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FOURIER p-ELEMENT FOR THE ANALYSIS OF BEAMS AND PLATES

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1. INTRODUCTION

The finite element method achieves an approximate solution by subdividing the domain of interest into a number of smaller sub-domains, called finite elements, and then approximating the solution by using local, piecewise continuous polynomial functions within each element. The accuracy of the solution may be improved in two ways. The first is the h-version to refine the finite element mesh and the second is the p-version to increase the order of polynomial shape functions for a fixed mesh. Zienkiewicz and Taylor [1] concluded that, in general, p convergence is more rapid per degree of freedom introduced. Central to the hierarchical concept is the ability to enrich the polynomial content of selected elements within the mesh. Polynomial functions are well known to be ill-conditioned, e.g., the computer can hardly find the difference between x^{10} and x^{11} within $0 < x < 1$. West *et al*. [2] showed recently that, by reference to an appropriate family of K-orthogonal polynomials, numerical rounding errors associated with floating point arithmetic prescribe the maximum available degree of polynomial enrichment. The principal source of these errors can be traced to the widely ranged coefficients that define a given K-orthogonal polynomial. They concluded that in h–p applications, one has to restrict the degree of polynomial enrichment to: (1) 24 or less in 1-D applications, (2) 14 or less in 2-D applications, and (3) 8 or less in 3-D applications. This severely limits the use of the p-version of finite elements.

Houmat [3] used a quintic polynomial plus some sine terms, i.e., $w(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4 + c_6x^5 + \Sigma_r\, c_{r+6} \sin r\pi x$ to study the free vibration of rectangular plates. Beslin and Nocolas [4] used the trigonometric sets $w_r(x) = \sin(a_rx + b_r)\sin(c_rx + d_r)$ as shape functions to study the same problem. Bardell *et al*. [5] suggested the use of mixing hermite cubics and trigonometric functions to analyse coplanar sandwich panels. All authors found that trigonometric functions are more effective in predicting the medium frequency natural modes than polynomials.

In this paper, the use of products of polynomials and Fourier series instead of polynomials alone in the p-element shape functions is recommended. Due to the fact that Fourier series are well behaved, the limitation of the polynomial functions disappears. When applied to the natural vibration analysis of structures, it is found as a bonus that higher modes converge much faster than when using polynomials alone. The concept is not new. Leung [6] enriched the standard beam finite element by means of eigenfunctions (beam functions) and predicted increased accuracy before the name p-version was used by Babuska *et al*. [7]. This involved the integration of products of polynomials and beams functions. Although closed form integration formulae can be found, e.g., Leung [8, 9], the complexity increased rapidly when the products of say three, beam functions are involved,

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for non-linear problems. However, Fourier series are well behaved as well as easy to integrate. This paper presents some preliminary results of the study on beams and plates by using the Fourier p-version of the finite element method.

2. FOURIER-VERSION C^0 SHAPE FUNCTIONS

There are two kinds of finite element shape functions with respect to the inter-element continuity requirements. C^0 shape functions refer to the minimum requirement of function continuity and C^1 shape functions refer to the requirement of function and first derivative continuity. C^0 shape functions are conveniently represented by area co-ordinates in 1-, 2-, and 3-D. Leung [10] formulated higher order C^0 shape functions in an explicit and programmable manner. To fix the idea, concentrate on the simplest C^0 shape functions, namely, the 1-D 2-node bar, $[\mathbf{N}(\xi)] = [1 - \xi, \xi]$, where $0 < \xi < 1$ is the non-dimensional length. The concept can be extended to higher dimensions and higher orders with the requirement that the Fourier series used must vanish at the nodes. For the particular 2-node element, the obvious enriching Fourier series is the sine series $\sin(i\xi\pi)$ which vanishes at $\xi = 0$ and $\xi = 1$ for all $i \in Z$, the set of natural numbers. Therefore, the Fourier enriched shape functions are $[\mathbf{N}(\xi)] = [1 - \xi, \xi, \sin(\xi\pi), \sin(2\xi\pi), \ldots]$. The sine functions represent internal degrees of freedom and are to be eliminated before assembling element

TABLE 1

Convergence of λ using cosine version with n terms

Mode	1	2	3	4	5	6
n	Free–free beam					
0	5·1800	9·5735				
1	5·1800	7·9495	20·9501			
2	4·7307	7·9495	13·1006	20·9501		
3	4·7307	7·8538	13·1006	14·5576	34·0701	
4	4·7301	7·8538	11·0079	14·5576	21·3524	34·0701
5	4·7301	7·8532	11·0079	14·1412	21·3116	21·3524
6	4·7300	7·8532	10·9968	14·1412	17·3141	21·3116
Exact	4·730	7·853	10·997	14·138	17·280	20·422
n	Hinged–free beam					
0	4·1886	8·3718				
1	4·1688	7·2755	17·7098			
2	3·9315	7·1132	11·5617	18·7348		
3	3·9315	7·0847	11·2851	14·1234	29·5235	
4	3·9276	7·0730	10·2462	13·5460	19·2050	30·5581
5	3·9276	7·0727	10·2462	13·3798	18·3093	21·3267
6	3·9269	7·0702	10·2190	13·3643	16·5760	20·0577
Exact	3·927	7·069	10·210	13·353	16·493	19·635
n	Clamped–free beam					
0	1·8796	5·8997				
1	1·8793	5·8991	8·3501			
2	1·8755	4·7086	7·9577	14·8457		
3	1·8755	4·7085	7·8800	14·0920	15·9468	
4	1·8752	4·6971	7·8643	11·0549	14·5386	24·1769
5	1·8752	4·6971	7·8641	11·0545	14·1664	21·0968
6	1·8751	4·6952	7·8588	11·0126	14·1561	17·3853
Exact	1·875	4·694	7·855	10·996	14·138	17·280

TABLE 2

Convergence of λ using sine version with n terms

Mode	1	2	3	4	5	6
n	Free–free beam					
0	5·1800	9·5735				
1	4·7331	9·5735	13·8613			
2	4·7331	7·8533	13·8613	18·2045		
3	4·7307	7·8533	10·9970	18·2045	22·5038	
4	4·7307	7·8532	10·9970	14·1428	22·5038	26·8155
5	4·7303	7·8532	10·9957	14·1428	17·2900	26·8155
6	4·7303	7·8532	10·9957	14·1378	17·2900	20·4388
Exact	4·730	7·853	10·997	14·138	17·280	20·422
n	Hinged–free beam					
0	4·1886	8·3718				
1	3·9396	8·1062	12·5779			
2	3·9279	7·1051	11·9101	16·7941		
3	3·9279	7·0812	10·2652	15·6884	21·0181	
4	3·9270	7·0753	10·2385	13·4290	19·4174	25·2504
5	3·9270	7·0721	10·2235	13·4002	16·5915	23·1238
6	3·9268	7·0710	10·2187	13·3739	16·5622	19·7550
Exact	3·927	7·069	10·210	13·352	16·493	19·635
n	Clamped–free beam					
0	1·8796	5·8997				
1	1·8755	4·7019	10·7863			
2	1·8752	4·6954	7·8755	15·3260		
3	1·8752	4·6954	7·8667	11·0293	19·8031	
4	1·8751	4·6945	7·8616	11·0222	14·1869	24·2010
5	1·8751	4·6945	7·8588	11·0094	14·1812	17·3436
6	1·8751	4·6943	7·8576	11·0057	14·1600	17·3394
Exact	1·875	4·694	7·855	10·996	14·138	17·280
n	Clamped–hinged beam					
0	4·5270					
1	3·9317	8·5582				
2	3·9295	7·0990	12·3920			
3	3·9273	7·0947	10·2738	16·1395		
4	3·9272	7·0761	10·2677	13·4504	19·8400	
5	3·9269	7·0760	10·2296	13·4423	16·6263	23·5111
6	3·9268	7·0717	10·2294	13·3862	16·6161	19·8005
Exact	3·927	7·069	10·210	13·352	16·493	19·635
n	Clamped–clamped beam					
1	4·7345					
2	4·7345	7·8826				
3	4·7312	7·8826	11·0551			
4	4·7312	7·8629	11·0551	14·2262		
5	4·7305	7·8629	11·0191	14·2262	17·3949	
6	4·7305	7·8575	11·0191	14·1770	17·3949	20·5610
Exact	4·730	7·853	10·997	14·138	17·280	20·422

matrices. Considering the axial vibration of a 2-node bar, the full mass and stiffness matrices are respectively given in close form by,

$$[\mathbf{M}] = \rho A L \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \left[\frac{1}{j\pi}\right] \\ \frac{1}{6} & \frac{1}{3} & \left[\frac{(-1)^{j+1}}{j\pi}\right] \\ \left\{\frac{1}{i\pi}\right\} & \left\{\frac{(-1)^{i+1}}{i\pi}\right\} & [\delta_{ij}/2] \end{bmatrix},$$

$$[\mathbf{K}] = EA/L \begin{bmatrix} 1 & -1 & [0] \\ -1 & 1 & [0] \\ \{0\} & \{0\} & [i^2\pi^2\delta_{ij}/2] \end{bmatrix},$$

where, $i, j = 1, 2, 3, \ldots$ Condensing all the internal degrees of freedom of $[\mathbf{K}] - \omega^2[\mathbf{M}]$ according to Leung [11] gives the dynamic stiffness matrix

$$[\mathbf{D}(\omega)] = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix},$$

TABLE 3

The square roots of the first 10 *natural frequencies of square plates having various boundary conditions using five terms of sine-version* (*the last row is supplied by the reviewer using* 10 *polynomial functions in each direction for comparison*)

Mode: Case	1	2	3	4	5	6	7	8	9	10
FFFF	0·0000	0·0000	0·0000	3·9787	4·7303	4·7303	6·2227	6·2227	7·8532	7·8532
FFHF	0·0000	2·8003	3·9270	5·2284	5·3043	7·0721	7·3636	7·9309	8·1189	9·7154
FFCF	1·9889	4·3945	4·4041	6·4137	7·3106	7·3117	8·7887	8·8049	10·3775	10·3777
FFCH	1·8751	3·1116	4·6945	5·3266	5·8086	7·6902	7·8588	8·1579	8·5727	9·9295
FFCC	2·4325	4·5248	5·0460	6·7983	7·3549	8·0513	9·0301	9·3382	10·3986	11·1416
HFHF	2·7352	5·1081	5·1634	7·1392	8·0847	8·0860	9·5101	9·5738	11·1578	11·1606
HFCF	3·9269	4·7084	6·5034	7·0760	7·6433	8·9342	9·0925	10·2296	10·6551	11·0878
HFCC	4·2639	6·1200	7·2611	8·5537	8·7042	10·3560	10·6096	11·3086	11·5900	12·9702
CFCF	5·2025	7·7904	7·8056	9·6546	10·7256	10·7318	12·1087	12·1189	13·8223	13·8248
CFCH	4·7305	5·2715	6·7914	7·8629	8·3227	9·0802	9·5817	11·0191	11·3848	11·4288
CFCC	4·8755	6·0655	7·9809	8·2600	8·8992	10·5837	10·9823	11·1117	11·8376	12·8074
CHCH	4·9461	6·4340	8·0089	8·8442	9·0956	10·9636	11·1265	11·6659	11·9556	13·3283
CHCC	5·6435	7·9691	8·4455	10·0706	10·8114	11·4466	12·3756	12·6764	13·8719	14·5397
CCCC	6·0020	8·5881	8·5881	10·4452	11·5083	11·5366	13·0111	13·0111	14·6429	14·6429
HFCH	4·1546	5·6917	7·2245	8·0878	8·3253	10·1960	10·3382	10·8933	11·1722	12·5442
	4·0963	5·5772	7·1686	8·0015	8·2175	10·0553	10·2751	10·8289	11·0777	12·4003

TABLE 4

Convergence study of a square CCCC plate for N number of sine terms and d.o.f. number of degrees of freedom

Mode: N	d.o.f.	1	2	3	4	5	6	7	8	9	10
1	1	6·0276									
2	3	6·0276	8·6356	8·6356							
3	6	6·0064	8·6356	8·6356	10·5260	11·5538	11·6216				
4	10	6·0064	8·5881	8·5881	10·5260	11·5538	11·6216	13·0111	13·0111	14·6429	14·6429
5	15	6·0020	8·5881	8·5881	10·4452	11·5083	11·5366	13·0111	13·0111	14·6429	14·6429
6	21	6·0020	8·5761	8·5761	10·4452	11·5083	11·5366	12·9102	12·9102	14·5654	14·5654
7	28	6·0004	8·5761	8·5761	10·4220	11·4889	11·5157	12·9102	12·9102	14·5654	14·5654
8	36	6·0004	8·5718	8·5718	10·4220	11·4889	11·5157	12·8767	12·8767	14·5379	14·5379
9	45	5·9997	8·5718	8·5718	10·4131	11·4808	11·5077	12·8767	12·8767	14·5379	14·5379
10	55	5·9997	8·5699	8·5699	10·4131	11·4808	11·5077	12·8628	12·8628	14·5259	14·5259
11	66	5·9994	8·5699	8·5699	10·4090	11·4769	11·5039	12·8628	12·8628	14·5259	14·5259
12	78	5·9994	8·5688	8·5688	10·4090	11·4769	11·5039	12·8560	12·8560	14·5198	14·5198
13	91	5·9992	8·5688	8·5688	10·4068	11·4748	11·5019	12·8560	12·8560	14·5198	14·5198
14	105	5·9992	8·5683	8·5683	10·4068	11·4748	11·5019	12·8524	12·8524	14·5163	14·5163
15	120	5·9990	8·5683	8·5683	10·4055	11·4736	11·5007	12·8524	12·8524	14·5163	14·5163
16	136	5·9990	8·5679	8·5679	10·4055	11·4736	11·5007	12·8502	12·8502	14·5143	14·5143
17	153	5·9990	8·5679	8·5679	10·4047	11·4728	11·5000	12·8502	12·8502	14·5143	14·5143
18	171	5·9990	8·5677	8·5677	10·4047	11·4728	11·5000	12·8488	12·8488	14·5130	14·5130
19	190	5·9989	8·5677	8·5677	10·4042	11·4723	11·4995	12·8488	12·8488	14·5130	14·5130
20	210	5·9989	8·5675	8·5675	10·4042	11·4723	11·4995	12·8479	12·8479	14·5121	14·5121
Leissa		5·9988	8·5670	8·5670	10·4027	11·4709	11·4979	12·8452	12·8452		

where

$$D_{11} = D_{22} = 1 - \frac{\omega^2}{3} - \sum_{i=1}^{\infty} \frac{2\omega^4}{i^2\pi^2(i^2\pi^2 - \omega^2)} = 1 - \frac{\omega^2}{3} - 2\omega^2 \sum_{i=1}^{\infty} \left[\frac{1}{i^2\pi^2 - \omega^2} - \frac{1}{i^2\pi^2}\right]$$

$$= 1 - 2\omega^2 \sum_{i=1}^{\infty} \frac{1}{i^2\pi^2 - \omega^2} = \omega \cot \omega,$$

and, similarly, $D_{12} = D_{21} = -\omega \csc \omega$, which is the exact dynamic stiffness matrix [12]. For the natural vibration of a hinged free rod, let $\omega \cot \omega = 0$ to obtain the non-dimensional frequencies ω with the non-dimensional factor $L(\rho/E)^{1/2}$, $\omega = (2m+1)/2$, $m = 1, 2, 3, \ldots$, which are also exact. This demonstrates the convergence of the method for single element models. For a bar with linearly varying cross-sectional area $A(\xi) = 1 - 0{\cdot}5\xi$, the non-dimensional mass and stiffness matrices become

$$[\mathbf{M}] = \int (1 - 0{\cdot}5\xi)[\mathbf{N}]^T[\mathbf{N}]\, d\xi = \begin{bmatrix} \frac{7}{24} & \frac{1}{8} & m_{13} \\ \frac{1}{8} & \frac{5}{24} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$[\mathbf{K}] = \int (1 - 0{\cdot}5\xi)[\mathbf{N}']^T[\mathbf{N}']\, d\xi = \begin{bmatrix} \frac{3}{4} & -\frac{3}{4} & k_{13} \\ -\frac{3}{4} & \frac{3}{4} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix},$$

where $m_{31} = m_{13}^T = \{(-1 + (-1)^j + j^2\pi^2)/(j^3\pi^3)\}$, $m_{32} = m_{23}^T = \{(2 - 2(-1)^j - (-1)^j j^2\pi^2)/(2j^3\pi^3)\}$, $m_{33}(i \neq j) = \{((1 - (-1)^{i+j})ij)/((-i^2 + j^2)2\pi^2)\}$, $m_{33}(i = j) = 3/8$, $k_{31} = k_{13}^T = \{(-1 + (-1)^j)/(2j\pi)\} = -k_{32} = -k_{23}^T$, $k_{33}(i \neq j) = \{((1 - (-1)^{i+j})ij(i^2 + j^2))/(2(-i^2 + j^2)^2)\}$, $k_{33}(i = j) = \{(3i^2\pi^2)/8\}$. With these matrices, one can find the natural modes of any combination of tapered and stepped bars.

3. FOURIER-VERSION C[1] SHAPE FUNCTIONS

For bending elements, the shape functions are required to give displacement and slope continuity at element interfaces. When considering the bending of a beam of unit length, the appropriate shape functions are $[\mathbf{N}] = [1 - 3\xi^2 + 2\xi^3, \xi(1 - 2\xi + \xi^2), 3\xi^2 - 2\xi^3, \xi(\xi^2 - \xi)]$. For C^1 requirement, the Fourier version is either $1 - \cos j\pi\xi$ or $(\xi - \xi^2)\sin j\pi\xi$, both functions and their first derivatives vanish at $\xi = 0$ and $\xi = 1$. For simplicity, we called the cosine version $[\mathbf{N}_c] = [1 - 3\xi^2 + 2\xi^3, \xi(1 - 2\xi + \xi^2), 3\xi^2 - 2\xi^3, \xi(\xi^2 - \xi), 1 - \cos j\pi\xi]$ and sine version $[\mathbf{N}_s] = [1 - 3\xi^2 + 2\xi^3, \xi(1 - 2\xi + \xi^2), 3\xi^2 - 2\xi^3, \xi(\xi^2 - \xi), (\xi - \xi^2)\sin j\pi\xi]$, respectively. The cosine version is the simplest. However, it produces zero shear forces at the nodes and is too flexible for shear connections. The cosine version is not recommended when the structure consists of just one element. Tables 1 and 2 show the convergence of the natural frequency parameter λ, where $\lambda^4 = \omega^2\rho AL^4/EI$ by using cosine and sine versions, respectively. The cosine version fails to predict the clamped–hinged and clamped–clamped modes. The convergence is excellent, e.g., six terms predict the sixth mode with about 0·5% relative error. The total number of degrees of freedom in the examples is equal to that of the usual finite element model plus the number of trigonometric terms.

4. VIBRATION OF SQUARE PLATES

With the mass and stiffness matrices of a Euler beam, the same matrices of a 16-degree-of-freedom rectangular plate in bending can easily be constructed. The first ten modes of a square plate of side L with various boundary conditions using five sine terms were computed and are tabulated in Table 3. The 15 boundary conditions are, respectively, FFFF, FFHF, FFCF, FFCH, FFCC, HFHF, HFCF, HFCC, CFCF, CFCH, CFCC, CHCH, CHCC, CCCC and HFCH. The non-dimensional frequency factor for the plate is λ, where $\lambda = \omega^{1/2}L(\rho h/D)^{1/4}$, h is the plate thickness, $D = Eh^3/[12(1 - \nu^2)]$ its flexural rigidity, and Poisson's ratio $\nu = 0{\cdot}3$. The reviewer supplied the last row of Table 3 for comparison with polynomial functions method. Table 4 gives the convergence study of the first 10 natural frequency parameters of a square CCCC plate by means of a different number of sine terms and compares them to the exact solution using the infinite determinant method [13]. It is shown that the five-term sine version produces extremely good results.

5. CONCLUSION

It is recommended that the product of polynomials and Fourier series are used instead of polynomials alone in the p-element shape functions. Due to the fact that Fourier series are the well behaved the limitation of the polynomial functions disappears. By using Fourier series instead of polynomials to formulate the p-version finite element, one can eliminate the inherent ill-conditioning problem of polynomials and orthogonal polynomials, which are perfectly good for low frequency analysis which in turn does not require many terms. When applied to the natural vibration analysis of structures it is found as a bonus that higher modes converge much faster than when using polynomials alone.

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