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SOME PARTICULAR SOLUTIONS WHICH DESCRIBE THE MOTION OF THE ROTOR

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1. INTRODUCTION

There are many papers dealing with the problems of free and self-excited rotor vibrations. The mathematical model of vibrations of a rotor is a second order non-linear differential equation with a complex deflection function;

$$\ddot{z} - 2\mathrm{i}\Omega_1 \dot{z} + \omega^2 z + z f(|z|) = f_1(z, \dot{z}), \tag{1}$$

Here z is the complex function, $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, $f(|z|)$ is the non-linear function, $|z|^2 = z\,\bar{z}$, $\bar{z}$ is the complex conjugate deflection function, Ω_1 is the angular velocity of the rotor, and $f_1(z, \dot{z})$ is a function of the deflection z and the complex velocity $\dot{z}$. In analyzing the motion of the rotor it is usually stated that "the equation of motion has an exact periodic solution,

$$z = A\,\mathrm{e}^{\mathrm{i}\Omega t}, \tag{2}$$

where A is a constant amplitude and Ω is the frequency of these vibrations" (see references [1–4]). The rotor system response is circular. There are no comments about the conditions for which this solution is evident.

In this letter a special case of differential equation (1) is considered; namely, it is supposed that the non-linearity is of cubic type. This type of non-linearity is physically realizable. For the case when $f(|z|) = k_3|z|^2$, where k_3 has a constant value, the differential equation of motion is then

$$\ddot{z} - 2\mathrm{i}\Omega_1 \dot{z} + \omega^2 z + k_3 z(|z|)^2 = 0. \tag{3}$$

The aim here is to obtain other particular solutions for equation (3) and to define the conditions for their existence.

2. EQUATIONS OF MOTION

The polar co-ordinates

$$z = \rho\,\mathrm{e}^{\mathrm{i}\theta}, \tag{4}$$

are introduced, where ρ and θ are the polar co-ordinates. The co-ordinate ρ defines the distance and θ the angular position of the rotor center in the fixed co-ordinate system which is connected with the undeformed rotor system. By substituting expression (4) and its first and second time derivatives into equation (3) and separating the real and imaginary terms a system of two conjugate equations is obtained:

$$\ddot{\rho} - \rho\dot{\theta}^2 + 2\Omega_1 \rho\dot{\theta} + \omega^2\rho + k_3\rho^3 = 0, \qquad \rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} - 2\Omega_1\dot{\rho} = 0. \tag{5, 6}$$

The second equation can be rewritten in the form

$$\mathrm{d}/\mathrm{d}t[\rho^2(\dot{\theta} - \Omega_1)] = 0, \tag{7}$$

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and after integration one has

$$\rho^2(\dot{\theta} - \Omega_1) = K, \tag{8}$$

where

$$K = \rho_0^2(\dot{\theta}_0 - \Omega_1), \tag{9}$$

and the initial values are $\rho(0) = \rho_0$, $\dot{\theta}(0) = \dot{\theta}_0$. By substituting

$$\dot{\theta} = (K/\rho^2) + \Omega_1, \tag{10}$$

into equation (5) an ordinary second order non-linear differential equation is obtained,

$$\ddot{\rho} - (K^2/\rho^3) + (\Omega_1^2 + \omega^2)\rho + k_3\rho^3 = 0: \tag{11}$$

i.e.,

$$\ddot{\rho} - \rho_0^4(\dot{\theta}_0 - \Omega_1)^2/\rho^3 + (\Omega_1^2 + \omega^2)\rho + k_3\rho^3 = 0. \tag{12}$$

To find the solution of equation (11) in closed form is impossible; only some particular solutions can be obtained.

3. PARTICULAR SOLUTIONS

3.1. *A first solution*

Suppose that

$$\dot{\theta} = \dot{\theta}_0 = \text{const.} \tag{13}$$

According to equation (10) one must have

$$\rho = \rho_0 = \text{const.} \tag{14}$$

Upon substituting equations (13) and (14) into equation (12) it is evident that the previous assumptions are correct only if the following condition between the initial values is satisfied:

$$\dot{\theta}_0 = \Omega_1 \pm \sqrt{\Omega_1^2 + \omega^2 + k_3\rho_0^2}. \tag{15}$$

Using the relations (13–15) yields the motion of the rotor as

$$z = \rho_0\, e^{i(\dot{\theta}_0 t + \theta_0)}, \tag{16}$$

where $\theta(0) = \theta_0$. This means that the particular solution (16) exists if the initial conditions satisfy the relation (15). Equation (16) describes vibrations with constant amplitude and constant circular velocity. The veolocity in the radial direction is zero. In Figure 1, the orbital motion of the rotor center is plotted. It is a circle.

3.2. *A second solution*

For the differential equation (3) the Hamiltonian is formed:

$$H = \dot{\rho}^2/2 + (\rho^2/2)(\Omega_1^2 + \omega^2) + \tfrac{1}{2}K^2/\rho^2 + \tfrac{1}{4}k_3\rho^4. \tag{17}$$

For the initial conditions $\rho(0) = \rho_0$ and $\dot{\rho}(0) = \dot{\rho}_0$ the value of the Hamiltonian is

$$H = \dot{\rho}_0^2/2 + (\rho_0^2/2)(\Omega_1^2 + \omega^2) + \tfrac{1}{2}K^2/\rho_0^2 + \tfrac{1}{4}k_3\rho_0^4 = \text{const.} \tag{18}$$

From equation (17) one has

$$\dot{\rho} = \sqrt{2H - \rho^2(\Omega_1^2 + \omega^2) - (K^2/\rho^2) - k_3\rho^4}. \tag{19}$$

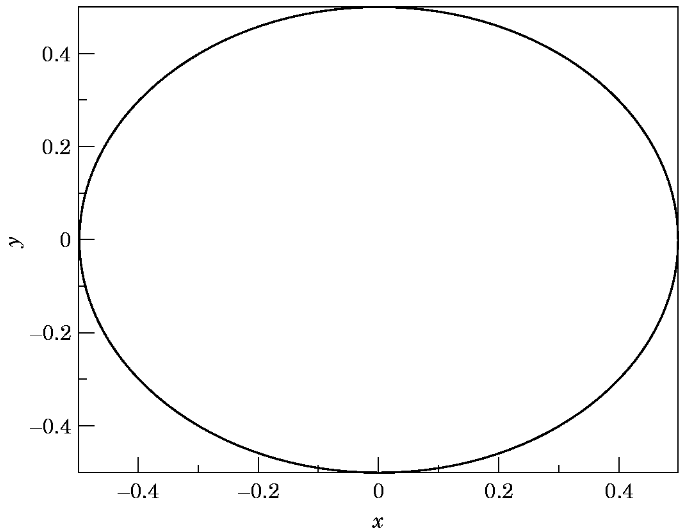

Figure 1. The orbital motion according to equation (16).

To find the solution of equation (11) it is necessary to integrate the relation (19). From reference [5] it is evident that the solution exists for

$$\rho < \rho(0),$$

and the initial condition

$$\rho(0) = \sqrt{a_5 - o/3}, \tag{20}$$

where

$$a_1 = -\tfrac{1}{3}ho - \tfrac{1}{2}\kappa - \tfrac{1}{27}o^3,$$

$$a_2 = -96h^3 - 12h^2o^2 + 108ho\kappa + 81\kappa^2 + 12\kappa o^3,$$

$$a_3 = \sqrt[3]{a_1 + \tfrac{1}{18}\sqrt{a_2}}, \qquad a_4 = (-\tfrac{2}{3}h - \tfrac{1}{9}o^2)/\sqrt[3]{a_1 + \tfrac{1}{18}\sqrt{a_2}},$$

$$a_5 = a_3 - a_4, \qquad a_6 = a_3 + a_4, \qquad h = H/k_3, \qquad o = (\Omega_1^2 + \omega^2)/k_3, \qquad \kappa = K^2/k_3.$$

By analyzing the parameters it can be seen that if $a_3 > 0$ then a_5 is also positive. Then one has

$$1/\sqrt[4]{\tfrac{9}{4}a_5^2 + \tfrac{3}{4}a_6^2}\,\mathrm{F}(\varphi, k) = 2t\sqrt{k_3}, \tag{21}$$

where $\mathrm{F}(\varphi, k)$ is the incomplete elliptic integral of the first kind,

$$\varphi = \arccos\left[\frac{\sqrt{\tfrac{9}{4}a_5^2 + \tfrac{3}{4}a_6^2} - \tfrac{1}{2}a_5 + \tfrac{2}{3}o + \rho^2}{\sqrt{\tfrac{9}{4}a_5^2 + \tfrac{3}{4}a_6^2} + \tfrac{3}{2}a_5 + \rho^2}\right], \tag{22}$$

$$k^2 = \frac{\sqrt{\tfrac{9}{4}a_5^2 + \tfrac{3}{4}a_6^2} + \tfrac{3}{2}a_5}{2\sqrt{\tfrac{9}{4}a_5^2 + \tfrac{3}{4}a_6^2}}\,. \tag{23}$$

It is evident that the modulus of the elliptic function does not depend on the position of the rotor center ρ. Solving equation (21) one obtains

$$\rho^2 = \tfrac{1}{2}a_5 - \tfrac{2}{3}o - \sqrt{\tfrac{9}{4}a_5^2 + \tfrac{3}{4}a_6^2} + \left(\sqrt{\tfrac{9}{4}a_5^2 + \tfrac{3}{4}a_6^2} + \tfrac{3}{2}a_5\right) \arccos\left[\mathrm{cn}\left(2t\sqrt{k_3^4\sqrt{\tfrac{9}{4}a_5^2 + \tfrac{3}{4}a_6^2}, k^2}\right)\right]_1 - \arccos\left[\mathrm{cn}\left(2t\sqrt{k_3^4\sqrt{\tfrac{9}{4}a_5^2 + \tfrac{3}{4}a_6^2}, k^2}\right)\right]. \tag{24}$$

Then the solution of equation (3) is

$$z = \rho\ \mathrm{e}^{\mathrm{i}\int_t (K/\rho^2 + \Omega_1)\,\mathrm{d}t + \theta_0},$$

for the relation (20). Note that due to the fact that h and κ are functions of the initial conditions $\rho(0)$, $\dot\rho(0)$ and $\dot\theta(0)$, the relation (20) gives the connection between them.

3.3. *A third solution*

For the case when $K = 0$, i.e.,

$$\dot\theta(0) = \dot\theta_0 = \Omega_1, \tag{25}$$

the differential equation (11) transforms to

$$\ddot\rho + (\Omega_1^2 + \omega^2)\rho + k_3\rho^3 = 0. \tag{26}$$

This is the Duffing differential equation which has an exact solution of the form

$$\rho = A\ \mathrm{cn}\,(\Omega t + \alpha, k^2), \tag{27}$$

where cn is the elliptic Jacobian function, and the frequency Ω and the modulus of the elliptic function k^2 are given by

$$\Omega^2 = \Omega_1^2 + \omega^2 + k_3A^2, \qquad k^2 = k_3A^2/2\Omega^2. \tag{28}$$

It can be seen that the frequency of the vibrations and the modulus of the vibrations are functions of the amplitude A. The amplitude A and phase angle α are constants obtained according to the initial conditions $\rho(0) = \rho_0$ and $\dot\rho(0) = \dot\rho_0$. By solving the equations

$$\rho_0 = A\ \mathrm{cn}\,(\alpha, k^2), \qquad \dot\rho_0 = -A\ \mathrm{sn}\,(\alpha, k^2)\ \mathrm{dn}\,(\alpha, k^2), \tag{29}$$

the amplitude of vibrations is obtained:

$$k_3A^6 + 2A^4(\Omega_1^2 + \omega^2) - A^2\rho_0^2(2\Omega_1^2 + 2\omega^2 + k_3\rho_0^2) - 2\dot\rho_0^2 = 0. \tag{30}$$

By substituting the first of equations (29) into equation (30) the phase angle is obtained as

$$\alpha = \mathrm{cn}^{-1}\left(\frac{\rho_0^2}{A^2}, \frac{k_3A^2}{2(\Omega_1^2 + \omega^2 + k_3A^2)}\right). \tag{31}$$

The solution of equation (3) is then

$$z = A\ \mathrm{cn}\,(\Omega t + \alpha, k^2)\ \mathrm{e}^{\mathrm{i}(\Omega_1 t + \theta_0)}, \tag{32}$$

where $\theta(0) = \theta_0$. This particular solution (32) exists for the relation (26): i.e., for the case when the circular velocity of vibration is proportional to the rotational speed of the rotor. The velocity of the vibrations is the sum of the vibrations in the radial and circular directions. Also, the deflection of rotor center is time dependent. In Figure 2, the orbital motion of the rotor center is plotted. It is a periodic function with period $4K(k)$, where

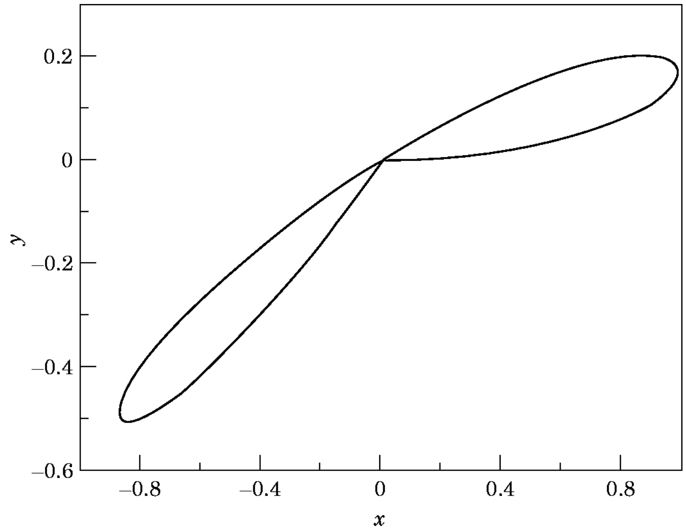

Figure 2. The orbital motion according to equation (32).

$K(k)$ is the complete elliptic integral of the first kind, depending on the modulus k as given in equations (29).

4. CONCLUSION

Some particular solutions of equation (3) have been obtained, which describe the vibrations of a rotor which rotates with angular velocity Ω_1. For all of them it is common that they are correct for some specific initial conditions. The particular solutions are periodic functions. In the literature usually only the first type of particular solution (16) is applied for discussing physical phenomena. But, monitoring the orbital motion of the rotor shows that other types of curves which differ from a circle can also occur (see, for example, references [6–8]). By using the suggested particular solutions one will be able to discuss some other phenomena in rotor dynamics.

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