAR ARCHES WITH VARIABLE CROSS-SECTIONS

X. TONG

Centre for Surface Transportation Technology, National Research Council Canada, Ottawa, Ontario, Canada K1A 0R6

N. MRAD

Institute for Aerospace Research, National Research Council Canada, Ottawa, Ontario, Canada K1A 0R6

AND

B. TABARROK

Department of Mechanical Engineering, University of Victoria, Victoria, British Columbia, Canada V8W 3P6

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Free and forced in-plane vibrations of circular arches with variable cross-sections are investigated. Using the Kirchhoff assumptions for thin beams and taking the neutral axis as inextensible, a closed form solution is obtained for circular arches of uniform cross-section. This exact solution is used for circular arches with stepped cross-sections and is applied to obtain an approximate solution for arches with non-uniform cross-sections. For free vibration, an analytic form of frequency equation is obtained by using the general solution expressed in terms of some initial parameters at one end of the arch; while for forced vibration, the system's response is obtained analytically by solving a set of algebraic equations with only three unknowns. Several examples are presented to illustrate the validity and accuracy of the method.

1. INTRODUCTION

Vibration analysis of arches under various kinds of loads has been the subject of numerous investigations [1–3] due to their important applications in many industrial fields. It is well known that the governing dynamic equation of inextensible Bernoulli–Euler arches with constant cross-sections is a sixth order differential equation with constant coefficients. The exact solutions for the free and forced vibrations of uniform Bernoulli–Euler arches can be found in references [1, 4]. However, it is more difficult to find general closed form solutions for the dynamic response of arches with arbitrarily varying cross-sections since the governing equations of such arches possess variable coefficients. Therefore, in the past many methods, such as the finite element method [5, 6], the Rayleigh–Ritz method [7–11], the cell discretization method [3, 12], and the correlation matrix method [13], have been proposed for investigating these arches' dynamic behaviour. Although these methods have been proven useful for vibration analysis of arches, they either require cumbersome computation as the number of discrete elements increase, or are restricted by their rate of convergence.

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In the following, a systematic approach is presented for investigating the free and forced vibrations of inextensible Bernoulli–Euler arches with arbitrarily varying cross-sections, using an approach, developed for non-uniform beams [14–16]. For this particular purpose, the arch with arbitrarily varying cross-sections is approximated by a number of stepped arches with constant cross-sections. For each stepped arch element, an analytic solution, which is expressed in terms of six initial parameters (deflections, rotation, bending moment, shear force and normal force) at one end of each stepped arch, may be obtained by solving the governing equation with constant coefficients. Then, the overall solution of the stepped arches can be expressed in terms of the end parameters at one end of the arch by satisfying the continuity and equilibrium conditions between adjacent elements. In the case of free vibration, the frequency equation under various boundary conditions is shown to have an analytic form in terms of some physical parameters; while in the case of forced vibration, the system's response can also be obtained analytically by solving a set of algebraic equations with only three unknowns, independent of the numbers of elements used in the computational model. As the number of stepped arches increases, a fast convergence to the exact solution of the original arch is obtained. Several examples illustrating the validity and accuracy of this method are presented.

2. GOVERNING EQUATIONS

Consider a thin circular arch with a variable cross-section, as shown in Figure 1. The equation of motion without taking into account the effects of shear deformation and rotary inertia are [1]

$$\frac{\partial T(\theta, t)}{\partial \theta} + N(\theta, t) + Rq_u(\theta, t) - \mu(\theta) R \frac{\partial^2 u(\theta, t)}{\partial t^2} = 0, \tag{1}$$

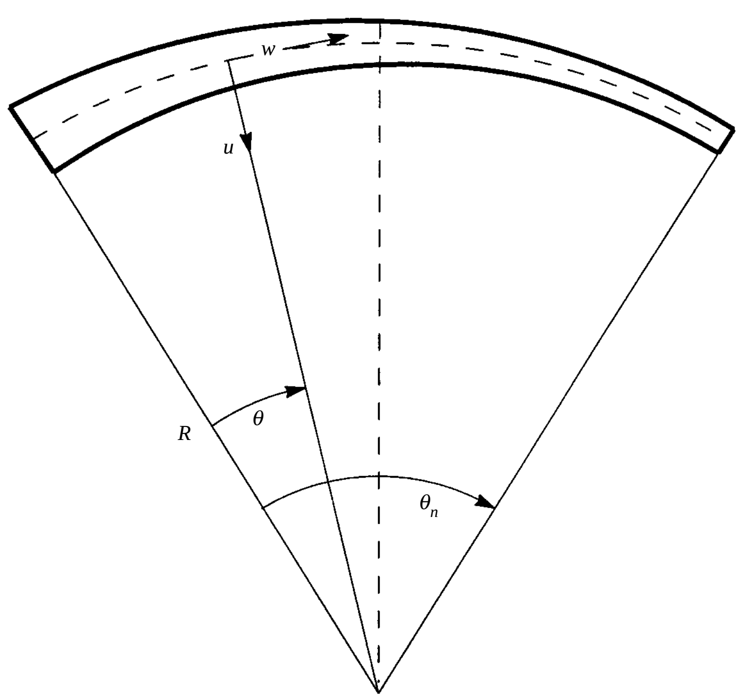

Figure 1. A circular arch with variable cross-section.

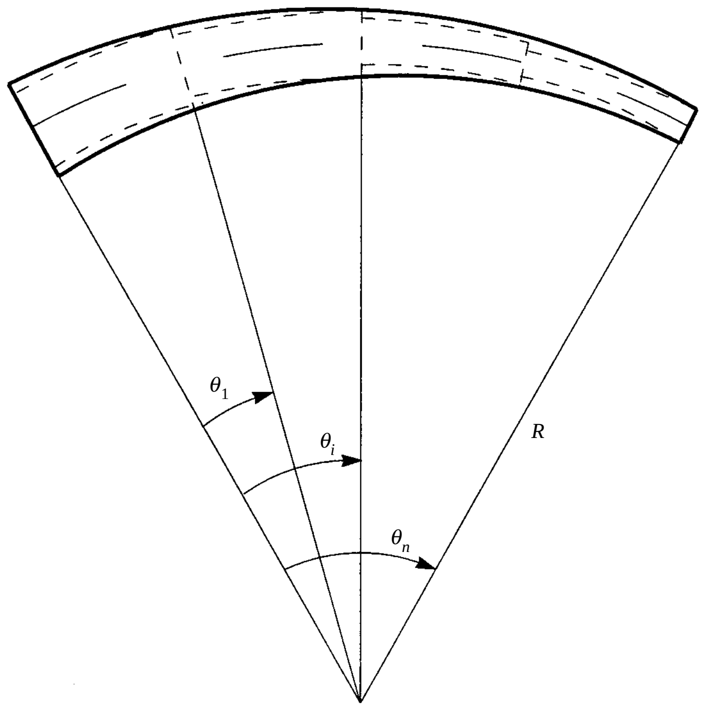

Figure 2. An arch represented by a number of stepped arches.

$$\frac{\partial N(\theta, t)}{\partial \theta} - T(\theta, t) + Rq_w(\theta, t) - \mu(\theta)R\frac{\partial^2 w(\theta, t)}{\partial t^2} = 0, \tag{2}$$

$$\frac{\partial M(\theta, t)}{\partial \theta} - RT(\theta, t) = 0, \tag{3}$$

where $T(\theta, t)$ denotes the shear force, $N(\theta, t)$ the normal force, $M(\theta, t)$ the bending moment, $\mu(\theta)$ the mass per unit length ($\rho A(\theta)$), and R the radius of the circular arch. The components of the external load in the normal and tangential directions are denoted by $q_u(\theta, t)$ and $q_w(\theta, t)$, respectively.The flexural deformations are more important than the axial deformation for the lowest modes of vibration, so that it is possible to neglect the extensibility of the arch's neutral axis. The inextensibility condition is written as

$$u = \frac{\partial w}{\partial \theta}, \tag{4}$$

whereas the bending moment can be expressed as

$$M(\theta, t) = -\frac{EI(\theta)}{R^2}\left(\frac{\partial^2 u}{\partial \theta^2} + u\right) = -\frac{EI(\theta)}{R^2}\left(\frac{\partial^3 w}{\partial \theta^3} + \frac{\partial w}{\partial \theta}\right), \tag{5}$$

where E is the Young modulus and $I(\theta)$ is the second moment of area. Substituting equation (5) into equation (3), one obtains the shear force as

$$T(\theta, t) = -\frac{1}{R^3}\frac{\partial}{\partial\theta}\left[EI(\theta)\left(\frac{\partial^3 w}{\partial\theta^3} + \frac{\partial w}{\partial\theta}\right)\right]. \tag{6}$$

From equations (6) and (1), the following relation identifying the normal force is obtained as

$$N(\theta, t) = -\frac{\partial T(\theta, t)}{\partial\theta} - Rq_u(\theta, t) + \mu(\theta)R\frac{\partial^2 u(\theta, t)}{\partial t^2}$$

$$= \frac{1}{R^3}\frac{\partial^2}{\partial\theta^2}\left[EI(\theta)\left(\frac{\partial^3 w}{\partial\theta^3} + \frac{\partial w}{\partial\theta}\right)\right] - Rq_u(\theta, t) + \mu(\theta)R\frac{\partial^3 w(\theta, t)}{\partial\theta\,\partial t^2}. \tag{7}$$

TABLE 1

Frequency equations under various boundary conditions

Types	Boundary conditions	Frequency equations
Clamped–clamped	$\theta = 0,\ W = W = \Psi = 0$ $\theta = \theta_n,\ W = W' = \Psi = 0$	$b_{14}b_{25}b_{36} + b_{24}b_{35}b_{16}$ $+b_{34}b_{26}b_{15} - b_{16}b_{25}b_{34}$ $-b_{26}b_{35}b_{14} - b_{36}b_{24}b_{15} = 0$
Hinged–hinged	$\theta = 0,\ W = W' = M = 0$ $\theta = \theta_n,\ W = W' = M = 0$	$b_{13}b_{25}b_{46} + b_{23}b_{45}b_{16}$ $+b_{43}b_{26}b_{15} - b_{16}b_{25}b_{43}$ $-b_{26}b_{45}b_{13} - b_{46}b_{23}b_{15} = 0$
Free–free	$\theta = 0,\ M = T = N = 0$ $\theta = \theta_n,\ M = T = N = 0$	$b_{41}b_{52}b_{63} + b_{51}b_{62}b_{43}$ $+b_{61}b_{53}b_{42} - b_{43}b_{52}b_{61}$ $-b_{53}b_{62}b_{41} - b_{63}b_{51}b_{42} = 0$
Hinged–clamped	$\theta = 0,\ W = W' = M = 0$ $\theta = \theta_n,\ W = W' = \Psi = 0$	$b_{14}b_{25}b_{46} + b_{24}b_{45}b_{16}$ $+b_{44}b_{26}b_{15} - b_{16}b_{25}b_{44}$ $-b_{26}b_{45}b_{14} - b_{46}b_{24}b_{15} = 0$
Hinged–free	$\theta = 0,\ W = W' = M = 0$ $\theta = \theta_n,\ M = T = N = 0$	$b_{11}b_{22}b_{43} + b_{21}b_{42}b_{13}$ $+b_{41}b_{23}b_{12} - b_{13}b_{22}b_{41}$ $-b_{23}b_{42}b_{11} - b_{43}b_{21}b_{12} = 0$
Clamped–hinged	$\theta = 0,\ W = W' = \Psi = 0$ $\theta = \theta_n,\ W = W' = M = 0$	$b_{13}b_{25}b_{36} + b_{23}b_{35}b_{16}$ $+b_{33}b_{26}b_{15} - b_{16}b_{25}b_{33}$ $-b_{26}b_{35}b_{13} - b_{36}b_{23}b_{15} = 0$
Clamped–free	$\theta = 0,\ W = W' = \Psi = 0$ $\theta = \theta_n,\ M = T = N = 0$	$b_{11}b_{22}b_{33} + b_{21}b_{32}b_{13}$ $+b_{31}b_{23}b_{12} - b_{13}b_{22}b_{31}$ $-b_{23}b_{32}b_{11} - b_{33}b_{21}b_{12} = 0$
Free–clamped	$\theta = 0,\ M = T = N = 0$ $\theta = \theta_n,\ W = W' = \Psi = 0$	$b_{44}b_{55}b_{66} + b_{54}b_{65}b_{46}$ $+b_{64}b_{56}b_{45} - b_{46}b_{55}b_{64}$ $-b_{56}b_{65}b_{44} - b_{66}b_{54}b_{45} = 0$
Free–hinged	$\theta = 0,\ M = T = N = 0$ $\theta = \theta_n,\ W = W' = M = 0$	$b_{43}b_{55}b_{66} + b_{53}b_{65}b_{46}$ $+b_{63}b_{56}b_{45} - b_{46}b_{55}b_{63}$ $-b_{56}b_{65}b_{43} - b_{66}b_{53}b_{45} = 0$

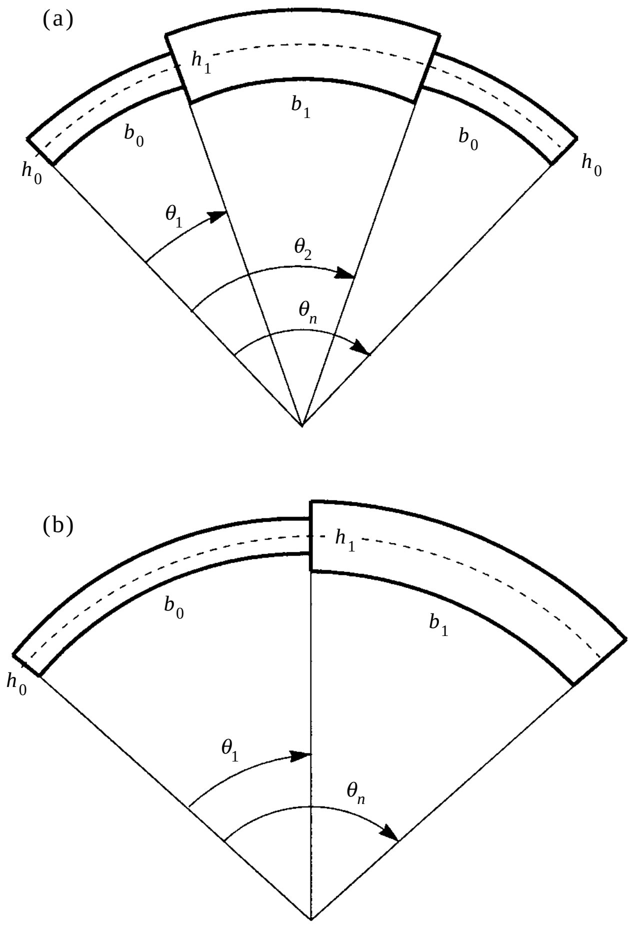

Figure 3. Two stepped arches ($\eta = h_1/h_0$ and $b_1 = b_0$): (a) symmetric stepped arch; (b) unsymmetric stepped arch.

Then, by substituting equations (6) and (7) into equation (12), the equation of motion for the deflection component w can be written as

$$\frac{\partial^3}{\partial\theta^3}\left[EI(\theta)\left(\frac{\partial^3 w(\theta,t)}{\partial\theta^3}+\frac{\partial w(\theta,t)}{\partial\theta}\right)\right]+\frac{\partial}{\partial\theta}\left[EI(\theta)\left(\frac{\partial^3 w(\theta,t)}{\partial\theta^3}+\frac{\partial w(\theta,t)}{\partial\theta}\right)\right]$$

$$+R^4\frac{\partial}{\partial\theta}\left[\mu(\theta)\frac{\partial^3 w(\theta,t)}{\partial\theta\,\partial t^2}\right]-\mu(\theta)R^4\frac{\partial w(\theta,t)}{\partial t^2}-R^4\frac{\partial q_u(\theta,t)}{\partial\theta}+R^4 q_w(\theta,t)=0, \qquad (8)$$

where the boundary conditions are:

(1) clamped,

$$u=0,\quad w=0,\quad \psi=0 \qquad \text{at} \qquad \theta=0 \quad \text{or} \quad \theta=\theta_n; \qquad (9)$$

(2) hinged,

$$u=0,\quad w=0,\quad M=0 \qquad \text{at} \qquad \theta=0 \quad \text{or} \quad \theta=\theta_n; \qquad (10)$$

(3) free,

$$M=0,\quad T=0,\quad N=0 \qquad \text{at} \qquad \theta=0 \quad \text{or} \quad \theta=\theta_n; \qquad (11)$$

and

$$u = \frac{\partial w}{\partial \theta}, \qquad \psi = \frac{1}{R}\left(\frac{\partial^2 w}{\partial \theta^2} + w\right),$$

$$M = -\frac{EI}{R^2}\left(\frac{\partial^3 w}{\partial \theta^3} + \frac{\partial w}{\partial \theta}\right), \qquad T = -\frac{1}{R^3}\frac{\partial}{\partial \theta}\left[EI(\theta)\left(\frac{\partial^3 w}{\partial \theta^3} + \frac{\partial w}{\partial \theta}\right)\right],$$

$$N = \frac{1}{R^3}\frac{\partial^2}{\partial \theta^2}\left[EI(\theta)\left(\frac{\partial^3 w}{\partial \theta^3} + \frac{\partial w}{\partial \theta}\right)\right] - Rq_u(\theta, t) + \mu(\theta)R\,\frac{\partial^3 w(\theta, t)}{\partial \theta\,\partial t^2}. \tag{12}$$

3. FREE VIBRATIONS

Consider the thin circular arch with an arbitrarily varying cross-section (Figure 1). In order to determine the solution of equation (8), one may divide this arch into a number of stepped arches with constant cross-sections, as illustrated in Figure 2. For the *i*th stepped arch element, the equation of motion (8) can be written as

$$\frac{\partial^6 w(\theta, t)}{\partial \theta^6} + 2\frac{\partial^4 w(\theta, t)}{\partial \theta^4} + \frac{\partial^2 w(\theta, t)}{\partial \theta^2} + \frac{\mu_{i-1}R^4}{EI_{i-1}}\frac{\partial^4 w(\theta, t)}{\partial \theta^2\,\partial t^2} - \frac{\mu_{i-1}R^4}{EI_{i-1}}\frac{\partial^2 w(\theta, t)}{\partial t^2}$$

$$= \frac{R^4}{EI_{i-1}}\frac{\partial q_{u_{i-1}}(\theta, t)}{\partial \theta} - \frac{R^4}{EI_{i-1}}\,q_{w_{i-1}}(\theta, t). \tag{13}$$

TABLE 2

First frequency coefficient of a symmetric stepped arch

θ_n (degrees)	Present method	χ_1 R-R [9]	F.E.M. [3]	C.D.M. [3]
$\eta = 0{\cdot}8$				
10	1844·84	1958·85		1840·9
20	459·662	489·30	456·31	458·68
30	203·157			202·72
40	113·392	121·874	113·195	113·15
45	89·175	96·1659		88·897
50	71·856			71·705
60	49·306			49·200
70	35·722			35·647
80	26·918			26·86
90	20·895	23·599		20·851
$\eta = 1{\cdot}2$				
10	2119·46	2082·9		2102·2
20	527·201	520·08	521·80	523·31
30	232·831			230·93
40	129·683	129·42	129·30	128·63
45	101·861	102·12		101·03
50	81·965			81·299
60	56·068			55·613
70	40·477			40·148
80	30·380			30·134
90	23·480	23·599		23·290

TABLE 3

First frequency coefficient of an unsymmetric stepped arch

θ_n (degrees)	χ_1 Present method	R-R [9]	F.E.M. [3]	C.D.M. [3]
(a) Clamped–clamped				
10	2277·412	2277·9		2264·9
20	567·170	567·10	566·86	564·05
30	250·472	250·37		249·10
40	139·647	139·62	139·72	138·88
50	88·372	88·439		87·887
60	60·538	60·540	60·604	60·206
(b) Hinged–hinged				
10	1458·852	1462·16		1456·0
20	362·609	363·32	362·667	361·92
30	159·625	160·128		159·33
40	88·601	88·7588	88·697	88·440
50	55·750	55·8865		55·651
60	37·926	37·989	38·007	37·862
(c) Hinged–clamped				
10	1853·663	1868·5		1848·4
20	461·342	464·76	461·15	460·03
30	203·520	205·03		202·95
40	113·014	114·16	113·36	112·98
50	71·563	72·103		71·363
60	48·910	49·269	48·978	48·775

Let $w(\theta, t)/R = W(\theta)\,\mathrm{e}^{j\omega t}$; the equation of motion (13) can be reduced to

$$\frac{\mathrm{d}^6 W}{\mathrm{d}\theta^6} + 2\frac{\mathrm{d}^4 W}{\mathrm{d}\theta^4} + \frac{\mathrm{d}^2 W}{\mathrm{d}\theta^2} - \chi_{i-1}^2 \frac{\mathrm{d}^2 W}{\mathrm{d}\theta^2} + \chi_{i-1}^2 W = 0, \tag{14}$$

for free vibration and

$$\frac{\mathrm{d}^6 W}{\mathrm{d}\theta^6} + 2\frac{\mathrm{d}^4 W}{\mathrm{d}\theta^4} + \frac{\mathrm{d}^2 W}{\mathrm{d}\theta^2} - \chi_{i-1}^2 \frac{\mathrm{d}^2 W}{\mathrm{d}\theta^2} + \chi_{i-1}^2 W = F_{i-1}(\theta), \tag{15}$$

for harmonic forced vibration where the non-dimensional frequency coefficient is

$$\chi_{i-1} = \sqrt{\mu_{i-1} R^4/EI_{i-1}}\,\omega, \tag{16}$$

the forcing function is

$$F_{i-1}(\theta)\,\mathrm{e}^{j\omega t} = \frac{R^3}{EI_{i-1}} \frac{\partial q_{u_{i-1}}(\theta, t)}{\partial \theta} - \frac{R^3}{EI_{i-1}} q_{w_{i-1}}(\theta, t), \tag{17}$$

and the continuity and equilibrium conditions at $\theta = \theta_i$ require that

$$\lim_{\epsilon\to 0} W(\theta_i - \epsilon) = W(\theta_i), \qquad \lim_{\epsilon\to 0} W'(\theta_i - \epsilon) = W'(\theta_i),$$
$$\lim_{\epsilon\to 0} \Psi(\theta_i - \epsilon) = \Psi(\theta_i), \qquad \lim_{\epsilon\to 0} M(\theta_i - \epsilon) = M(\theta_i),$$
$$\lim_{\epsilon\to 0} T(\theta_i - \epsilon) = T(\theta_i), \qquad \lim_{\epsilon\to 0} N(\theta_i - \epsilon) = N(\theta_i). \tag{18}$$

In the ith stepped arch, the solution of free vibration (14) can be expressed in terms of the initial parameters (deflections, rotation, bending moment, shear force and normal force) at $\theta = \theta_{i-1}$, as

$$\{\delta(\theta)\} = \mathbf{A}^i(\theta, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}})\{\delta(\theta_{i-1})\}, \qquad \theta_{i-1} \leqslant \theta < \theta_i, \tag{19}$$

where

$$\{\delta(\theta)\} = \left\{W(\theta), W'(\theta), \Psi(\theta), \frac{M(\theta)R}{EI_0}, \frac{T(\theta)R^2}{EI_0}, \frac{N(\theta)R^2}{EI_0}\right\}^T, \tag{20}$$

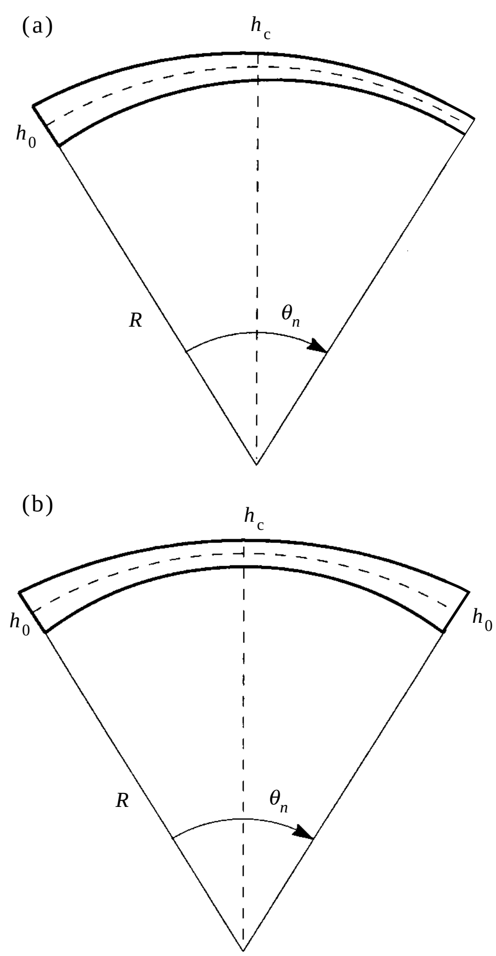

Figure 4. Two tapered arches: (a) unsymmetric tapered arch; (b) symmetric stepped arch.

$$\mathbf{A}^i = \begin{bmatrix} f_1 & g_1 & f_2 & -u_{i-1}g_2 & -u_{i-1}f_3 & u_{i-1}g_3 \\ f_1' & g_1' & f_2' & -u_{i-1}g_2' & -u_{i-1}f_3' & u_{i-1}g_3' \\ f_1 + f_1'' & g_1 + g_1'' & f_2 + f_2'' & -u_{i-1}(g_2 + g_2'') & -u_{i-1}(f_3 + f_3'') & u_{i-1}(g_3 + g_3'') \\ -(f_1' + f_1''')/u_{i-1} & -(g_1' + g_1''')/u_{i-1} & -(f_2' + f_2''')/u_{i-1} & g_2' + g_2''' & f_3' + f_3''' & -(g_3' + g_3''') \\ -(f_1'' + f_1^{iv})/u_{i-1} & -(g_1'' + g_1^{iv})/u_{i-1} & -(f_2'' + f_2^{iv})/u_{i-1} & g_2'' + g_2^{iv} & f_3'' + f_3^{iv} & -(g_3'' + g_3^{iv}) \\ (f_1''' + f_1^{V} - \chi_{i-1}^2 f_1')/u_{i-1} & (g_1''' + g_1^{V} - \chi_{i-1}^2 g_1')/u_{i-1} & (f_2''' + f_2^{V} - \chi_{i-1}^2 f_2')/u_{i-1} & -(g_2''' + g_2^{V} - \chi_{i-1}^2 g_2') & -(f_3''' + f_3^{V} - \chi_{i-1}^2 f_3') & g_3''' + g_3^{V} - \chi_{i-1}^2 g_3' \end{bmatrix}, \qquad (21)$$

TABLE 4

First frequency coefficient of a clamped–clamped unsymmetric tapered arch

	$\eta = 0{\cdot}1$			$\eta = 0{\cdot}4$		
N	Scheme (a)	Scheme (b)	Scheme (c)	Scheme (a)	Scheme (b)	Scheme (c)
10	54·1564	53·0791	53·6178	53·9509	49·4777	51·7143
20	53·8796	53·3408	53·6102	52·7034	50·4603	51·5818
30	53·7882	53·4291	53·6087	52·3020	50·8055	51·5538
40	53·7429	53·4735	53·6081	52·1050	50·9824	51·5437
50	53·7157	53·5001	53·6079	51·9880	51·0898	51·5389
60	53·6975	53·5180	53·6077	51·9016	51·1621	51·5363
70	53·6846	53·5307	53·6077	51·8556	51·2140	51·5348
80	53·6750	53·5403	53·6077	51·8145	51·2531	51·5338
90	53·6674	53·5478	53·6076	51·7826	51·2836	51·5331
100	53·6614	53·5537	53·6076	51·7572	51·3080	51·5326

$$u_i = 1 + \sum_{k=1}^{i} \{\theta - \theta_k\}^0 \left[\frac{EI_0}{EI_k} - \frac{EI_0}{EI_{k-1}} \right], \tag{22}$$

and the Heaviside function

$$\{\theta - \theta_i\}^0 = \begin{cases} 1 & \text{if} \quad \theta \geqslant \theta_i, \\ 0 & \text{if} \quad \theta < \theta_i. \end{cases} \tag{23}$$

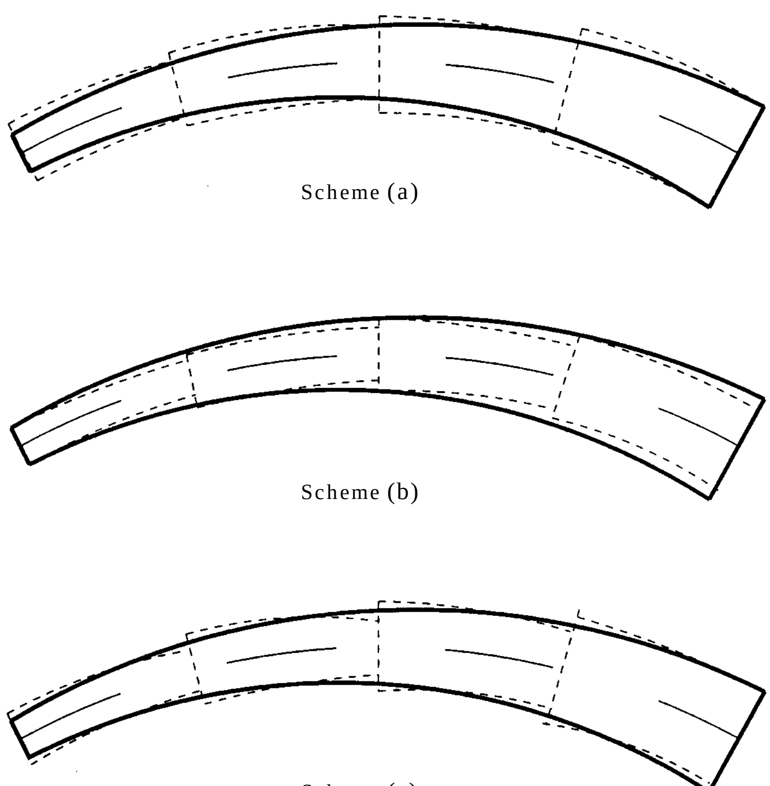

Figure 5. Three discretization schemes.

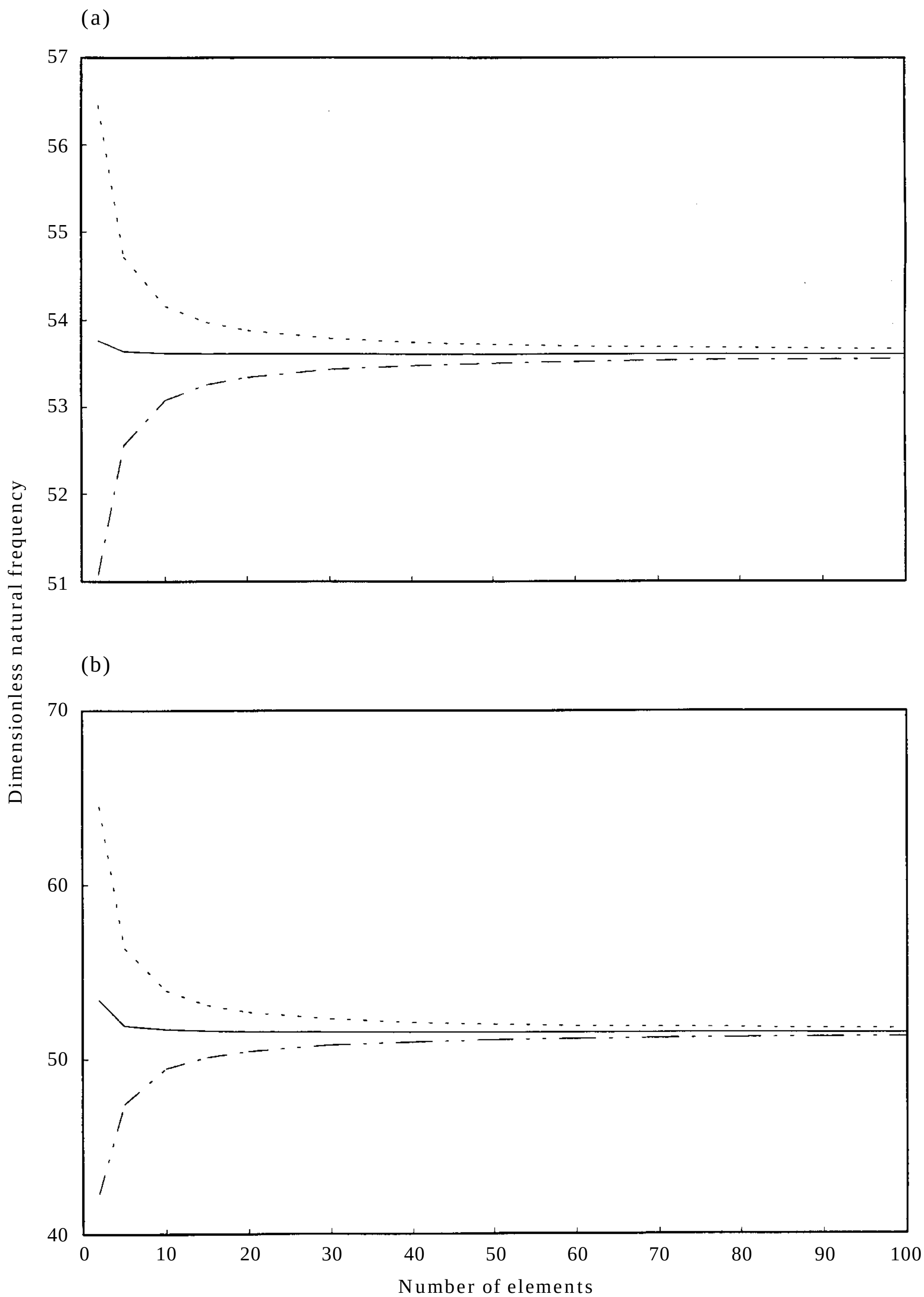

Figure 6. The natural frequencies as functions of number of elements: - - - -, scheme (a); – · – · –, scheme (b); ——, (c). (a) $\eta = 0{\cdot}1$; (b) $\eta = 0{\cdot}4$.

The detailed derivation of the functions f_i and g_i ($i = 1, 2, 3$) is presented in the Appendix. However, it can be verified that

$$\mathbf{A}^i(\theta, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}}) = \mathbf{A}^i(0, \theta - \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}}),$$
$$\mathbf{A}^i(\theta_{i-1}, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}}) = \mathbf{I}. \tag{24}$$

Thus the solution of the arch may be written in terms of the initial parameters at the starting end $\theta = 0$, as

$$\{\delta(\theta)\} = \mathbf{B}^i(\theta)\{\delta(\theta)\}, \qquad \theta_{i-1} \leqslant \theta < \theta_i, \tag{25}$$

where

$$\mathbf{B}^i(\theta) = \mathbf{A}^i(\theta, \theta_{i-1}) + \mathbf{A}^i(\theta, \theta_{i-1}) \sum_{k=1}^{i-1} \{\theta - \theta_k\}^0 \mathbf{F}^k, \tag{26}$$

and the constant matrix

$$\mathbf{F}^k = \varDelta \mathbf{A}^k(\theta_k, \theta_{k-1}) + \varDelta \mathbf{A}^k(\theta_k, \theta_{k-1}) \sum_{m=1}^{k-1} \{\theta - \theta_m\}^0 \mathbf{F}^m \tag{27}$$

is determined by satisfying the continuity and equilibrium conditions (18), where

$$\varDelta \mathbf{A}^k = \mathbf{A}^k(\theta_k, \theta_{k-1}) - \mathbf{A}^k(\theta_{k-1}, \theta_{k-1}). \tag{28}$$

At the finishing end $\theta = \theta_n$, the solution of the arch can be written as

$$\{\delta(\theta_n)\} = \mathbf{B}^n(\theta_n)\{\delta(0)\}. \tag{29}$$

TABLE 5

First frequency coefficient of a hinged–hinged symmetric tapered arch

θ_n (degrees)	Present method	χ_1 R-R [8]	C.D.M. [3]	SAP90 [3]
$\eta = 0{\cdot}1$				
20	1357·63	1299·0	1354·4	
40	337·517	322·86	336·70	
60	148·646	142·15	148·25	
80	82·581	78·890	82·31	
$\eta = 0{\cdot}2$				
20	1420·650	1315·1	1416·1	1418·8
40	353·219	326·88	352·08	352·79
60	155·584	143·90	155·05	155·39
80	86·452	79·875	86·105	86·325
$\eta = 0{\cdot}3$				
20	1482·695	1340·7	1476·2	1478·2
40	368·677	333·20	367·05	367·72
60	162·414	146·70	161·66	161·99
80	90·262	81·434	89·799	90·006

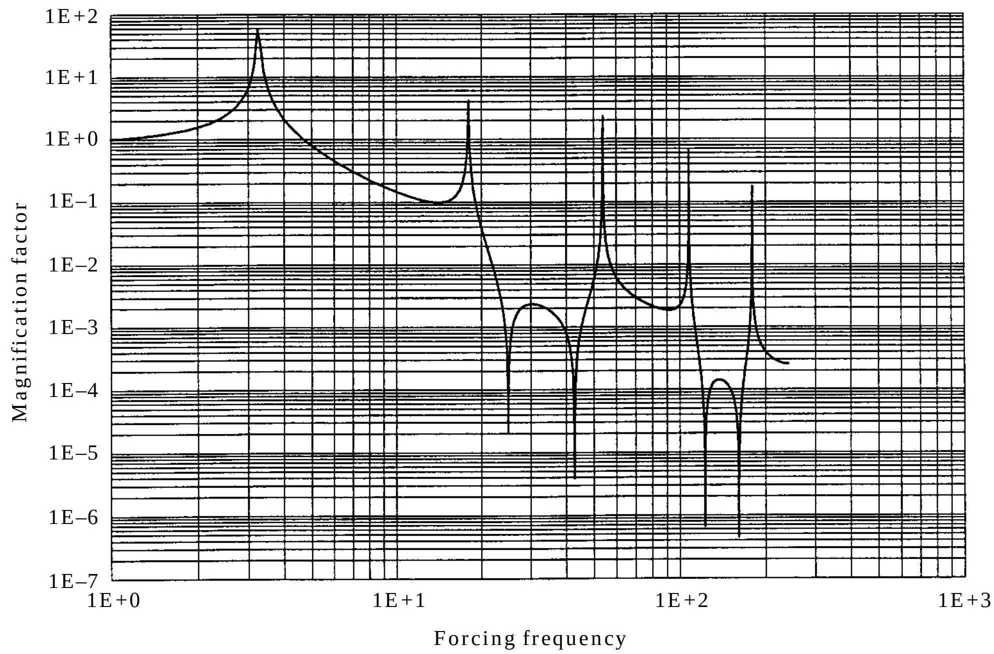

Figure 7. The normal deflection response of the tip in the frequency domain ($\eta = 0{\cdot}1$).

Upon substitution of the boundary conditions into equation (29), one obtains the frequency equation, which is in analytic form for the stepped arches. The natural frequencies can be determined by finding the roots of the frequency equation. The frequency equations under various boundary conditions are listed in Table 1.

4. HARMONIC FORCED VIBRATION

For the ith element, the solution of harmonic forced vibration (15) can be expressed in terms of the initial parameters at $\theta = \theta_{i-1}$, as

$$\{\delta(\theta)\} = \mathbf{A}^i(\theta, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}})\{\delta(\theta_{i-1})\} + \{\mathbf{p}_e^i(\theta)\}, \qquad \theta_{i-1} \leqslant \theta < \theta_i, \tag{30}$$

where the first term on the right-hand side is the homogeneous solution of the free vibration, as indicated in equation (21), and the second term is the particular solution of equation (15), which can be expressed as

$$\mathbf{p}_{e_1}^i = \int_{\theta_{i-1}}^{\theta} g_3(\theta - \alpha, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}})F_{i-1}(\alpha)\,\mathrm{d}\alpha + \frac{R^3 q_u(\theta_{i-1})}{EI_0} u_{i-1} g_3(\theta, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}}), \tag{31}$$

$$\mathbf{p}_{e_2}^i = \int_{\theta_{i-1}}^{\theta} f_3(\theta - \alpha, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}})F_{i-1}(\alpha)\,\mathrm{d}\alpha + \frac{R^3 q_u(\theta_{i-1})}{EI_0} u_{i-1} f_3(\theta, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}}), \tag{32}$$

$$\mathbf{p}^i_{e_3} = \int_{\theta_{i-1}}^{\theta} g_2(\theta - \alpha, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}}) F_{i-1}(\alpha)\,\mathrm{d}\alpha + \frac{R^3 q_u(\theta_{i-1})}{EI_0} u_{i-1} g_2(\theta, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}}), \tag{33}$$

$$\mathbf{p}^i_{e_4} = -\frac{1}{u_{i-1}} \int_{\theta_{i-1}}^{\theta} f_2(\theta - \alpha, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}}) F_{i-1}(\alpha)\,\mathrm{d}\alpha - \frac{R^3 q_u(\theta_{i-1})}{EI_0} f_2(\theta, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}}), \tag{34}$$

$$\mathbf{p}^i_{e_5} = -\frac{1}{u_{i-1}} \int_{\theta_{i-1}}^{\theta} g_1(\theta - \alpha, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}}) F_{i-1}(\alpha)\,\mathrm{d}\alpha - \frac{R^3 q_u(\theta_{i-1})}{EI_0} g_1(\theta, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}}), \tag{35}$$

$$\mathbf{p}^i_{e_6} = \frac{1}{u_{i-1}} \int_{\theta_{i-1}}^{\theta} f_1(\theta - \alpha, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}}) F_{i-1}(\alpha)\,\mathrm{d}\alpha + \frac{R^3 q_u(\theta_{i-1})}{EI_0} f_1(\theta, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}}) - \frac{R^3 q_u(\theta)}{EI_0}. \tag{36}$$

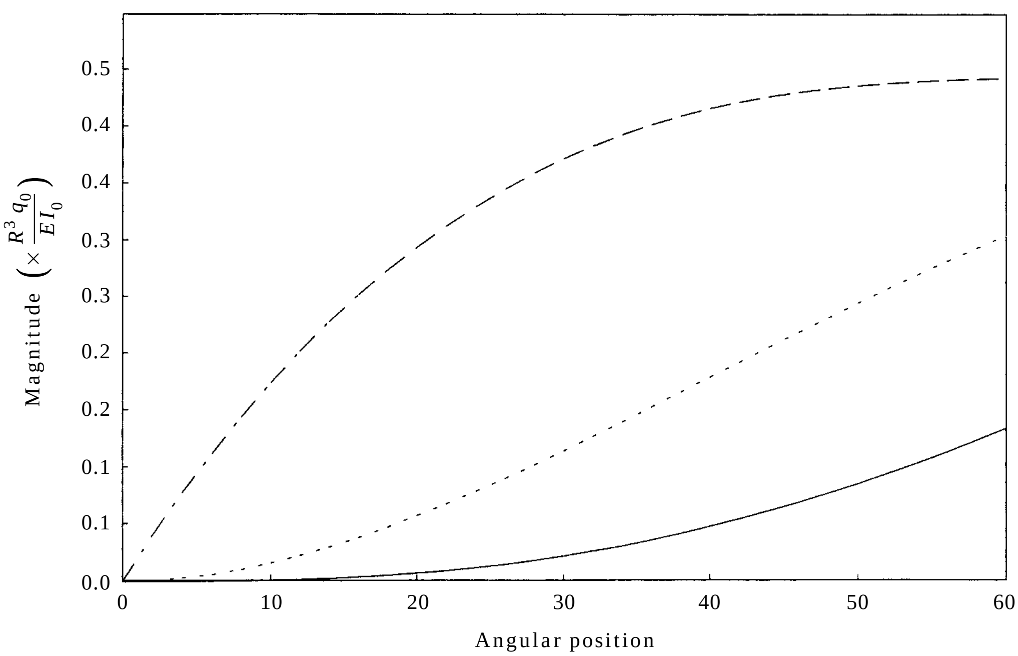

Figure 8. Deflection and rotation responses of the arch for the driving frequency ($\chi = 2{\cdot}5$): – – –, rotation; - - - -, normal deflection; ——, tangential deflection.

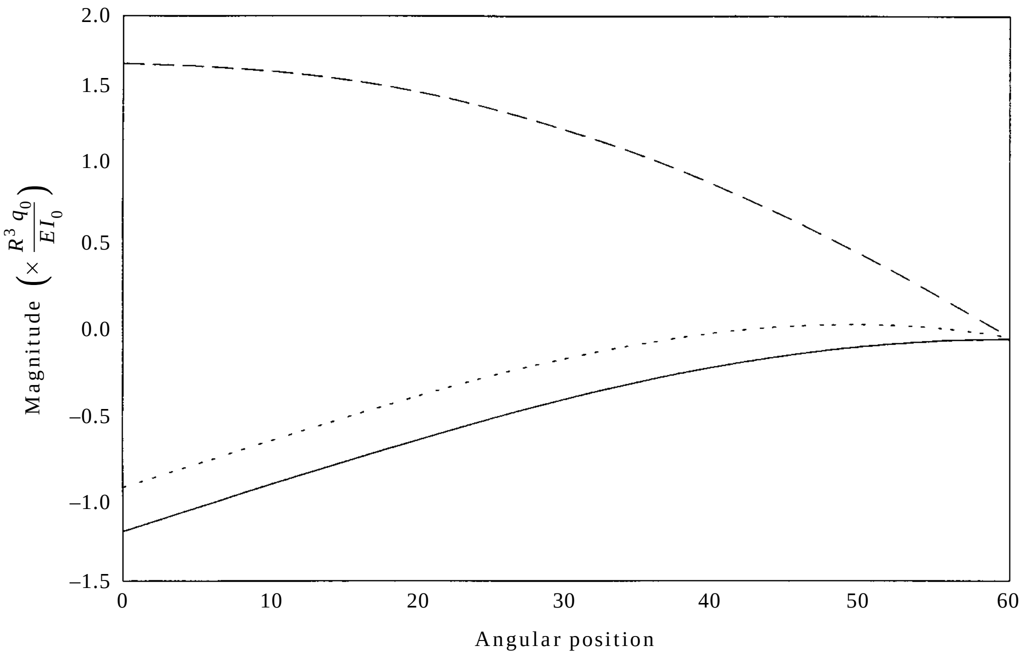

Figure 9. Bending moment, shear force and normal force responses of the arch for the driving frequency ($\chi = 2{\cdot}5$): – – –, shear; - - - -, normal force; ——, bending moment.

Now, the solution of the forced vibration of the arch may be written in terms of the initial parameters at the starting end $\theta = 0$ as

$$\{\delta(\theta)\} = \mathbf{B}^i(\theta)\{\delta(0)\} + \{\mathbf{p}_g^i\}, \qquad \theta_{i-1} \leqslant \theta < \theta_i, \tag{37}$$

where

$$\{\mathbf{p}_g^i\} = \{\mathbf{p}_e^i(\theta, \theta_{i-1}, n_{1_{i-1}}, n_{2_{i-1}}, n_{3_{i-1}})\} + \mathbf{A}^i(\theta, \theta_{i-1}) \sum_{k=1}^{i-1} \{\theta - \theta_k\}^0\{\mathbf{h}^k\}, \tag{38}$$

and the constant vector

$$\{\mathbf{h}^k\} = \{\mathbf{p}_e^k(\theta_k, \theta_{k-1}, n_{1_{k-1}}, n_{2_{k-1}}, n_{3_{k-1}})\} + \Delta\mathbf{A}^k(\theta_k, \theta_{k-1}) \sum_{m=1}^{k-1} \{\theta - \theta_k\}^0\{\mathbf{h}^m\}. \tag{39}$$

is determined by satisfying the continuity and equilibrium conditions between adjacent stepped arches (18). At the finishing end $\theta = \theta_n$, the solution of the arch can be expressed as

$$\{\delta(\theta_n)\} = \mathbf{B}^n(\theta_n)\{\delta(0)\} + \{\mathbf{p}^n(\theta_n)\}. \tag{40}$$

Upon substitution of the boundary conditions into equation (40), one obtains a set of algebraic equations to determine the three remaining unknowns in the vector $\{\delta(0)\}$. Then the dynamic response of the arch under harmonic loading can be obtained from equation (37).

5. NUMERICAL EXAMPLES

5.1. FREE VIBRATION OF STEPPED ARCHES

Consider the stepped arches with rectangular cross-sections, as shown in Figure 3. The first non-dimensional frequency $\chi_1 = \sqrt{\mu_0 R^4/EI_0}\omega_1$ is given in Tables 2 and 3. In Table 2, a clamped–clamped symmetric stepped arch with $\theta_1 = 0{\cdot}3\theta_n$, $\theta_2 = 0{\cdot}7\theta_n$, is considered for two different values of $\eta = 0{\cdot}8$ and 1·2. In Table 3, an unsymmetric stepped arch with $\theta_1 = 0{\cdot}5\theta_n$ for $\eta = 0{\cdot}8$ is considered for three different boundary conditions, that is the clamped–clamped, the hinged–hinged, and the hinged–clamped. For purposes of comparison, the results obtained by the Rayleigh–Ritz method [9], the cell discretization method [3], and the finite element package SAP IV [3] are also presented in these tables. Using the present method, the exact solutions for these stepped arches are obtained.

5.2. FREE VIBRATION OF ARCHES WITH LINEARLY VARYING CROSS-SECTION

Two rectangular cross-section tapered arches with the heights varying linearly, are shown in Figure 4. In Figure 4(a), the height of the arch's cross-section varies linearly from h_0 at one end to h_c at the crown, i.e.,

$$h(\theta) = h_c(1 - \eta + 2\eta\theta/\theta_n), \qquad 0 \leqslant \theta \leqslant \theta_n, \tag{41}$$

whereas in Figure 4(b), the height of the arch's cross-section varies linearly from h_0 at both ends to h_c at the crown, i.e.,

$$h(\theta) = \begin{cases} h_c(1 + \eta - 2\eta\theta/\theta_n), & 0 \leqslant \theta \leqslant \theta_n/2, \\ h_c(1 - \eta + 2\eta\theta/\theta_n)\,, & \theta_n/2 < \theta \leqslant \theta_n. \end{cases} \tag{42}$$

Table 4 shows the fundamental non-dimensional frequency ($\chi_1 = \sqrt{\mu_0 R^4/EI_0}\omega_1$) of a clamped–clamped unsymmetric tapered arch (Figure 4(a)) obtained by three different schemes for producing approximate stepped arches (see Figure 5). As can be seen from Figure 6, if the arch is approximated by scheme (a), the natural frequencies approach the exact values from above, and by scheme (b), the natural frequencies approach the exact values from below. If the arch is approximated by scheme (c), the rate of convergence improves significantly. For scheme (c), as shown in Table 4, 20 elements provide satisfactory results while 40 elements give more accurate results. Table 5 shows the fundamental non-dimensional frequency ($\chi_1 = \sqrt{\mu_0 R^4/EI_0}\omega_1$) of a hinged–hinged symmetric tapered arch (Figure 4(b)) obtained by using 40 elements divided in terms of scheme (c). For comparison, the results obtained by the Rayleigh–Ritz method [8], the cell discretization method [3], and the finite element package SAP 90 [3] are also presented in Table 5. It is interesting to notice that from Table 1, no matter how many elements are used, the present method needs only to solve the determinant of a 3×3 matrix to determine the natural frequencies.

5.3. FORCED VIBRATION OF A CLAMPED–FREE TAPERED ARCH

A clamped–free unsymmetric tapered arch subjected to a harmonic uniform distribution load, $p_w(\theta, t) = 0$, $p_u(\theta, t) = p_0\,e^{i\omega t}$, is considered. The initial displacement and velocity are set to zero. The normal deflection response of the tip of the arch in the frequency domain for $\eta = 0{\cdot}1$ is shown in Figure 7. The horizontal axis is the non-dimensional driving frequency $x = \sqrt{\mu_0 R^4/EI_0}\omega$ and the vertical axis is the magnification factor u/u_{st}. Figure 8 shows the deflection and rotation responses of the whole arch for $\chi = 2{\cdot}5$ at $t = \pi/2\omega$, and Figure 9 shows the bending moment, shear force, and normal force responses of the whole arch for $\chi = 2{\cdot}5$ at $t = \pi/2\omega$.

6. CONCLUSIONS

In this paper, a simple and efficient method for free and forced vibrations of inextensible Bernoulli–Euler arches with arbitrarily varying cross-section is presented. As an approximation, such an arch is divided by a number of stepped arches with constant cross-sections. Then the closed form solution of both free and forced vibrations for the stepped arches can be obtained in terms of the initial parameters (deflections, rotation, bending moment, shear force and normal force) at one end of the arch. As the number of the stepped arches increased, the fast convergence to the exact solutions of the original arch was observed. The method proposed in this paper makes it more convenient to use symbolic programming in conjunction with the conventional numerical programming. As a result, it can provide more efficient and accurate evaluation of dynamic responses of non-uniform arches, as well as great physical insight into the vibration of such arches.

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APPENDIX A: SOLUTIONS OF FREE VIBRATION OF A CIRCULAR ARCH

For a circular arch with constant cross-section, the governing equation of free vibration is [1]

$$\frac{\partial^6 w(\theta, t)}{\partial \theta^6} + 2\frac{\partial^4 w(\theta, t)}{\partial \theta^4} + \frac{\partial^2 w(\theta, t)}{\partial \theta^2} + \frac{\mu R^4}{EI}\frac{\partial^4 w(\theta, t)}{\partial \theta^2\, \partial t^2} - \frac{\mu R^4}{EI}\frac{\partial^2 w(\theta, t)}{\partial t^2} = 0. \quad \text{(A1)}$$

Let $w(\theta, t)/R = W(\theta)\,\mathrm{e}^{\mathrm{j}\omega t}$, we have

$$\frac{\mathrm{d}^6 W}{\mathrm{d}\theta^6} + 2\frac{\mathrm{d}^4 W}{\mathrm{d}\theta^4} + \frac{\mathrm{d}^2 W}{\mathrm{d}\theta^2} - \chi^2\frac{\mathrm{d}^2 W}{\mathrm{d}\theta^2} + \chi^2 W = 0, \quad \text{(A2)}$$

where $x = \sqrt{\mu R^4/EI}\omega$ is the non-dimensional frequency parameter.

Assuming

$$W(\theta) = C\,\mathrm{e}^{\mathrm{j}n\theta}, \quad \text{(A3)}$$

where $\mathrm{j} = \sqrt{-1}$, and substituting equation (A3) into equation (A2), we obtain the characteristic equation

$$n^6 - 2n^4 + (1 - \chi^2)n^2 - \chi^2 = 0. \quad \text{(A4)}$$

The general solution of equation (A2) may be expressed as

$$W(\theta) = A_1 \cos n_1\theta + A_2 \cos n_2\theta + A_3 \cos n_3\theta + B_1 \sin n_1\theta + B_2 \sin n_2\theta + B_3 \sin n_3\theta, \quad \text{(A5)}$$

where $\pm n_i$ $(i = 1, 2, 3)$ are the roots of the characteristic equation (A4), and A_i and B_i $(i = 1, 2, 3)$ are constants of integration, which can also be expressed in terms of the initial parameters (deflections, rotation, bending moment, shear force and normal force) at $\theta = 0$, that is

$$W(0), \quad U(0) = W'(0), \qquad \Psi(0) = W(0) + W''(0),$$

$$M(0) = -\frac{EI}{R}[W'(0) + W'''(0)], \qquad T(0) = -\frac{EI}{R^2}[W''(0) + W^{iv}(0)],$$

$$N(0) = \frac{EI}{R^2}[W'''(0) + W^{v}(0)] - \mu R^2\omega^2 W'(0). \quad \text{(A6)}$$

Substituting equation (A5) into equations (A6), one obtains

$$W(0) = A_1 + A_2 + A_3, \qquad W'(0) = n_1 B_1 + n_2 B_2 + n_3 B_3,$$

$$\Psi(0) = (1 - n_1^2)A_1 + (1 - n_2^2)A_2 + (1 - n_3^2)A_3,$$

$$M(0) = -\frac{EI}{R}[n_1(1 - n_1^2)B_1 + n_2(1 - n_2^2)B_2 + n_3(1 - n_3^2)B_3],$$

$$T(0) = -\frac{EI}{R^2}[-n_1^2(1 - n_1^2)A_1 - n_2^2(1 - n_2^2)A_2 - n_3^2(1 - n_3^2)A_3],$$

$$N(0) = \frac{EI}{R^2}[-n_1^3(1 - n_1^2)B_1 - n_2^3(1 - n_2^2)B_2 - n_3^3(1 - n_3^2)B_3],$$

$$-\mu R^2\omega^2(n_1 B_1 + n_2 B_2 + n_3 B_3). \quad \text{(A7)}$$

Solving the above equation for A_i and B_i ($i = 1, 2, 3$), we obtain

$$A_1 = \frac{1}{D}\left\{[n_2^2 - n_3^2 + n_3^4 - n_2^4 + n_2^2 n_3^2 (n_2^2 - n_3^2)]W(0)\right.$$

$$\left. + (n_3^2 - n_2^2 + n_2^4 - n_3^4) - (n_2^2 - n_3^2)\frac{T(0)R^2}{EI}\right\},$$

$$A_2 = \frac{1}{D}\left\{[n_3^2 - n_1^2 + n_1^4 - n_3^4 + n_3^2 n_1^2 (n_3^2 - n_1^2)]W(0)\right.$$

$$\left. + (n_1^2 - n_3^2 + n_3^4 - n_1^4)\Psi(0) - (n_3^2 - n_1^2)\frac{T(0)R^2}{EI}\right\},$$

$$A_3 = \frac{1}{D}\left\{[n_1^2 - n_2^2 + n_2^4 - n_1^4 + n_1^2 n_2^2 (n_1^2 - n_2^2)]W(0)\right.$$

$$\left. + (n_2^2 - n_1^2 + n_1^4 - n_2^4)\Psi(0) - (n_1^2 - n_2^2)\frac{T(0)R^2}{EI}\right\},$$

$$B_1 = \frac{1}{n_1 D}\left\{[n_2^2 - n_3^2 + n_3^4 - n_2^4 + n_2^2 n_3^2 (n_2^2 - n_3^2) - \chi^2 (n_3^2 - n_2^2)]W'(0)\right.$$

$$\left. - (n_3^2 - n_2^2 + n_2^4 - n_3^4)\frac{M(0)R}{EI} + (n_2^2 - n_3^2)\frac{N(0)R^2}{EI}\right\},$$

$$B_2 = \frac{1}{n_2 D}\left\{[n_3^2 - n_1^2 + n_1^4 - n_3^4 + n_3^2 n_1^2 (n_3^2 - n_1^2) - \chi^2 (n_1^2 - n_3^2)]W'(0)\right.$$

$$\left. - (n_1^2 - n_3^2 + n_3^4 - n_1^4)\frac{M(0)R}{EI} + (n_3^2 - n_1^2)\frac{N(0)R^2}{EI}\right\},$$

$$B_3 = \frac{1}{n_3 D}\left\{[n_1^2 - n_2^2 + n_2^4 - n_1^4 + n_1^2 n_2^2 (n_1^2 - n_2^2) - \chi^2 (n_2^2 - n_1^2)]W'(0)\right.$$

$$\left. - (n_2^2 - n_1^2 + n_1^4 - n_2^4)\frac{M(0)R}{EI} + (n_1^2 - n_2^2)\frac{N(0)R^2}{EI}\right\}, \qquad \text{(A8)}$$

where

$$D = n_1^2 n_2^2 (n_1^2 - n_2^2) + n_2^2 n_3^2 (n_2^2 - n_3^2) + n_3^2 n_1^2 (n_3^2 - n_1^2). \qquad \text{(A9)}$$

Substituting equations (A8) into equation (A5), one may express

$$W(\theta) = W(0)f_1(\theta) + W'(0)g_1(\theta) + \Psi(0)f_2(\theta) - \frac{M(0)R}{EI} g_2(\theta) - \frac{T(0)R^2}{EI} f_3(\theta) + \frac{N(0)R^2}{EI} g_3(\theta), \quad \text{(A10)}$$

where

$$f_1(\theta) = \frac{1}{D} \{[n_2^2 - n_3^2 + n_3^4 - n_2^4 + n_2^2 n_3^2(n_2^2 - n_3^2)] \cos n_1\theta + [n_3^2 - n_1^2 + n_1^4 - n_3^4 + n_3^2 n_1^2(n_3^2 - n_1^2)] \cos n_2\theta + [n_1^2 - n_2^2 + n_2^4 - n_1^4 + n_1^2 n_2^2(n_1^2 - n_2^2)] \cos n_3\theta\},$$

$$f_2(\theta) = \frac{1}{D} \{(n_3^2 - n_2^2 + n_2^4 - n_3^4) \cos n_1\theta + (n_1^2 - n_3^2 + n_3^4 - n_1^4) \cos n_2\theta + (n_2^2 - n_1^2 + n_1^4 - n_2^4) \cos n_3\theta\},$$

$$f_3(\theta) = \frac{1}{D} \{(n_2^2 - n_3^2) \cos n_1\theta + (n_3^2 - n_1^2) \cos n_2\theta + (n_1^2 - n_2^2) \cos n_3\theta\},$$

$$g_1(\theta) = \frac{1}{D} \left\{\frac{1}{n_1} [n_2^2 - n_3^2 + n_3^4 - n_2^4 + n_2^2 n_3^2(n_2^2 - n_3^2) - \chi^2(n_3^2 - n_2^2)] \sin n_1\theta + \frac{1}{n_2} [n_3^2 - n_1^2 + n_1^4 - n_3^4 + n_3^2 n_1^2(n_3^2 - n_1^2) - \chi^2(n_1^2 - n_3^2)] \sin n_2\theta + \frac{1}{n_3} [n_1^2 - n_2^2 + n_2^4 - n_1^4 + n_1^2 n_2^2(n_1^2 - n_2^2) - \chi^2(n_2^2 - n_1^2)] \sin n_3\theta \right\}$$

$$g_2(\theta) = \frac{1}{D} \left\{\frac{1}{n_1} (n_3^2 - n_2^2 + n_2^4 - n_3^4) \sin n_1\theta + \frac{1}{n_2} (n_1^2 - n_3^2 + n_3^4 - n_1^4) \sin n_2\theta + \frac{1}{n_3} (n_2^2 - n_1^2 + n_1^4 - n_2^4) \sin n_3\theta \right\},$$

$$g_3(\theta) = \frac{1}{D} \left\{\frac{1}{n_1} (n_2^2 - n_3^2) \sin n_1\theta + \frac{1}{n_2} (n_3^2 - n_1^2) \sin n_2\theta + \frac{1}{n_3} (n_1^2 - n_2^2) \sin n_3\theta \right\}. \quad \text{(A11)}$$