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LETTERS TO THE EDITOR

ON THE STOCHASTIC AVERAGING METHOD OF ENERGY ENVELOPE

C. W. S. To

Department of Mechanical Engineering, University of Nebraska,
104N Walter Scott Engineering Center, Lincoln, Nebraska 68588-0656, *U.S.A.*

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1. INTRODUCTION

The stochastic averaging method of energy envelope (SAMEE) was introduced by Stratonovich [1], and subsequently applied and improved by Roberts [2–5], Dimentberg [6, 7], Zhu and Lei [8, 9], Zhu and Lin [10], Red-Horse and Spanos [11–13]. One major advantage of SAMEE's over the method of classical stochastic averaging (CSA) of Stratonovich [1] is that the influence of the non-linear restoring forces in the governing equation of motion on the joint probability density function of displacement and velocity is present in the energy relation of the system. However, the SAMEE's quoted thus far require, in general, the evaluation of the period of free oscillation of the non-linear system of interest and they are difficult, if not impossible, to apply to systems with both non-linear stiffness and non-linear damping which is a function of both displacement and velocity.

Consequently, in this paper a SAMEE is presented. It has the advantages that: (a) its standard form equations, drift and diffusion coefficients are defined in line with those of the CSA and therefore it can be easily employed; (b) it does not require the evaluation of the period of free oscillation of the system of interest; and (c) it can be applied to systems with both small linear and non-linear dampings which are functions of both displacement and velocity, and large non-linear restoring forces under small random excitations.

2. GENERAL THEORY

Consider the following equation of motion for a non-linear single-degree-of-freedom (sdof) system,

$$\ddot{x} + \epsilon h(x, \dot{x}) + g(x) = \sqrt{\epsilon}\xi(t), \tag{1}$$

where $h(x, \dot{x})$ is a non-linear function of displacement x and velocity $\dot{x} = \mathrm{d}x/\,dt$, $g(x)$ is a non-linear function of displacement x, ϵ is a small parameter, $\xi(t)$ is a zero mean stationary random process, and the over dot and double dot denote, respectively, the first derivative and second derivative with respect to time t.

By multiplying equation (1) with the velocity $\dot{x}$ one has

$$\dot{x}\ddot{x} + \epsilon\dot{x}h(x, \dot{x}) + \dot{x}g(x) = \sqrt{\epsilon}\dot{x}\xi(t). \tag{2}$$

The energy envelope or total energy of the system may be defined as

$$U = \tfrac{1}{2}\dot{x}^2 + \int g(x)\,\mathrm{d}x = \tfrac{1}{2}\dot{x}^2 + G(x). \tag{3}$$

The limits of integration in equation (3) are not identified as the reference level of the potential energy may be selected arbitrarily. This flexibility seems to have an added

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advantage in selecting an appropriate co-ordinate transformation for a wide variety of non-linear systems.

Differentiation of equation (3) with respect to time t results in

$$\dot{U} = \dot{x}\ddot{x} + \dot{x}g(x).$$

Substituting equation (2) into the last equation and re-arranging terms gives

$$\dot{U} = -\epsilon\dot{x}h(x, \dot{x}) + \sqrt{\epsilon}\dot{x}\xi(t). \tag{4}$$

By making use of equation (4), and a co-ordinate transformation for the displacement, $x = R_1(U, \Phi)$, and velocity $\dot{x} = R_2(U, \Phi)$, where $R_1(U, \Phi)$ and $R_2(U, \Phi)$ are functions of energy U and phase angle Φ, one can obtain the state space equation as

$$\mathrm{d}Z_j / \mathrm{d}t = \epsilon f_j(Z, t) + \epsilon^{1/2} g_{jr}(Z, t)\xi_r(t), \qquad j = 1, 2; r = 1, 2, \tag{5}$$

where in general f_j and g_{jr} are non-linear functions, $\mathbf{Z}$ is a two-dimensional random vector process whose elements are $z_1 = U$ and $z_2 = \Phi$, ϵ is a small positive parameter, and $\xi_r(t)$ is the rth element of the stationary random excitation vector $\mathscr{E}$. The elements of the latter vector are of zero mean and have cross-correlation matrix $\mathbf{\Gamma}(\tau)$ whose elements are $\mathbf{\Gamma}_{rv}(\tau) = \langle \xi_r(t)\ \xi_v(t+\tau) \rangle$. In the latter the angular brackets denote mathematical expectation. The choice of the co-ordinate transformation is such that

$$U = U(R_1(U, \Phi), R_2(U, \Phi)). \tag{6}$$

That is, the total energy of the system is a function of the transformed co-ordinates. This choice of co-ordinate transformation seems to be the most general one. Other choices, such as those in references [2–9, 11–13], are special cases to equation (6). These special cases are difficult to use for systems with both non-linear stiffness and non-linear damping which is a function of both displacement and velcoity. This may explain the fact that non-linear damping as functions of velocity only were considered in references [3, 10] for example.

If the correlation times of the random excitations are all smaller than the relaxation time of the system, then it can be shown [14] that the state vector $\mathbf{Z}$ weakly converges to a diffusive Markov vector $\mathbf{Z}^{(0)}$ with transition probability density $p(\mathbf{Z}^{(0)}, t|\mathbf{Z}_0^{(0)}, t_0)$ or simply p, where the subscript 0 denotes at time t_0. The Fokker-Planck-Kolmogorov (FPK) equation is

$$\partial p/\partial t = -\epsilon\partial(a_j p)/\partial z_j^{(0)} + (\epsilon/2)(\partial^2(b_{jk} p)/\partial z_j^{(0)}\partial z_k^{(0)}), \tag{7}$$

where the drift coefficient a_j and the diffusion coefficient b_{jk} are given, respectively, as

$$a_j(Z^{(0)}) = T_s^{av}\left\{\langle f_j(\mathbf{Z}^{(0)}, s)\rangle + \int_{-\infty}^{0}\left\langle \frac{\partial g_{jr}(\mathbf{Z}^{(0)}, s)}{\partial z_k^{(0)}} g_{kv}(\mathbf{Z}^{(0)}, s+\tau)\ \xi_r(s)\xi_v(s+\tau)\right\rangle \mathrm{d}\tau\right\}, \tag{8}$$

$$b_{jk}(\mathbf{Z}^{(0)}) = T_s^{av}\left\{\int_{-\infty}^{\infty}\langle g_{jr}(\mathbf{Z}^{(0)}, s)\, g_{kv}(\mathbf{Z}^{(0)}, s+\tau)\ \xi_r(s)\xi_v(s+\tau)\rangle\, \mathrm{d}\tau\right\}, \tag{9}$$

in which the notation $T_s^{av}\{\cdot\}$ is given by

$$T_{t_0}^{av}\{\cdot\} = \lim_{T(E)\to\infty}\frac{1}{T(E)}\int_{t_0}^{t_0+T(E)}\{\cdot\}\,\mathrm{d}t, \tag{10}$$

where the integration is performed over explicit time t, and E is the energy envelope. That is, $z_1^{(0)} = U^{(0)} = E$. If the quantities in equations (8) and (9) are periodic, with period $T_0(E)$ for example, then equation (10) becomes

$$T_{t_0}^{av}\{\cdot\} = \frac{1}{T_0(E)} \int_{t_0}^{t_0 + T_0(E)} \{\cdot\}\, dt \tag{11}$$

and the results are independent of t_0.

It should be emphasized that if $g(x)$ is a linear function the above SAMEE reduces to the CSA method in reference [1]. However, $g(x)$ in equation (1) is a non-linear function and, therefore, the phase angle $z_2^{(0)} = \Phi^{(0)} = \theta$ is not a slowly varying random process. This complicates the determination of the drift and diffusion coefficients defined by equations (8) and (9). To circumvent this difficulty, various expressions for $T_0(E)$ have been given in reference [2–13] such that determination of the drift and diffusion coefficients then requires either approximation to the phase angle θ [2–5, 11] or the assumption that the random energy E and the random phase θ are deterministic with respect to the mathematical expectation operator [10].

A strategy adopted in the present analysis is to eliminate the rapid oscillations and rapidly varying components of the phase angle θ in every cycle. Applying this strategy, which is consistent with that adopted by Stratonovich [15] and which is closely parallel to that presented in reference [3], no evaluation of the period of oscillation $T_0(E)$ is necessary.

Thus, the phase angle is now written as $\theta = \Theta(t) + \varphi$, where φ is the slowly varying random phase angle, while $\Theta(t)$ is the integral, with respect to time t, of the rapidly varying term on the right side of the first order differential equation for the phase angle θ. In other words, the term $\Theta(t) \approx \omega(E) T_0(E)$, where $\omega(E)$ is the energy dependent frequency of oscillation and $T_0(E)$ is the period of oscillation. Superficially, both sides of the approximately equal sign are inconsistent as the left side is a function of time t, whereas the right side is a function of total energy E. However, as Roberts pointed out in reference [3], the error incurred in this approximation can be assessed by a Taylor expansion of $\omega(E)$. He reasoned that if the assumption is made that the stationary probability density function exists, which implies the existence of stationary motion, $\omega(E)$ is indeed approximately time-independent within every cycle. Thus, the stationary probability density function for the total energy E is given by [1–3, 6, 10]

$$p(E) = C T_0(E)/b_{11}(E)\, e^{2\int_0^E [a_1(v)/b_{11}(v)]\, dv}, \tag{12}$$

where C is the normalization constant. From the signal processing point of view, this strategy may be regarded as low-pass filtering in which the high frequency components in the response are eliminated so that only low frequency or slowly varying components are retained.

The stationary joint probability density function of x and $\dot{x}$ can be shown to be

$$p(x, \dot{x}) = p(E)/T_0(E) = \frac{C}{b_{11}(E)}\, e^{2\int_0^E [a_1(v)/b_{11}(v)]\, dv}. \tag{13}$$

3. APPLICATIONS

Two examples are included in the following for illustration of the technique introduced in section 2 above. They have important applications in many areas of engineering and physics.

3.1. *Van der Pol–Duffing oscillator*

A Van der Pol–Duffing oscillator under white noise excitation is considered here. This non-linear oscillator has been considered by the author in reference [16]. The only difference here is that the non-linear damping term and external white noise excitation of the governing equation of motion contain a small parameter ϵ. That is, the equation of motion consider here becomes

$$\ddot{x} + \epsilon(\alpha + \beta x^2)\dot{x} + \gamma x + \delta x^3 = \sqrt{\epsilon} w(t), \tag{14}$$

where α, β, γ, and δ are real constants of order 1; $w(t)$ is a zero mean white noise process such that $< w(t)w(t+\tau) > = 2\pi S\delta(\tau) = I\delta(\tau)$, in which $\delta(\tau)$ is the Dirac delta function.

To begin with the following co-ordinate transformation is assumed:

$$\dot{x} = \sqrt{2U}\sin \Phi, \qquad x = \sqrt{\sqrt{(4U/\delta)}\cos \Phi - (\gamma/\delta)}, \tag{15a, b}$$

such that the total energy of the system becomes

$$U = \dot{x}^2/2 + (\delta/4)(x^2 + \gamma/\delta)^2. \tag{16}$$

Without loss of generality U and Φ will be replaced by E and θ henceforth. Thus,

$$\dot{x} = \sqrt{2E}\sin\theta, \qquad x = \sqrt{\sqrt{(4E/\delta)}\cos\theta - (\gamma/\delta)} \tag{17a, b}$$

and

$$E = \dot{x}^2/2 + (\delta/4)(x^2 + \gamma/\delta)^2. \tag{18}$$

By making use of equations (14), (17), (18), and equation (4) one can obtain the equation in standard form for the energy E as

$$\dot{E} = -\epsilon(2E)[(\alpha - \beta\gamma/\delta)\sin^2\theta + (\beta\sqrt{4E/\delta})\sin^2\theta\cos\theta] + \sqrt{\epsilon}(\sqrt{2E}\sin\theta)w(t). \tag{19}$$

The last equation can be defined as

$$\mathrm{d}E/\mathrm{d}t = \epsilon f_1(E, \theta) + \epsilon^{1/2} g_{11}(E, \theta)w, \tag{20}$$

where

$$f_1(E, \theta) = -2E[(\alpha + \beta\gamma/\delta)\sin^2\theta + (\beta\sqrt{4E/\delta})\sin^2\theta\cos\theta], \tag{21a}$$

and

$$g_{11}(E, \theta) = \sqrt{2E}\sin\theta. \tag{21b}$$

After some lengthy algebraic manipulation, the complementary equation to equation (20) can be reduced to

$$\mathrm{d}\varphi/\mathrm{d}t = \epsilon f_{24}(E, \theta) + \sqrt{\epsilon} g_{22}(E, \theta)w, \tag{22}$$

where

$$f_{24} = -\tfrac{1}{2}[\alpha + \beta(\sqrt{4E/\delta}\cos\theta - \gamma/\delta)]\sin 2\theta, \qquad g_{22}(E, \theta) = \cos\theta/\sqrt{2E}. \tag{23a, b}$$

Equations (20) and (22) are a pair of first order differential equations which can be applied to describe completely the behaviour of the oscillator governed by equation (14).

Upon application of equations (8), (21) and (23), it reduces to

$$a_1 = T_s^{av}\{<f_1(E,\theta)>\} + T_s^{av}\left\{\int_{-\infty}^{0}\frac{\partial g_{11}}{\partial E}\,g_{11}(E,\theta,s+\tau) < w(s)w(s+\tau) > \mathrm{d}\tau\right\}$$

$$+ T_s^{av}\left\{\int_{-\infty}^{0}\frac{\partial g_{11}}{\partial \theta}\,g_{22}(E,\theta,s+\tau) < w(s)w(s+\tau) > \mathrm{d}\tau\right\}$$

$$= T_s^{av}\{<f_1(E,\theta)>\} + T_s^{av}\left\{\tfrac{1}{2}\int_{-\infty}^{\infty}(2\pi S)\delta(\tau)\,\mathrm{d}\tau\right\}$$

$$= T_s^{av}\{<f_1(E,\theta)>\} + T_s^{av}\{I/2\}. \tag{24}$$

By making use of the relation

$$\frac{1}{T_0(E)}\int_0^{T_0(E)} f(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta, \tag{25}$$

and substituting equation (21a) into equation (24), one has

$$a_1 = -(\alpha - \beta\gamma/\delta)E + I/2. \tag{26}$$

Application of equation (9) gives

$$b_{11} = T_s^{av}\left\{\int_{-\infty}^{\infty}(2E)\sin\theta\,\sin(\theta+\tau)I\delta(\tau)\,\mathrm{d}\tau\right\}$$

$$= \frac{EI}{\pi}\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta = EI. \tag{27}$$

Clearly, the first order differential equation for the energy envelope E is uncoupled from that for the phase angle φ. Thus, the FKP equation for E is

$$\partial p/\partial t = -\epsilon\partial/\partial E\{[-(\alpha - \beta\gamma/\delta)E + I/2]p\} + (\epsilon/2)(\partial^2/\partial E^2)(EIp). \tag{28}$$

To obtain the stationary probability density function $p(E)$ one may apply equation (12) or equation (28) directly with its left side set to zero to give

$$p(E) = CT_0(E)/I\,\mathrm{e}^{-(1/2\pi S)(\alpha - (\beta\gamma/\delta))[\dot{x}^2 + (\delta/2)(x^2 + \gamma/\delta)^2]}. \tag{29}$$

Upon application of equation (13), the stationary joint probability density of displacement and velocity becomes

$$p(x,\dot{x}) = (C/I)\,\mathrm{e}^{-(1/2\pi S)(\alpha - (\beta\gamma/\delta))[\dot{x}^2 + (\delta/2)(x^2 + \gamma/\delta)^2]}. \tag{30}$$

This result agrees with that given in reference [16] in which the statistical non-linearization technique was applied. The result given in reference [16] holds for all time t whereas equation (30) is valid in a slow time ϵt. Of course, the limiting case is $\alpha = \beta\gamma/\delta$ and when $\alpha < \beta\gamma/\delta$ the constant C in equation (30) cannot be normalized. This is the case where bifurcation occurs.

It may be appropriate to emphasize that the solution above for the system described by equation (14) cannot be obtained by other SAMEE quoted in this paper.

3.2. *Non-linear ship rolling in random seas*

This model is similar to that investigated in reference [17] except that for the present problem the small parameter ϵ in the non-linear damping and the square root of this small parameter in the external random excitation are included. Thus, the equation of motion for this model becomes

$$\ddot{x} + \epsilon\beta|\dot{x}^2|\,\mathrm{sgn}\,(\dot{x}) + \gamma x + \eta x^3 = \sqrt{\epsilon}w(t), \tag{31}$$

where β, γ, and η are constants of order 1. The last equation, with due modification to the signs of the constant parameters, can be employed to model and analyse the non-linear rolling motion of a ship in random seas [3].

To evaluate the stationary joint probability density function of x and $\dot{x}$ in the above equation one applies the co-ordinate transformation

$$\dot{x} = \sqrt{2E}\sin\theta, \qquad \sqrt{G(x)} = \sqrt{E}\cos\theta, \tag{32a, b}$$

in which

$$E = \tfrac{1}{2}\dot{x}^2 + G(x), \quad G(x) = \int g(x)\,\mathrm{d}x, \quad g(x) = \gamma x + \eta x^3. \tag{32c–e}$$

By making use of equations (31), (32), and equation (4) one can obtain the first equation in standard form for the energy E as

$$\mathrm{d}E/\,\mathrm{d}t = \epsilon f_1(E, \theta) + \epsilon^{1/2} g_{11}(E, \theta)w, \tag{33}$$

where

$$f_1(E, \theta) = -\beta(2E)^{3/2}|\sin\theta|\sin^2\theta, \quad g_{11}(E, \theta) = \sqrt{2E}\sin\theta. \tag{34a, b}$$

With some lengthy algebraic manipulation the complementary equation to equation (33) can be obtained as

$$\mathrm{d}\varphi/\,\mathrm{d}t = \epsilon f_2(E, \theta) + \epsilon^{1/2} g_{22}(E, \theta)w, \tag{35}$$

where

$$f_2(E, \theta) = -(\beta/2)\sqrt{2E}\sin 2\theta|\sin\theta|, \qquad g_{22}(E, \theta) = \cos\theta/\sqrt{2E}. \tag{36a, b}$$

Equations (33) and (35) are a pair of first order differential equations which can be applied to describe completely the behaviour of the oscillator governed by equation (31).

Upon application of equations (8), (34) and (36), one has

$$\begin{aligned} a_1 &= T_s^{av}\{<f_1(E, \theta)>\} + T_s^{w}\left\{\int_{-\infty}^{0} \frac{\partial g_{11}}{\partial E}\, g_{11}(E, \theta, s+\tau) < w(s)w(s+\tau) > \mathrm{d}\tau\right\} \\ &\quad + T_s^{av}\left\{\int_{-\infty}^{0} \frac{\partial g_{11}}{\partial \theta}\, g_{22}(E, \theta, s+\tau) < w(s)w(s+\tau) > \mathrm{d}\tau\right\} \\ &= T_s^{av}\{<f_1(E, \theta)>\} + T_s^{w}\left\{\tfrac{1}{2}\int_{-\infty}^{\infty} (2\pi S)\delta(\tau)\,\mathrm{d}\tau\right\} \\ &= T_s^{av}\{<f_1(E, \theta)>\} + T_s^{w}\{I/2\}. \end{aligned} \tag{37}$$

By making use of equation (25) and substituting equation (33) into equation (37), leads to

$$a_1 = -(8\beta/3\pi)\,(2E)^{3/2} + I/2. \tag{38}$$

By making use of equation (9) gives

$$b_{11} = T_s^{qw}\left\{\int_{-\infty}^{\infty}(2E)\sin\theta\sin(\theta+\tau)I\delta(\tau)\,\mathrm{d}\tau\right\}$$

$$= \frac{EI}{\pi}\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta = EI. \tag{39}$$

Note that the first order differential equation for the energy envelope E is uncoupled from that for the phase angle φ. Therefore, the FKP equation for the energy envelope E is

$$\partial p/\partial t = -(\epsilon\partial/\partial E)\{[-(8\beta/3\pi)\,(2E)^{3/2} + I/2]p\} + (\epsilon/2)\,(\partial^2/\partial E^2)\,(EIp). \tag{40}$$

The stationary probability density function $p(E)$ may be obtained by applying either equation (12) or equation (40) directly with its left side set to zero giving

$$p(E) = C[T_0\,(E)/I]\,\mathrm{e}^{-(8\beta/9\pi^2 S)\,(\dot{x}^2+\gamma x^2+\frac{1}{2}\eta x^4)^{3/2}}. \tag{41}$$

The stationary joint probability density function of displacement and velocity, from equation (13), thus is

$$p(x,\dot{x}) = (C/I)\,\mathrm{e}^{-(8\beta/9\pi^2 S)\,(\dot{x}^2+\gamma x^2+\frac{1}{2}\eta x^4)^{3/2}}. \tag{42}$$

Equation (42) agrees with that presented in reference [17] except that it is valid in the slow time ϵt while the result in reference [17] is valid for all time t.

It is also important to mention that equation (42) agrees with equation (5.61) in reference [18] in which equation (5.61) was verified by Monte Carlo simulation. Thus, one can conclude that the present SAMEE can give correct solutions.

4. CONCLUSION

In the foregoing a stochastic averaging method of energy envelope (SAMEE) has been presented. Two non-linear oscillators have been included to demonstrate the usefulness of the proposed SAMEE. The solution for the first example agrees with that in reference [16]. This particular example cannot be solved by previous SAMEE available in the literature. The solution for the second example agrees with that in references [17] and [18] in which Monte Carlo simulation data were used to verify the solution. This, in turn, indicates that the present SAMEE is able to provide correct solutions.

Furthermore, the SAMEE proposed in the foregoing has the advantages that: (a) its standard form equations, drift and diffusion coefficients are defined in line with those of the CSA and therefore it is easy to apply; (b) it does not require the evaluation of the period of free oscillation of the system of interest; and (c) it is general in that it can be applied to systems with small non-linear dampings which are functions of both displacement and velocity, and large non-linear restoring forces under small stationary random excitations.

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